STUDIA MATHEMATICA, T. XLII. (1972)

Bases in weakly sequentially complete Banach spaces

by

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Abstract. A theorem of Zippin states that if every basis of a Banach space X is boundedly-complete, then X is reflexive; we here obtain a similar characterization of weakly sequentially complete Banach spaces. A basis is called β-complete if it induces a β-perfect sequence space; the main result of this paper is that a Banach space with a basis is weakly sequentially complete if and only if every basis is β-complete

Let X be a Banach space, let (x_n) be a Schauder basis of X and let (f_n) denote the dual sequence of continuous linear functionals on X; thus for each $x \in X$

$$x = \sum_{n=1}^{\infty} f_n(x) x_n.$$

Then we say that (x_n) is *shrinking* if (f_n) is a basis of X^*, and that (x_n) is *boundedly-complete* if whenever (a_n) is a sequence of scalars such that

$$\sup_n \Big\| \sum_{i=1}^{n} a_i x_i \Big\| < \infty,$$

then $\sum_{i=1}^{\infty} a_i x_i$ converges.

These classical definitions lead to the following two well-known theorems.

THEOREM 1. (James [2]). *X is reflexive if and only if (x_n) is shrinking and boundedly-complete.*

THEOREM 2. (Zippin [9]). *If X has a basis then the following are equivalent:*

(i) *X is reflexive.*

(ii) *Every basis of X is shrinking.*

(iii) *Every basis of X is boundedly-complete.*

The theory of bases is related to the theory of sequence spaces

* While engaged on this research, the author was in part supported by an S.R.C. Resettlement Fellowship.

for we may define the sequence space

$$\lambda_x = \Big\{(a_n);\ \sum_{n=1}^{\infty} a_n x_n \text{ converges}\Big\}.$$

For a given sequence space μ we define (see Garling [1])

$$\mu^\beta = \Big\{(a_n);\ \sum_{n=1}^{\infty} a_n b_n \text{ converges for all } (b_n)\,\epsilon\,\mu\Big\},$$

$$\mu^\gamma = \Big\{(a_n);\ \sup_m \Big|\sum_{n=1}^{m} a_n b_n\Big| < \infty \text{ for all } (b_n)\,\epsilon\,\mu\Big\}.$$

Then it may be easily seen that (x_n) is boundedly-complete if $\lambda_x = \lambda_x^{\gamma\gamma}$ (for other results of a similar nature, see Ruckle [7]). One naturally asks what is obtained if one only assumes $\lambda_x = \lambda_x^{\beta\beta}$ (a weaker assumption, see [1]). This is equivalent to the following definition (see [3]):

DEFINITION 1. (x_n) *is β-complete if whenever* $\big(\sum_{i=1}^{n} a_i x_i\big)_{n=1}^{\infty}$ *is a weakly Cauchy sequence then* $\sum_{i=1}^{\infty} a_i x_i$ *converges.*

This leads to a modification of Theorem 1.

THEOREM 3. ([3]). *X is reflexive if and only if (x_n) is shrinking and β-complete.*

In this paper we establish a result related to Zippin's Theorem 2, characterizing spaces in which every basis is β-complete. It is obvious that if X is weakly sequentially complete then any basis of X is β-complete; thus, for example, the space l^1 of absolutely convergent sequences is not reflexive but every basis of l^1 is β-complete. The main theorem of this paper will demonstrate that the property of having every basis β-complete characterizes weak sequential completeness.

The proof will depend on a refinement of a very useful lemma discovered by Zippin [9]; we shall call a sequence (y_n) *semi-normalised* (this terminology follows Pełczyński [6]; the term "normalised" has been used by the author for the equivalent property in a locally convex space) if $0 < \inf\|y_n\| < \sup\|y_n\| < \infty$.

PROPOSITION 1. *Let (x_n) be a basis of X and suppose that (p_n) is an increasing sequence of integers with $p_0 = 0$ and $p_n - p_{n-1} \geqslant 1$; let $X_n = \operatorname{lin}(x_{p_{n-1}+1}, \ldots, x_{p_n})$. Let $u_n \,\epsilon\, X_n$ and $\varphi_n \,\epsilon\, X_n^*$ be two sequences with $\varphi_n(u_n) = 1$ and $\sup_n \|u_n\|\ \|\varphi_n\| < \infty$; then there is a basis (y_n) of X with $y_{p_n} = u_n$ and $\varphi_n(y_i) = 0$ for $p_{n-1}+1 \leqslant i \leqslant p_n - 1$.*

Proof. The proof is essentially that of Zippin. Let $Z_n = \varphi_n^{-1}(0)$ and $Y_n = \operatorname{lin}(x_{p_{n-1}+1}, \ldots, x_{p_n-1})$; then by a lemma of Zippin there is a linear isomorphism $T_n\colon Y_n \to Z_n$ such that

$$\|T_n\|\cdot\|T_n^{-1}\| \leqslant 9.$$

Then let

$$y_i = T_n x_i, \qquad i = p_{n-1}+1, \ldots, p_n - 1,$$
$$y_{p_n} = u_n.$$

It is clear that (y_k) is fundamental in X and it remains to be shown that (y_k) is a basic sequence. There is a constant $K > 0$ (the *basis constant*) such that for all sequences of scalars (c_k) and $r < s$

$$\Big\|\sum_{i=1}^{r} c_i x_i\Big\| \leqslant K \Big\|\sum_{i=1}^{s} c_i x_i\Big\|.$$

Now let $x = \sum_{i=1}^{p_n} c_i y_i = \sum_{i=1}^{p_n} d_i x_i$ and suppose $r < p_n$. If $r = p_m$ for some m

$$\Big\|\sum_{i=1}^{r} c_i y_i\Big\| = \Big\|\sum_{i=1}^{r} d_i x_i\Big\| \leqslant K\|x\|.$$

If $p_{m-1}+1 \leqslant r \leqslant p_m - 1$

$$\begin{aligned}
\Big\|\sum_{i=1}^{r} c_i y_i\Big\| &\leqslant \Big\|\sum_{i=1}^{p_{m-1}} c_i y_i\Big\| + \Big\|\sum_{p_{m-1}+1}^{r} c_i y_i\Big\| \\
&\leqslant K\|x\| + \|T_m\|\,\Big\|\sum_{p_{m-1}+1}^{r} c_i x_i\Big\| \\
&\leqslant K\|x\| + K\|T_m\|\,\Big\|\sum_{p_{m-1}+1}^{p_m-1} c_i x_i\Big\| \\
&\leqslant K\|x\| + K\|T_m\|\cdot\|T_m^{-1}\|\,\Big\|\sum_{p_{m-1}+1}^{p_m-1} c_i y_i\Big\| \\
&\leqslant K\|x\| + 9K\Big(\Big\|\sum_{p_{m-1}+1}^{p_m} c_i y_i\Big\| + |c_{p_m}|\,\|u_m\|\Big) \\
&\leqslant K\|x\| + 9K(1+\|\varphi_m\|\ \|u_m\|)\,\Big\|\sum_{p_{m-1}+1}^{p_m} c_i y_i\Big\| \\
&\leqslant \|x\|\big(K + 18K^2(1+\|\varphi_m\|\ \|u_m\|)\big).
\end{aligned}$$

Hence for all r

$$\Big\|\sum_{i=1}^{r} c_i y_i\Big\| \leqslant C\|x\|,$$

where $C = K + 18K^2(1+\sup_m \|\varphi_m\|\ \|u_m\|)$.

Thus (y_n) is a basic sequence.

Zippin's lemma in its original form states that if $u_n \neq 0$ and $u_n \in X_n$ then there is a basis (y_n) with $y_{p_n} = u_n$. This follows by choosing φ_n such that $\varphi_n(u_n) = 1$ and $\|\varphi_n\| \cdot \|u_n\| = 1$.

PROPOSITION 2. *Under the same assumptions as Proposition 1, suppose $u_n \in X_n$ and $v_n \in X_n$ are two semi-normalised sequences. Then there is a basis (y_n) of X with $y_{p_n} = u_n$ and $y_{p_n-1} = v_n$ if and only if*

$$\inf_n \inf_c \|u_n + cv_n\| = \sigma > 0.$$

Proof. Suppose there is a basis (y_n) with $y_{p_n-1} = v_n$ and $y_p = u_n$. Then there is a constant K such that if $r < s$

$$\Big\|\sum_{i=1}^{r} c_i y_i\Big\| \leqslant K \Big\|\sum_{i=1}^{s} c_i y_i\Big\|$$

for all sequences (c_k). Thus, as

$$\|u_n\| \leqslant \|u_n + cv_n\| + \|cv_n\| \leqslant (1+K)\|u_n + cv_n\| \quad \text{for all } c$$

if

$$0 < \inf_n \|u_n\| \leqslant (\inf_n \inf_c \|u_n + cv_n\|)(1+K)$$

so that $\sigma > 0$.

Conversely suppose $\delta > 0$. Then the linear functional ψ_n on $\mathrm{lin}(u_n; v_n)$ given by $\psi_n(u_n) = 1$ and $\psi_n(v_n) = 0$ satisfies

$$\|\psi_n\| \leqslant \delta^{-1} \quad \text{for } \psi_n(au_n + bv_n) = a \text{ and } \|au_n + bv_n\| \geqslant |a|\,\delta.$$

Extend ψ_n to a linear functional φ_n on X_n with $\|\varphi_n\| \leqslant \delta^{-1}$, by the Hahn–Banach Theorem. Then by Lemma 1, there is a basis (z_n) of X with $z_{p_n} = u_n$ and $\varphi_n(z_i) = 0$, $i = p_{n-1}+1, \ldots, p_n - 1$. Clearly $v_n \in \mathrm{lin}(z_{p_{n-1}+1}, \ldots, z_{p_n-1})$ and by applying Zippin's lemma in its original form there is a basis (y_n) of X with $y_{p_n-1} = v_n$ and $y_{p_n} = u_n$.

Before proceeding with the proof of the main theorem, it is useful to introduce a further concept intermediate to β-completeness and boundedly-completeness.

DEFINITION 2. *A basis (x_n) is said to be totally β-complete if whenever $\big(\sum_{i=1}^{p_n} a_i x_i\big)_{n=1}^{\infty}$ is a weakly Cauchy sequence for some sequence $\{p_n\}$ with $p_n \to \infty$, then $\sum_{i=1}^{\infty} a_i x_i$ converges.*

PROPOSITION 3. *A boundedly-complete basis is totally β-complete.*

Proof. If $\sum_{i=1}^{p_n} a\, x_i$ is a weakly Cauchy sequence, then

$$\sup_n \Big\|\sum_{i=1}^{p_n} a_i x_i\Big\| = M.$$

Let K be the basis constant; then

$$\sup_n \Big\|\sum_{i=1}^{n} a_i x_i\Big\| \leqslant KM.$$

As (x_n) is boundedly-complete, $\sum_{i=1}^{\infty} a_i x_i$ converges.

PROPOSITION 4. *If X is a Banach space with a basis, and every basis of X is β-complete, then every basis of X is totally β-complete.*

Proof. Let (x_n) be a basis of X and suppose that the sequence

$$\Big(\sum_{i=1}^{p_n} a_i x_i\Big)_{n=1}^{\infty}$$

is weakly Cauchy, where $p_n > p_{n-1}$ for all n. It may be assumed that an infinite number of $(a_i)_{i=1}^{\infty}$ are non-zero, and that, for each n, $\sum_{p_{n-1}+1}^{p_n} a_i x_i \neq 0$. Then by Proposition 1, there is a basis (y_n) of X with $y_{p_n} = \sum_{p_{n-1}+1}^{p_n} a_i x_i$; then $\big(\sum_{n=1}^{m} y_{p_n}\big)_{m=1}^{\infty}$ is weakly Cauchy and as (y_n) is β-complete, $\sum_{n=1}^{\infty} y_{p_n}$ exists and

$$\sum_{n=1}^{\infty} y_{p_n} = \lim_{m\to\infty} \sum_{i=1}^{p_m} a_i x_i \quad \text{weakly}.$$

Clearly

$$\sum_{n=1}^{\infty} y_{p_n} = \sum_{i=1}^{\infty} a_i x_i.$$

Two further ideas from [4] and [8] will be required for the main theorem. If (x_n) is a Schauder basis of X with dual sequence (f_n) then we say a subsequence (x_{p_n}) is a *type P subsequence* if

$$\inf_n \|x_{p_n}\| \neq 0 \quad \text{and} \quad \sup_n \Big\|\sum_{i=1}^{n} x_{p_i}\Big\| < \infty$$

and is a *type P^* subsequence* if

$$\sup_n \|x_{p_n}\| < \infty \quad \text{and} \quad \sup_n \Big\|\sum_{i=1}^{n} f_{p_i}\Big\| < \infty.$$

Then the following result is proved in [4].

PROPOSITION 5. *If (x_{p_n}) is a type P subsequence of (x_n) then the sequence (y_n) given by $y_i = x_i$, $i \neq p_n$, and $y_{p_n} = \sum_{i=1}^{n} x_{p_i}$ is a basis of X. If (x_{p_n}) is a type P^* subsequence of (x_n) then the sequence (y_n) given by $y_i = x_i$, $i \neq p_n$ and $y_{p_n} = x_{p_n} - x_{p_{n-1}}$ (where $x_{p_0} = 0$) is a basis of X.*

THEOREM 4. *Let X be a Banach space with a basis and suppose every basis of X is β-complete; then X is weakly sequentially complete.*

Proof. Let (x_n) be a basis of X; we shall prove three lemmas under the assumption that every basis of X is β-complete.

LEMMA 1. *Let (z_n) be a sequence of the form*

$$z_n = \sum_{i=p_{n-1}+1}^{p_n} a_i x_i, \quad \text{where } p_0 = 0 < p_1 < p_2 \ldots$$

If (z_n) is weakly Cauchy then $\lim_{n\to\infty} z_n = 0$ weakly.

Proof. We may assume $z_n \neq 0$ for all n; if the lemma is false there exists $\varphi \in X^*$ with $\lim_{n\to\infty} \varphi(z_n) = 1$. We further assume (discarding, if necessary, a finite number of the sequence (z_n)) that $\varphi(z_n) \neq 0$ for all n. Then let $u_n = \frac{1}{\varphi(z_n)} z_n$, and let φ_n be the restriction of φ to $X_n = \mathrm{lin}(x_{p_{n-1}+1}, \ldots$ $\ldots, x_{p_n})$. We have

$$\inf_n |\varphi(z_n)| = \delta > 0$$

and as (z_n) is a weakly Cauchy sequence

$$\sup_n \|z_n\| = M < \infty,$$

so that

$$\sup_n \|u_n\| \leqslant M\delta^{-1} < \infty,$$

and hence

$$\sup_n \|\varphi_n\| \; \|u_n\| \leqslant M\delta^{-1}\|\varphi\|.$$

Then by Proposition 1, there is a basis (y_n) of X with $y_{p_n} = u_n$ and $\varphi(y_i) = 0$ for $p_{n-1} < i < p_n$. Let (g_n) be the sequence dual to (y_n) in X^*; in the weak*-topology of X^*,

$$\varphi = \sum_{n=1}^{\infty} g_{p_n}$$

so that $\sup \|\sum_{i=1}^{n} g_{p_i}\| < \infty$.

Certainly $\|y_{p_n}\| = \|u_n\| \leqslant M\delta^{-1}$.

Thus (y_{p_n}) is a subsequence of (y_n) of type P^* and the sequence (w_n) given by

$$w_i = y_i \quad (i \neq p_n), \qquad w_{p_n} = y_{p_n} - y_{p_{n-1}}$$

is a basis of X (Proposition 5).

Then

$$\sum_{i=1}^{n} w_{p_i} = y_{p_n} = u_n = \frac{1}{\varphi(z_n)} z_n$$

is a weakly Cauchy sequence. As (w_n) is β-complete

$$\sum_{i=1}^{\infty} w_{p_i} = \lim_{n\to\infty} \frac{1}{\varphi(z_n)} z_n = \lim_{n\to\infty} z_n \quad \text{exists weakly.}$$

But for all k

$$f_k(\lim_{n\to\infty} z_n) = 0$$

so that $\sum_{i=1}^{\infty} w_{p_i} = 0$ contradicting the fact that (w_n) is a basis of X.

LEMMA 2. *Suppose that for a sequence $(a_n)_{n=1}^{\infty}$ of scalars*

$$\Big\| \sum_{p_{n-1}+1}^{p_n} a_i x_i \Big\| \geqslant \delta > 0 \quad \text{where } p_0 = 0 < p_1 < p_2 \ldots$$

and that $v_n \in X_n = \mathrm{lin}(x_{p_{n-1}+1}, \ldots, x_{p_n})$; then the sequence $z_n = v_n + \sum_{i=1}^{p_{n-1}} a_i x_i$ is not a weakly Cauchy sequence.

Proof. Suppose (z_n) is weakly Cauchy; then

$$\sup \|z_n\| = M < \infty$$

and if K is the basis constant of (x_n), $\|\sum_{i=1}^{p_n} a_i x_i\| \leqslant KM$ for all n and $\|v_n\| \leqslant (1+K)M$.

Since by Proposition 4, (x_n) is totally β-complete, no subsequence of $\big(\sum_{i=1}^{p_n} a_i x_i;\ n = 1, 2 \ldots\big)$ is weakly Cauchy and it follows, that since (z_n) is weakly Cauchy,

$$\liminf_{n\to\infty} \|v_n\| \neq 0, \quad \liminf_{n\to\infty} \Big\| \sum_{i=p_{n-1}+1}^{p_n} a_i x_i - v_n \Big\| \neq 0.$$

Hence there exists k_0 and $\varepsilon > 0$ such that for $n \geqslant k_0$

$$\|v_n\| \geqslant \varepsilon, \quad \|u_n\| \geqslant \varepsilon, \quad \text{where } u_n = \Big(\sum_{i=p_{n-1}+1}^{p_n} a_i x_i\Big) - v_n.$$

Suppose now that for some $k \geqslant k_0$ and $\gamma > 0$ $\|u_n + cv_n\| \geqslant \gamma$ whenever $n \geqslant k$ and c is any scalar. Then by Proposition 2, there is a basis (y_n) of X with $y_{p_n} = u_n$ and $y_{p_n - 1} = v_n$ for $n \geqslant k$; the sequence

$$w_n = y_{p_k - 1} + \sum_{i=k}^{n-1} (y_{p_i} + y_{p_{i-1}})$$

is then weakly Cauchy, since

$$w_n = v_n + \sum_{p_{k-1}+1}^{p_{n-1}} a_i x_i = z_n - \sum_{i=1}^{p_{k-1}} a_i x_i.$$

Thus as (y_n) is totally β-complete, (w_n) and hence (z_n) converges weakly. But $f_k(\lim z_n) = a_k$ so that $\lim z_n = \sum_{k=1}^{\infty} a_k x_k$ contradicting the fact that

$$\Big\| \sum_{p_{n-1}}^{p_n} a_i x_i \Big\| \geqslant \delta \quad \text{for all } n.$$

Hence we conclude that

$$\liminf_{n\to\infty} \inf_{c} \|u_n + cv_n\| = 0.$$

There exists a sequence $\lambda_n \to \infty$ and c_n, real, such that $\|u_{\lambda_n} + c_n v_{\lambda_n}\| \to 0$ and as it is easily seen that $\sup_n |c_n| < \infty$, we may assume that $\lim_{n\to\infty} c_n = c$ exists. Thus $\|u_{\lambda_n} + cv_{\lambda_n}\| \to 0$ and hence

$$\Big\| z_{\lambda_n}(c-1) + \sum_{i=1}^{p_{\lambda_n}} a_i x_i - c \sum_{i=1}^{p_{\lambda_n-1}} a_i x_i \Big\| \to 0.$$

If $c = 1$, then $\lim \Big\| \sum_{i=p_{\lambda_{n}-1}+1}^{p_{\lambda_n}} a_i x_i \Big\| = 0$. But

$$\delta \leqslant \Big\| \sum_{i=p_{\lambda_n-1}}^{p_{\lambda_n}} a_i x_i \Big\|.$$

Hence $c \neq 1$ and the sequence $\sum_{i=1}^{p_{\lambda_n}} a_i x_i - c \sum_{i=1}^{p_{\lambda_n-1}} a_i x_i$ is weakly Cauchy.

By Proposition 2, there is a basis (t_n) of X with $t_{p_n} = \sum_{p_{n-1}+1}^{p_n} a_i x_i$. Then $\|t_{p_n}\| \geqslant \delta$ and

$$\Big\| \sum_{i=1}^{n} t_{p_i} \Big\| \leqslant KM$$

so that (t_{p_n}) is a subsequence of type P. Hence there is a basis (s_n) with $s_{p_n} = \sum_{i=1}^{p_n} a_i x_i$ (Proposition 5). Then $s_{p_{\lambda_n}} - cs_{p_{\lambda_n-1}}$ is a weakly Cauchy sequence, and by applying Lemma 1 to the subsequence $s_{p_{\lambda_{2n}}} - cs_{p_{\lambda_{2n}-1}}$ we obtain that

$$\lim_{n\to\infty} \sum_{i=1}^{p_{\lambda_n}} a_i x_i - c \sum_{i=1}^{p_{\lambda_n-1}} a_i x_i = 0 \quad \text{weakly}$$

and hence $\lim_{n\to\infty} (c-1) z_{p_{\lambda_n}} = 0$ weakly.

As $c \neq 1$ $\lim_{n\to\infty} z_{p_{\lambda_n}} = 0$ weakly. But $\lim_{n\to\infty} f_k(z_{p_{\lambda_n}}) = a_k$ and so $a_k = 0$ for all n, contradicting the fact that,

$$\Big\| \sum_{p_{n-1}+1}^{p_n} a_i x_i \Big\| \geqslant \delta.$$

Lemma 3. *If (z_n) is a weakly Cauchy sequence and $\lim_{n\to\infty} f_k(z_n) = a_k$, then $\sum_{k=1}^{\infty} a_k x_k$ converges.*

Proof. If the lemma is false, there exists $\delta > 0$ such that for any n, m there exists $N \geqslant m$ such that

$$\Big\| \sum_{i=n+1}^{N} a_i x_i \Big\| \geqslant \delta.$$

Now choose $(m_n)_{n=1}^{\infty}$ and $(p_n)_{n=0}^{\infty}$ thus: let $p_0 = 0$ and suppose that $(p_n)_{n<k}$ and $(m_n)_{n<k}$ have been chosen (where $k \geqslant 1$). Then choose

(i) $m_k > m_{k-1}$ such that $\Big\| \sum_{i=1}^{p_{k-1}} a_i x_i - \sum_{i=1}^{p_{k-1}} f_i(z_{m_k}) x_i \Big\| \leqslant \frac{1}{2^{k+1}}$,

(ii) $p_k > p_{k-1}$ such that $\Big\| \sum_{p_k+1}^{\infty} f_i(z_{m_k}) x_i \Big\| \leqslant \frac{1}{2^{k+1}}$ and $\Big\| \sum_{p_{k-1}+1}^{p_k} a_i x_i \Big\| \geqslant \delta$.

Then the sequence

$$u_n = \sum_{i=1}^{p_n} a_i x_i + \sum_{i=p_{n-1}+1}^{p_n} f_i(z_{m_n}) x_i$$

is weakly Cauchy and a contradiction is obtained by Lemma 2.

We are now in a position to prove Theorem 4; let (z_n) be a weakly Cauchy sequence which does not converge weakly. Then if $\lim_{n\to\infty} f_k(z_n) = a_k$, by Lemma 3 there exists $z \in X$ with $z = \sum_{k=1}^{\infty} a_k x_k$. Then the series $(z_n - z)$ is weakly Cauchy and $\lim_{n\to\infty} f_k(z_n - z) = 0$.

We then determine increasing sequences $(p_n)_{n=0}^{\infty}$ and $(m_n)_{n=1}^{\infty}$ inductively. Let $p_0 = 0$ and suppose $(p_n)_{n<k}$ and $(m_n)_{n<k}$ have been determined; then choose

(i) $m_k > m_{k-1}$ such that $\Big\| \sum_{i=1}^{p_{k-1}} f_i(z_{m_k}) x_i - \sum_{i=1}^{p_{k-1}} a_i x_i \Big\| \leqslant 1/2^{k+1}$,

(ii) $p_k > p_{k-1}$ such that $\Big\| \sum_{i=p_k+1}^{\infty} f_i(z_{m_k}) x_i \Big\| \leqslant 1/2^{k+1}$.

Then consider the sequence $u_n = \sum_{i=p_{n-1}+1}^{p_n} f_i(z_{m_n})x_i$

$$\Big\| u_n + \Big(\sum_{i=1}^{p_{n-1}} a_i x_i\Big) - z_{m_n} \Big\| \leqslant 1/2^n$$

and hence $(u_n + \sum_{i=1}^{p_{n-1}} a_i x_i)_{n=1}^{\infty}$ is weakly Cauchy. It follows that $(u_n)_{n=1}^{\infty}$ is weakly Cauchy and by Lemma 1 $\lim_{n\to\infty} u_n = 0$ weakly.

Therefore

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} z_{m_n} = \lim_{n\to\infty} \sum_{i=1}^{p_{n-1}} a_i x_i = z$$

and (z_n) is a weakly convergent sequence.

Next we apply Theorem 4, using a result due to Pełczyński (see [5]) to obtain a characterization of all weakly sequentially complete Banach spaces.

PROPOSITION 6. *If X is a Banach space in which every closed subspace with a basis is weakly sequentially complete, then X is weakly sequentially complete.*

Proof. This is essentially the first half of the proof of Theorem 2 of [5].

THEOREM 5. *If X is a Banach space and every basic sequence in X is β-complete then X is weakly sequentially complete.*

Proof. By Theorem 4 and Proposition 6.

We conclude with two remarks. First it should be observed that if X possesses a β-complete unconditional basis then X is weakly sequentially complete: for it is easily seen that the basis is in fact boundedly-complete. Secondly it is possible for a Banach space to possess a totally β-complete (or indeed boundedly-complete basis) and yet fail to be weakly sequentially complete. For let X be the non-reflexive space of James [2]; then X^* possesses a boundedly-complete basis, but if X^* is weakly sequentially complete, then X^{**} would be inseparable, contradicting the fact that X has codimension one in X^{**}.

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Received November 12, 1970 (272)