[5] R. Jajte, *On stable distributions in Hilbert space*, Studia Math. 30 (1964), pp. 63–71.
[6] J. Kuelbs and V. Mandrekar, *Harmonic analysis on certain vector spaces*, Trans. Amer. Math. Soc. 14a (1970), pp. 213–231.
[7] — — *Harmonic analysis on F-spaces with a basis*, (ibidem, to appear).
[8] M. Loève, *Probability Theory*, 3rd ed., New York 1963.
[9] K. S. Parthasarathy, *Probability measures in metric spaces*, 1967.
[10] K. Urbanik, *Generalized convolutions*, Studia Math. 23 (1964), pp. 217–245.
[11] S. R. S. Varadhan, *Limit theorems for sums of independent random variables with values in a Hilbert space*, Sankhya 24 (1962), pp. 213–238.
[12] A. C. Zaanen, *Linear Analysis*, New York, 1953.

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Unitary representations induced from compact subgroups

by

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Abstract. It is shown, for the case in which the subgroup is compact, that the induced representations of Mackey and Mautner can be defined in terms of certain Hilbert module tensor products, or, alternatively, certain spaces of Hilbert–Schmidt intertwining operators. These definitions are used to derive basic properties of induced representations, and the connection with Blattner's approach in terms of positive type measures is discussed.

Let H be a compact subgroup of a locally compact group G. Mackey ([13], [14], [15]) and Mautner ([18], [19]), using definitions involving certain spaces of measurable vector valued functions, showed how to induce representations of H up to G. (Mackey, in fact, treated the more general case in which H need not be compact). In the present paper we show how these induced representations can be defined in terms of certain Hilbert module tensor products, or, alternatively, in terms of certain spaces of Hilbert–Schmidt intertwining operators. Such definitions enable us to give convenient derivations of the basic properties of induced representations along lines which follow fairly closely the theory of induced representations as it is developed for finite groups (for which see [2]). Our approach is also quite similar to that for induced Banach space representations which we gave in [22].

The exposition is organized in the following way. In Section 1 we consider the basic properties of the Hilbert space tensor product. The principal result is that this tensor product provides the left adjoint for the construction of spaces of Hilbert–Schmidt operators. We believe that this result is new, although the interconnection between the Hilbert space tensor product and Hilbert–Schmidt operators is found implicitly in a number of papers. In Section 2 the results of Section 1 are extended to the setting of Hilbert spaces which are modules over sets, and in Section

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3 the special case in which these sets are groups or algebras is considered. Section 4, the heart of the paper, contains the definitions of induced representations and the derivation of their basic properties, such as the Frobenius reciprocity theorem for compact groups. In the process we also show that if H is a compact subgroup of a locally compact group G which is not compact, then the restriction functor from unitary representations of G to unitary representations of H has neither a left nor a right adjoint. Section 5 is devoted to showing that the induced representations studied in Section 4 are the same as those of Mautner and the special case of those of Mackey in which the subgroup is compact. In the course of doing this we show that the two types of induced representations studied in Section 4 (analogues of the left and right adjoints of the restriction functor) are naturally equivalent, in analogy with the situation for finite groups and Frobenius extensions of algebras [20]. Finally, in Section 6 we show how our definitions of induced representations are related to the approach to induced representations developed by Blattner [1] using positive definite measures.

It would, of course, be very desirable to be able to define the induced representations of Mackey in terms of some kind of tensor product in the more general case in which the subgroup need not be compact. But we have had no success in trying to find such a definition. At the time of writing this paper we suspect that some construction more general than a tensor product will be needed in order to be able to do this.

In a subsequent paper we will use many of the results and ideas of the present paper to discuss unitary representations of a compact Lie group, G, which are induced directly from representations of subalgebras of the complexified Lie algebra of G. This will permit, among other things, a quite simple approach to a theorem of Borel and Weil giving a concrete realization for the irreducible representation of G of given highest weight in terms of holomorphic sections of an appropriate line bundle.

1. Hilbert space tensor products and Hilbert–Schmidt operators. In this section we recall the definition and basic properties of the tensor product of Hilbert spaces (see [4], [25] or [26] for details), and we show that this tensor product provides the left adjoint for the functor consisting of forming spaces of Hilbert–Schmidt operators.

Let V and W be Hilbert spaces (all vector spaces will be over the complex numbers), and let $V\otimes W$ be their algebraic tensor product. There is a unique inner product on $V\otimes W$ whose value on elementary tensors is given by

$$\langle v\otimes w,\ v'\otimes w'\rangle = \langle v,\ v'\rangle\langle w,\ w'\rangle.$$

The completion of $V\otimes W$ with respect to this inner product is called the Hilbert space tensor product of V and W. Throughout the rest of this paper the symbol $\otimes$ will always denote this Hilbert space tensor product.

The Hilbert space tensor product is commutative and associative, and it is distributative with respect to direct sums. That is, for Hilbert spaces V, W and X there are natural (unitary) isomorphisms

$$V\otimes W \cong W\otimes V,$$

$$V\otimes(W\otimes X) \cong (V\otimes W)\otimes X,$$

and

$$V\otimes(W\oplus X) \cong (V\otimes W)\oplus(V\otimes X).$$

Furthermore, if $\{e_i\}$ and $\{e_j'\}$ are orthonormal bases for V and W respectively, then $\{e_i\otimes e_j'\}$ is an orthonormal basis for $V\otimes W$.

Finally, we recall that if V, V', W, and W' are Hilbert spaces, and if $f\colon V\to V'$, $g\colon W\to W'$ are bounded linear operators, then there is a unique bounded linear operator

$$f\otimes g\colon V\otimes V'\to W\otimes W'$$

whose value on elementary tensors is given by

$$(f\otimes g)(v\otimes v') = (fv)\otimes(gv').$$

Furthermore, $\|f\otimes g\| = \|f\|\,\|g\|$, $(f\otimes g)^* = f^*\otimes g^*$, and, if one also has bounded operators $f'\colon V'\to V''$, $g'\colon W'\to W''$, then

$$(f'\otimes g')(f\otimes g) = (f'f)\otimes(g'g).$$

Most of the various tensor products which have been defined in the mathematical literature satisfy a universal property with respect to appropriate bilinear maps, and also provide the left adjoint of an appropriate Hom functor. This is true in particular of the Hilbert space tensor product, and we now proceed to indicate the way in which this is the case. Special cases of the properties which we now describe are certainly implicit in a number of works involving the Hilbert space tensor product beginning with the work of Murray and von Neumann in which this tensor product was first defined (page 17 of [21]), but we have not seen these properties stated explicitly before.

The properties which we describe are closely related to the theory of Hilbert–Schmidt operators (for the basic properties of which we refer the reader to [6]). For this reason we establish:

NOTATION. 1.1. If V and W are Hilbert spaces, then $\mathrm{Hs}(V, W)$ will denote the Hilbert space of all Hilbert–Schmidt operators from V to W, with the Hilbert–Schmidt norm, which we will denote by $\|\ \|_2$.

DEFINITION. 1.2. Let V, W and X be Hilbert spaces. A continuous bilinear map, b, from $V\times W$ into X will be said to be a *Hilbert–Schmidt* *bilinear map* if there exist orthonormal bases $\{e_i\}$ and $\{e'_j\}$ for V and W respectively such that

$$\|b\|_2 = \Big(\sum_{i,j=1}^{\infty} \|b(e_i, e'_j)\|^2\Big)^{1/2} < \infty.$$

We will let $\mathrm{Bhs}(V, W; X)$ denote the space of Hilbert-Schmidt bilinear maps from $V\times W$ into X, with the indicated norm.

It will be clear from the proof of the next theorem that $\|b\|_2$ does not depend on the choice of bases, and that $\mathrm{Bhs}(V, W; X)$ is itself a Hilbert space. This next theorem is the statement of the universal property of the Hilbert space tensor product with respect to bilinear maps.

THEOREM. 1.3. *Let V, W and X be Hilbert spaces, and let d denote the bounded bilinear map of $V\times W$ into $V\otimes W$ defined by $d(v, w) = v\otimes w$. Then the map $f\to f\circ d$ defined for $f\in \mathrm{Hs}(V\otimes W, X)$ establishes a natural unitary isomorphism*

$$\mathrm{Bhs}(V, W; X) \cong \mathrm{Hs}(V\otimes W, X).$$

Proof. If $f\in \mathrm{Hs}(V\otimes W, X)$, then it is easily verified that $f\circ d\in \mathrm{Bhs}(V, W; X)$, and that $\|f\circ d\|_2 = \|f\|_2$.

Conversely, let $b\in\mathrm{Bhs}(V, W; X)$ be given, and let $\{e_i\}$ and $\{e'_j\}$ be orthonormal bases for V and W respectively such that

$$\sum_{i,j=1}^{\infty} \|b(e_i, e'_j)\|^2 < \infty.$$

Then we can define an operator, f_b, from $V\otimes W$ into X by defining its values on the elements of the orthonormal basis $\{e_i\otimes e'_j\}$ for $V\otimes W$ to be

$$f_b(e_i\otimes e'_j) = b(e_i, e'_j).$$

Then it is easily verified that $f_b\in\mathrm{Hs}(V\otimes W, X)$, that $\|f_b\|_2 = \|b\|_2$ and that $b = f_b\circ d$. From this observation, together with the basic facts about Hilbert-Schmidt operators, it is easily seen that the definition of f_b does not depend on the choice of bases, and neither does $\|b\|_2$.

We remark that the canonical bilinear map of $V\times W$ into $V\otimes W$ given by $(v, w)\to v\otimes w$ is *not* a Hilbert–Schmidt bilinear map if either V or W is infinite dimensional. Because of this, many proofs from the theory of algebraic tensor products do not carry over directly to the present setting. This is true, for example, of the usual proof of the uniqueness of tensor products. Nevertheless it is still true that abstract Hilbert space tensor products (for their definition see Definition 2.3) are unique (up to unitary equivalence). This will be shown in the next section (Proposition 2.5) in the more general context of tensor products of Hilbert modules.

We will now use the universal property of the Hilbert space tensor product with respect to Hilbert–Schmidt bilinear maps to prove the important adjointness relation satisfied by this tensor product. The appropriate setting for the adjointness relation is the category of Hilbert spaces in which the morphisms are taken to be the Hilbert–Schmidt operators. We remark that, strictly speaking, this category is not a category, since the identity map on an infinite dimensional Hilbert space is not a Hilbert–Schmidt operator. This fact will have the effect that again many proofs from the theory of algebraic tensor products can not be carried over to the present setting. However, we feel that we are quite justified in considering the collection of Hilbert spaces with Hilbert–Schmidt morphisms to be a category because of the fact that in this setting the Hilbert space tensor product does provide the left adjoint for the functor $\mathrm{Hs}(X, \cdot)$. We remark further that since $\mathrm{Hs}(V, W)$ is again a Hilbert space, this category should be considered to be an autonomous category (see [11]).

THEOREM 1.4. *Let V, W and X be Hilbert spaces. Then*

$$\mathrm{Hs}(X\otimes V, W) \cong \mathrm{Hs}\big(V, \mathrm{Hs}(X, W)\big)$$

in which the isomorphism consists of assigning to any f in $\mathrm{Hs}(X\otimes V, W)$ the element f' of $\mathrm{Hs}\big(V, \mathrm{Hs}(X, W)\big)$ defined by $f'(v)(x) = f(x\otimes v)$. This isomorphism is unitary and natural.

Proof. In view of THEOREM 1.3 it suffices to show that there is a natural isomorphism

$$\mathrm{Bhs}(X, V; W) \cong \mathrm{Hs}\big(V, \mathrm{Hs}(X, W)\big).$$

But it is easily verified that such an isomorphism is given by $b\to g_b$ for any $b\in\mathrm{Bhs}(X, V; W)$ where g_b is defined by

$$g_b(v)(x) = b(x, v), \qquad x\in X,\ v\in V.$$

For any Hilbert spaces V and W the spaces $V\otimes W$ and $\mathrm{Hs}(V, W)$ will have the same dimension, and so will be isomorphic. But there is in general no natural isomorphism between them. However, if V is $L^2(E)$ for some measure space E (we will suppress mention of the measure involved), then one has the following result, which is certainly well known although we have not found a good reference for it (part of it can be found in Theorem 3.1 of [7]).

THEOREM 1.5. *Let E be a positive measure space and let W be a Hilbert space. Let $L^2(E, W)$ denote the Hilbert space of W-valued Bochner square-integrable functions on E. Then there are natural unitary isomorphisms*

$$L^2(E)\otimes W \cong L^2(E, W) \cong \mathrm{Hs}\big(L^2(E), W\big).$$

Under the first of these isomorphisms an elementary tensor $f \otimes w$ corresponds to the function $e \to f(e)w$, whereas under the second the operator, T_F, which corresponds to $F \in L^2(E, W)$ is defined by

$$T_F(f) = \int_E f(e)F(e)\,de.$$

Proof. The first isomorphism is obtained by first lifting to the algebraic tensor product, $L^2(E) \otimes W$, the bilinear map from $L^2(E) \times W$ into $L^2(E, W)$ which carries the pair (f, w) to the function $e \to f(e)w$. This bilinear map is not a Hilbert–Schmidt bilinear map. But by using the Gram–Schmidt process to ensure that the elements of W which appear in any finite sum of elementary tensors are orthogonal, one sees easily that this lifted map is isometric on the algebraic tensor product, and so extends to an isometry of the Hilbert tensor product, $L^2(E) \otimes W$, into $L^2(E, W)$. The fact that this isometry is surjective follows from the fact that the simple functions in $L^2(E, W)$ (which are, of course, dense in $L^2(E, W)$) are easily seen to be the images under this isometry of finite sums of elementary tensors.

To prove the second assertion we state a lemma which is the basis for Mackey's definition of the tensor product of Hilbert spaces (see the comments immediately following Theorem 5.1 of [15]). This lemma does not seem to be a consequence of the universal property of the Hilbert tensor product.

LEMMA. 1.6. *Let V and W be Hilbert spaces. Then there is a natural unitary isomorphism*

$$V \otimes W \cong \mathrm{Hs}(V^*, W)$$

(where V^* denotes the dual of V, which we do not identify with V). Under this isomorphism the operator, $T_{v\otimes w}$, which corresponds to an elementary tensor, $v \otimes w$, is defined by $T_{v\otimes w}(v^*) = \langle v, v^*\rangle w$.

Proof. It is easily seen that every operator of finite rank is the image under the indicated mapping of a finite sum of elementary tensors, and that the mapping is isometric on this dense set of tensors so that it extends to an isometry defined on all of $V \otimes W$. But the operators of finite rank are dense among the Hilbert–Schmidt operators, and so the isometry is surjective.

We conclude the proof of Theorem 1.5. From Lemma 1.6 it follows that $L^2(E) \otimes W \cong \mathrm{Hs}\big((L^2(E))^*, W\big)$. But there is a natural *linear* isomorphism of $L^2(E)$ onto $\big(L^2(E)\big)^*$ which assigns to each $f \in L^2(E)$ the linear functional h_f defined by

$$h_f(g) = \int_E g(e)f(e)\,de, \qquad g \in L^2(E)$$

(with no complex conjugate sign). What is being used here is the fact that the complex vector space $L^2(E)$ has a canonical real form, namely the space of real-valued functions which it contains. Thus $L^2(E) \otimes W \cong \mathrm{Hs}\big(L^2(E), W\big)$ as desired. There is no difficulty in following the isomorphisms in this proof to verify that the second isomorphism of Theorem 1.5 has the form indicated in the statement of the theorem.

2. Hilbert module tensor products. In this section we define what we will mean by the tensor product of Hilbert spaces which in addition are modules, and we derive the basic properties of this tensor product. The exposition is very similar to that which we gave for the tensor product of Banach modules in [22], the most essential difference being that, instead of using the projective tensor product as was done in [22], we now use the Hilbert space tensor product which was discussed in the previous section. As a consequence, the proofs of some of the results of this section differ from the proofs of the corresponding results in [22]. As was the case in [16] and [22], we find it convenient to consider the general case of modules over an arbitrary set.

DEFINITION. 2.1. Let S be a set. By a *Hilbert S-module* we mean a Hilbert space, V, together with an assignment to every element of S of a bounded linear operator on V. For $s \in S$, $v \in V$, we will denote the action of s on v by sv or vs. If V and W are Hilbert S-modules, then $\mathrm{Hom}_S(V, W)$ will denote the collection of bounded linear operators, f, from V to W which are S-module homomorphisms, that is, for which $f(sv) = sf(v)$ for all $s \in S$ and $v \in V$. Furthermore, $\mathrm{Hs}_S(V, W)$ will denote the closed subspace of $\mathrm{Hs}(V, W)$ consisting of the S-module homomorphisms.

We remark that if S is a group then $\mathrm{Hs}_S(V, W)$ is the space of strong intertwining operators which Mackey describes on page 118 of [15].

DEFINITION. 2.2. Let V and W be Hilbert S-modules. A bilinear map, b, from $V \times W$ into a Hilbert space X is said to be S-balanced if $b(sv, w) = b(v, sw)$ for all $s \in S$, $v \in V$, $w \in W$. We will let $\mathrm{Bhs}_S(V, W; X)$ denote the closed subspace of $\mathrm{Bhs}(V, W; X)$ consisting of the S-balanced Hilbert–Schmidt bilinear maps from $V \times W$ into X.

DEFINITION. 2.3. Let V and W be Hilbert S-modules. Then a *Hilbert space tensor product over S of V and W* is a pair, (U, d), consisting of a Hilbert space, U, and a bounded S-balanced bilinear map, d, of $V \times W$ into U such that for any Hilbert space X the map $f \mapsto f \circ d$ defined for every $f \in \mathrm{Hs}(U, X)$ establishes a unitary isomorphism between $\mathrm{Hs}(U, X)$ and $\mathrm{Bhs}_S(V, W; X)$.

We show first that such tensor products always exist.

THEOREM. 2.4. *Let V and W be Hilbert S-modules, and let $V \otimes_S W$ be the quotient Hilbert space $(V \otimes W)/K$, where K is the (closed) subspace* *of $V \otimes W$ which is spanned by all elements of the form $sv \otimes w - v \otimes sw$, $s \in S$, $v \in V$, $w \in W$. Then $V \otimes_S W$ (with the obvious bilinear map of $V \times W$ into $V \otimes_S W$) is a Hilbert space tensor product over S of V and W.*

The proof of this theorem, using Theorem 1.3, is straightforward (and very similar to the proof of Theorem 2.3 of [22]) and so we will not include the proof here.

We remark that a somewhat similar construction of a tensor product has been given by Grove [8] for the case of H^*-algebras, the principal difference being that he requires K to be a two-sided *ideal* rather than simply a subspace.

For $v \in V$ and $w \in W$ we will denote the image in $V \otimes_S W$ of the element $v \otimes w$ of $V \otimes W$ again by $v \otimes w$. Then the S-balanced bounded bilinear map from $V \times W$ into $V \otimes_S W$ given by $(v, w) \mapsto v \otimes w$ need not be a Hilbert–Schmidt bilinear map. Because of this, many proofs from the theory of algebraic tensor products do not work in the present setting. This is true, for example, of the usual proof of the uniqueness of tensor products. However, we can still prove this uniqueness by a different method which takes advantage of the reflexivity of Hilbert spaces.

THEOREM. 2.5. *If (U, d) and (U', d') are two tensor products over S of S-modules V and W, then there is a unitary transformation, J, from U onto U' such that $d' = J \circ d$.*

Proof. In Definition 2.3 let $X = C$ (the complex number field) and denote by I and I' the unitary transformations $f \mapsto f \circ d$ and $f \mapsto f \circ d'$ of $\mathrm{Hs}(U, C)$ and $\mathrm{Hs}(U', C)$ respectively onto $\mathrm{Bhs}_S(V, W; C)$. Then $I^{-1} \circ I'$ is a unitary transformation of $(U')^*$ (the dual of U') onto U^*. Let $J = (I^{-1} \circ I')^*$, so that J maps U into U' (because Hilbert spaces are reflexive). We show that J is the desired isomorphism. Now J is unitary since I and I' are. Thus all that needs to be shown is that $d' = J \circ d$. To show this it suffices to show that

$$\langle (J \circ d)(v \times w), h' \rangle = \langle d'(v \times w), h' \rangle$$

for every $h' \in U'^*$, $v \in V$ and $w \in W$. Now the left hand side is equal to

$$(*) \qquad \langle (I^{-1})^* \big(d(v \times w)\big), I'(h') \rangle = \langle d(v \times w), I^{-1}(h' \circ d') \rangle.$$

But from the definition of I it is clear that for any $b \in \mathrm{Bhs}_S(V, W; C)$ we have $I^{-1}(b)\big(d(v \times w)\big) = b(v \times w)$. From this fact we see that the right hand side of $(*)$ is equal to $\langle d'(v \times w), h' \rangle$ as desired.

In view of this result we will from now on denote the Hilbert tensor product of Hilbert S-modules V and W by $V \otimes_S W$.

COROLLARY. 2.6. (THE COMMUTATIVITY OF THE TENSOR PRODUCT). *There is a natural unitary isomorphism of $V \otimes_S W$ onto $W \otimes_S V$ which carries $v \otimes w$ to $w \otimes v$ for all $v \in V$ and $w \in W$.*

The Hilbert module tensor product behaves properly with respect to intertwining operators. In view of the important role played up to this point by Hilbert–Schmidt operators it might seem that attention should be restricted to Hilbert–Schmidt intertwining operators, as will be necessary in several important places later. However, in the present situation it is crucial not to make such a restriction. The reason for this is that we have never required the operators by which a set S acts on a Hilbert S-module V to be Hilbert–Schmidt operators. Thus it will be necessary to be able to consider arbitrary (bounded) intertwining operators in order to be able to conclude later that the tensor product of a Hilbert module with a Hilbert bimodule is again a Hilbert module.

PROPOSITION. 2.7. *If V, V', W, and W' are Hilbert S-modules, and if $f \in \mathrm{Hom}_S(V, V')$ and $g \in \mathrm{Hom}_S(W, W')$ then there is a unique operator, $f \otimes g$, from $V \otimes_S W$ to $V' \otimes_S W'$ such that*

$$(f \otimes g)(v \otimes w) = f(v) \otimes g(w)$$

for all $v \in V$ and $w \in W$. Furthermore $\|f \otimes g\| \leqslant \|f\| \ \|g\|$.

Proof. The usual proof from the theory of algebraic tensor products does not work in this context because the bilinear map $(v, w) \mapsto f(v) \otimes g(w)$ need not be a Hilbert–Schmidt map. Instead we make use of the well-known fact, mentioned near the beginning of the previous section, that the proposition is true for ordinary Hilbert space tensor products (that is with S empty), so that $f \otimes g$, viewed as a map from $V \otimes W$ to $V' \otimes W'$, is well defined. If this $f \otimes g$ is then composed with the projection of $V' \otimes W'$ onto $V' \otimes_S W'$, it is easily verified that this composed map contains K (of Theorem 2.4) in its kernel, and so lifts to the desired map of $V \otimes_S W$ into $V' \otimes_S W'$.

COROLLARY. 2.8. *If in addition to the hypotheses of* PROPOSITION 2.7 *we have S-modules V'' and W'', and if $f' \in \mathrm{Hom}_S(V', V'')$ and $g' \in \mathrm{Hom}_S(W', W'')$, then*

$$(f' \otimes g')(f \otimes g) = (f'f) \otimes (g'g).$$

The Hilbert module tensor product, of course, again provides the left adjoint for an appropriate Hom functor. The statement of this adjointness relation in its most general form involves bimodules.

DEFINITION. 2.9. Let S and T be sets. Then by a *Hilbert S-T-bimodule* we mean a Hilbert space Z which is simultaneously a Hilbert S-module and a Hilbert T-module such that the actions of S and T on Z commute, that is, $s\big(t(z)\big) = t\big(s(z)\big)$ for all $s \in S$, $t \in T$ and $z \in Z$.

It is important to note that if Z is a Hilbert S-T-bimodule and if V is a Hilbert T-module, then both $\mathrm{Hs}_T(Z, V)$ and $Z \otimes_T V$ become Hilbert S-modules, when the action of S is defined by

$$(sf)(z) = f(sz), \qquad s \in S,\ f \in \mathrm{Hs}_T(Z, V),\ z \in Z$$

and

$$s(z\otimes v) = (sz)\otimes v, \qquad s\in S,\ z\in Z,\ v\in V,$$

respectively. (Proposition 2.7 is needed for the verification of this fact for the case of $Z\otimes_T V$). A similar statement holds with the roles of Z and V reversed.

We now state the basic adjointness relation for the Hilbert module tensor product.

THEOREM. 2.10. *Let S and T be sets, and let X be a Hilbert S-module, Z be a Hilbert S-T-bimodule, and Y be a Hilbert T-module. Then*

$$\mathrm{Hs}_T(Z\otimes_S X,\ Y) \cong \mathrm{Hs}_S\big(X,\ \mathrm{Hs}_T(Z,\ Y)\big),$$

where the isomorphism consists of assigning to any f in $\mathrm{Hs}_T(Z\otimes_S X, Y)$ the element f' of $\mathrm{Hs}_S\big(X, \mathrm{Hs}_T(Z, Y)\big)$ defined by $\big(f'(x)\big)(z) = f(z\otimes x)$, for all $x\in X$, $z\in Z$. This isomorphism is unitary and natural.

The proof of this theorem using the universal property of the Hilbert module tensor product is straightforward (and very similar to the proof of Theorem 2.12 of [22]) and so we will not include a proof here.

The Hilbert module tensor product satisfies the usual associativity law for tensor products. However, once again the usual proof from the theory of algebraic tensor products does not work in the present setting. Instead we give a proof which uses the reflexivity of Hilbert spaces (as did the proof of Theorem 2.5) and which also uses the adjointness relation stated just above.

THEOREM. 2.11. *Let S and T be sets, and let X be a Hilbert S-module, Z be a Hilbert S-T-bimodule, and Y be a Hilbert T-module. Then*

$$X\otimes_S(Z\otimes_T Y) \cong (X\otimes_S Z)\otimes_T Y,$$

where the isomorphism carries a tensor of the form $x\otimes(z\otimes y)$ to the tensor $(x\otimes z)\otimes y$. This isomorphism is unitary and natural.

Proof. For any Hilbert space W we have

$$\begin{aligned}\mathrm{Hs}\big(X\otimes_S(Z\otimes_T Y),\ W\big) &\cong \mathrm{Hs}_S\big(Z\otimes_T Y,\ \mathrm{Hs}(X,\ W)\big)\\ &\cong \mathrm{Hs}_T\Big(Y,\ \mathrm{Hs}_S\big(Z,\ \mathrm{Hs}(X,\ W)\big)\Big),\end{aligned}$$

while

$$\begin{aligned}\mathrm{Hs}\big((X\otimes_S Z)\otimes_T Y,\ W\big) &\cong \mathrm{Hs}_T\big(Y,\ \mathrm{Hs}(X\otimes_S Z,\ W)\big)\\ &\cong \mathrm{Hs}_T\Big(Y,\ \mathrm{Hs}_S\big(Z,\ \mathrm{Hs}(X,\ W)\big)\Big).\end{aligned}$$

Letting W be the complex numbers, it follows that the duals of the two spaces considered in the statement of the theorem are isomorphic, and so by reflexivity the spaces themselves are isomorphic. It is not difficult to verify that the isomorphism acts on elementary tensors as stated in the theorem.

As a final basic property of the tensor product of Hilbert modules we show that it commutes with Hilbert space direct sums. We recall that if $\{V_i\}$ is a family of Hilbert spaces, then the direct sum of this family, denoted by $\oplus V_i$, is defined to be the Hilbert space of all collections $\{v_i\}$ such that $v_i\in V_i$ for each i and $\sum\|v_i\|^2<\infty$. There are obvious isometric injections of the V_i into this direct sum, but the direct sum does not satisfy the usual universal property for direct sums (as described for example on page 31 of [10], where direct sums are called coproducts) if there is an infinite number of summands. It is clear that if all the V_i are modules over a given set S, then $\oplus V_i$ becomes an S-module under the evident coordinate-wise action provided that the operator norms of each element of S viewed as an operator on the different V_i are uniformly bounded (for example if all these operators are unitary). With this assumption in force we have:

THEOREM. 2.12. *Let $\{V_i\}$ be a family of S-modules with the action of each element of S uniformly bounded on the V_i, and let Z be an S-T-bimodule. Then there are natural unitary T-module isomorphisms*

$$Z\otimes_S(\oplus V_i) \cong \oplus(Z\otimes_S V_i), \qquad \mathrm{Hs}_S(\oplus V_i,\ Z) \cong \oplus\mathrm{Hs}_S(V_i,\ Z),$$

and

$$\mathrm{Hs}_S(Z,\ \oplus V_i) \cong \oplus\mathrm{Hs}_S(Z,\ V_i).$$

Proof. Since the Hilbert space direct sum does not satisfy the usual universal property for direct sums, the usual proof from the purely algebraic setting can not be used in the present setting. We begin by indicating the proof of the second isomorphism. The proof of the third isomorphism is quite similar and so we will omit it. If T is an element of $\mathrm{Hs}_S(\oplus V_i, Z)$, then its composition, T_i, with the natural injection of V_i into $\oplus V_i$ is easily seen to be an element of $\mathrm{Hs}_S(V_i, Z)$, and it easily verified that the map $T\mapsto\{T_i\}$ yields the desired isomorphism.

We now turn to the proof of the first isomorphism. We will show that $\oplus(Z\otimes_S V_i)$ satisfies the universal property of the tensor product $Z\otimes_S(\oplus V_i)$. The isomorphism of these two spaces will then follow from the uniqueness of the tensor product (Theorem 2.5), and it is then easy to verify that the isomorphism respects the action of T. But let Y be any Hilbert space. Then, by applying the adjointness relation (though disregarding the action of T) together with the second isomorphism whose proof we have indicated immediately above, we obtain

$$\begin{aligned}\mathrm{Hs}\big(\oplus(Z\otimes_S V_i),\ Y\big) &\cong \oplus\mathrm{Hs}(Z\otimes_S V_i,\ Y)\\ &\cong \oplus\mathrm{Hs}_S\big(V_i,\ \mathrm{Hs}(Z,\ Y)\big) \cong \mathrm{Hs}_S\big(\oplus V_i,\ \mathrm{Hs}(Z,\ Y)\big)\\ &\cong \mathrm{Hs}\big(Z\otimes_S(\oplus V_i),\ Y\big) \cong \mathrm{Bhs}_S(Z,\ \oplus V_i;\ Y)\end{aligned}$$

and so the desired universal property is established.

3. Hilbert modules over groups and algebras. We begin this section with a brief summary of some of the notation which we will use. This notation is the same as that used in [22], to which we refer the reader if he feels the need for a more complete explanation.

Let A be a Banach algebra. By a left *Hilbert A-module* we mean a Hilbert space V which is a left module over A in the usual algebraic sense, and for which there is a constant, k, such that $\|av\| \leqslant k\|a\|\,\|v\|$ for all $a \in A$, $v \in V$. We will say that V is an *essential* A-module if in addition AV is dense in V. If A is a Banach algebra with involution, then we will say that an essential A-module V is a *unitary* A-module if the corresponding representation of A is a *-representation. Similar definitions apply for right Hilbert A-modules.

If G is a locally compact group, then $M(G)$ will denote the measure algebra of G, and $L(G)$ will denote the group algebra of G, usually viewed as a two-sided ideal in $M(G)$. By a left (right) *Hilbert G-module* we will mean a Hilbert space V together with a strongly continuous uniformly bounded representation (anti-representation) of G on V. If, in addition, the action of G on V is by unitary operators, then we will call V a *unitary* G-module. It is well-known that the category of Hilbert G-modules with continuous (not necessarily Hilbert–Schmidt) module homomorphisms is isomorphic in the evident way to the category of essential Hilbert $L(G)$-modules, and that the category of unitary G-modules is isomorphic to the category of essential unitary $L(G)$-modules.

The definitions of Hilbert bimodules over Banach algebras or locally compact groups are analogous to the definitions of Hilbert modules given above.

Let A be a Banach algebra and S be a set. If V is an S-module and Z is an A-S-bimodule which as an A-module is a left A-module, then it is easily seen that with the actions defined just after Definition 2.9 $Z \otimes_S V$ becomes a left A-module while $\mathrm{Hs}_S(Z, V)$ becomes a right A-module. Similar statements hold if Z is a right A-module, or if the roles of Z and V are reversed, or if we replace A by a group.

Proposition. 3.1. *Let A be a Banach algebra with bounded approximate identity, and let S be a set. Let Z be a Hilbert A-S-bimodule, and let V be a Hilbert S-module. If Z is essential as an A-module, then so is $Z \otimes_S V$.*

The proof is entirely analogous to the proof of Theorem 3.9 of [22] and so it will be omitted.

On the other hand, in sharp contrast to the situation for Hom which was illustrated by Examples 3.12 and 3.13 of [22], we have:

Proposition. 3.2. *Let A be a Banach algebra with bounded approximate identity, and let S be a set. Let Z be a Hilbert A-S-bimodule, and let V be a Hilbert S-module. If Z is essential as an A-module, then so is $\mathrm{Hs}_S(Z, V)$.*

Proof. It suffices to show that $\mathrm{Hs}(Z, V)$ (disregarding S) is an essential A-module. For if this is shown, then it follows by a straightforward argument (which can be found in the proof of Proposition 3.4 of [22]) that $i_j f$ converges to f for any bounded approximate identity $\{i_j\}$ for A and any f in $\mathrm{Hs}(Z, V)$. Since $i_j f$ is in $\mathrm{Hs}_S(Z, V)$ if f is in $\mathrm{Hs}_S(Z, V)$, it follows that $\mathrm{Hs}_S(Z, V)$ itself is essential.

Now every element of $\mathrm{Hs}(Z, V)$ can be approximated in $\mathrm{Hs}(Z, V)$ by operators of finite rank, and so it suffices to consider only such operators. But if f is an operator of finite rank, then there exist a finite number of orthonormal vectors, $v_1, \ldots, v_n$, of V and elements, $\tilde{z}_1, \ldots, \tilde{z}_n$, of Z^* (the dual of Z) such that

$$f(z) = \sum_i \big(\tilde{z}_i(z)\big) v_i$$

for every $z \in Z$. Then for any $a \in A$ we have

$$(af)(z) = \sum_i \big(\tilde{z}_i(az)\big) v_i = \sum_i (a\tilde{z}_i)(z) v_i,$$

where $a\tilde{z}_i$ is defined by the dual action of A on Z^*. Furthermore, it is easily verified that

$$(\|af-f\|_2)^2 = \sum_i \|a\tilde{z}_i - \tilde{z}_i\|^2.$$

But by Proposition 8.7 of [22] the dual of an essential module which is reflexive as a Banach space is again an essential module. Thus we can choose an element, a, of an approximate identity for A such that $\|a\tilde{z}_i - \tilde{z}_i\|$ is arbitrarily small simultaneously for all i, and hence such that $\|af-f\|_2$ is arbitrarily small.

We remark that if Z is in fact unitary as an A-module, then a much simpler proof of the above proposition can be given by using:

Proposition. 3.3. *Let A be a Banach algebra with involution, and let S be a set. Let Z be a Hilbert A-S-bimodule, and let V be a Hilbert S-module. If Z is unitary as an A-module, then so are $Z \otimes_S V$ and $\mathrm{Hs}_S(Z, V)$.*

We omit the straightforward proof.

Proposition. 3.4. *Let G be a locally compact group and let S be a set. Let Z be a Hilbert G-S-bimodule and let V be a Hilbert S-module. Then $Z \otimes_S V$ and $\mathrm{Hs}_S(Z, V)$ are both Hilbert G-modules (that is, the obvious action of G on these spaces is strongly continuous). If Z is unitary as a G-module, then so are $Z \otimes_S V$ and $\mathrm{Hs}_S(Z, V)$.*

Proof. Since Z is a G-module it is an $M(G)$-module which is essential as an $L(G)$-module. Thus $Z \otimes_S V$ and $\mathrm{Hs}_S(Z, V)$ are essential as $L(G)$-modules, and so the action of G on each of them is strongly continuous. The statement about unitary modules is easily verified.

In conclusion, we state the analogue of Theorem 3.14 of [22]. We omit the proof since it is virtually the same as the proof of that theorem.

THEOREM. 3.5. *Let V and W be Hilbert G-modules, so that they are also Hilbert $L(G)$-modules and $M(G)$-modules. Then*

$$V \otimes_G W \cong V \otimes_{L(G)} W \cong V \otimes_{M(G)} W,$$

the isomorphisms being natural and unitary.

Because of this theorem we will not distinguish between these three tensor products in the sequal, and we will denote all three by $V \otimes_G W$ when we find it convenient to do so.

4. Induced representations. Throughout this section G will denote a locally compact group and H will denote a (usually compact) subgroup of G. Following the notation of [22] we will let μ and β denote left Haar measures on G and H respectively. Let $L^2(G)$ denote the usual Hilbert space of complex-valued square-integrable functions on G with respect to μ, viewed as a right-H-unitary-left-G-bimodule under the actions

$$(xf)(y) = f(x^{-1}y), \qquad x, y \in G, f \in L^2(G)$$

and

$$(fs)(y) = f(ys^{-1}), \qquad s \in H, f \in L^2(G), y \in G.$$

We remark that if H is compact, then the modular function of G restricted to H has constant value 1, so that $L^2(G)$ is also unitary as an H-module. We will let $\tilde{L}^2(G)$ denote the same Hilbert space as $L^2(G)$, but viewed as a left-H-unitary-right-G-bimodule under the actions

$$(fx)(y) = f(xy), \qquad x, y \in G, f \in \tilde{L}^2(G)$$

and

$$(sf)(y) = f(ys), \qquad s \in H, f \in \tilde{L}^2(G), y \in G.$$

(As discussed in the second section of [24], $\tilde{L}^2(G)$ can be considered to be the dual of $L^2(G)$ with the dual action.) Without further comment we will also consider $L^2(G)$ and $\tilde{L}^2(G)$ to be the corresponding $L(G)$-$L(H)$-bimodules.

We are now in a position to define induced representations. The definitions which we give are quite similar to those usually given for finite groups, and also to those which we gave in [22]. The principal difference is that by using $L^2(G)$ instead of the group algebra of G and by using the Hilbert tensor product instead of the projective tensor product we ensure that the induced represention which are obtained are unitary representations, and not just Banach representations as was the case in [22].

DEFINITION. 4.1. Let V be a left Hilbert H-module. The unitary left G-modules

$${}^G V = L^2(G) \otimes_H V$$

and

$$V^G = \mathrm{Hs}_H(\tilde{L}^2(G), V)$$

will be called respectively the adjoint and coadjoint unitary modules obtained by inducing V from H to G.

We remark that ${}^G V$ and V^G are unitary even if V is not, as can be seen from Proposition 3.4. Our terminology is motivated by that for adjoint functors which is given in [17]. We will show presently that, at least when G is compact, ${}^G V$ and V^G are in fact the adjoint and coadjoint of the obvious restriction functor from the category of unitary G-modules to the category of Hilbert H-modules.

We also remark that Dieudonné, on page 165 of [3], has given a definition of induced representations which is somewhat similar to that which we have given above for V^G. He assumes that G is compact and consider only finite dimensional representations, V, of H (and so does not need to introduce Hilbert-Schmidt operators) and defines the corresponding induced representation to be $\mathrm{Hom}_H(V, L^2(G))$ (which is, of course, a contravariant functor contrary to our definition). But he does not seem to take full advantage of this definition in deriving the basic properties of induced representations.

In the rest of this section we will derive the basic properties of the induced representations defined in Definition 4.1. In the following section we will then show the relation between these induced representations and those which were defined earlier by Mackey ([13], [14], [15]) and Mautner ([18], [19]).

The principal fact which is needed for the derivation of the basic properties of the induced representations defined above states essentially that if $H = G$ (compact) then ${}^G V = V = V^G$. This is the analogue of Theorems 4.4 and 4.5 of [22], but it fails to be true when G is not compact (Proposition 4.7). (The failure of this theorem when G is not compact is closely related to the fact that the Frobenius reciprocity theorem in its ordinary (non-infinitesimal) form does not hold when G is not compact.) This difference in comparison to the results of [22] is related to the fact that, in the present context, bounded approximate identities are not available to the extent that they were is Theorems 4.4 and 4.5 of [22]. As a consequence, the method of proof must be quite different from that in [22], and generalization of the present results to Hilbert modules over general algebras becomes fairly cumbersome. We will further discuss this last point immediately after we have stated the theorem. In preparation for the statement of the theorem we mention that if Z is a G-module and if $f \in L^2(G)$, then, since G is now compact, $f \in L(G)$, so that fz is well-defined for $z \in Z$. Also, we define $\tilde{f}$ by $\tilde{f}(x) = f(x^{-1})$ for $x \in G$.

THEOREM. 4.2. *Let G be a compact group (with normalized Haar measure),* *let S be a set, and let Z be a Hilbert S-G-bimodule. If Z as a G-module is a unitary left G-module, and if $L^2(G)$ and $\tilde{L}^2(G)$ are viewed as G-G-bimodules, then*

$$L^{\sim 2}(G)\otimes_G Z \cong Z \quad \text{and} \quad \mathrm{Hs}_G(L^2(G), Z) \cong Z,$$

as S-G-bimodules, the isomorphisms being unitary and natural. Under the first of these isomorphisms an elementary tensor $f\otimes z$ is identified with $\tilde{f}z\in Z$, whereas under the second isomorphism the operator T_z which corresponds to $z\in Z$ is defined by $T_z(f) = fz$. If Z as a G-module is a unitary right G-module, then, as S-G-bimodules

$$Z\otimes_G L^2(G) \cong Z \quad \text{and} \quad \mathrm{Hs}_G(\tilde{L}^2(G), Z) \cong Z,$$

with properties analogous to those just stated for left modules.

We remark that because G is compact, $L^2(G)$ is an H^*-algebra [12]. But the analogue of the above theorem is not true for modules over arbitrary H^*-algebras, or even over ones in which all the minimal two-sided ideals are finite dimensional, although this latter condition is necessary. The additional condition which must be satisfied by an H^*-algebra before the analogue of the above theorem will hold is that the norm of each minimal central idempotent in the algebra should be the square-root of the dimension of the minimal two-sided ideal which it generates (equivalently, the formal dimension of each irreducible representation of the H^*-algebra, as defined in Definition 6.2 of [23], should be equal to its actual dimension, or, the H^*-algebra should be "normal" as defined on page 84 of [7]). This fact will appear only implicitly in the proof which we give below for Theorem 4.2, because we prefer to give a proof for the particular case of groups which is comparatively elementary in the sense that it does not depend on the structure theory of H^*-algebras or of $L^2(G)$. But the above considerations certainly indicate that one can not expect to obtain a proof which is as elementary as the proofs of Theorems 4.4 and 4.5 of [22].

The main step in the proof of Theorem 4.2 is:

LEMMA. 4.3. *If as a G-module Z is a unitary left G-module, then*

$$\mathrm{Hs}_G(L^2(G), Z) \cong Z,$$

the isomorphism being unitary and natural. Under this isomorphism the operator, T_z, which corresponds to $z\in Z$ is defined by $T_z(f) = fz$.

Proof. For any $z\in Z$ the operator T_z is defined by

$$T_z(f) = \int_G f(x)xz\,d\mu(x), \quad f\in L^2(G).$$

Since G is compact and $x\mapsto xz$ is continuous, it is clear that $x\mapsto xz$ is a function in $L^2(G, Z)$, and so T_z is in $\mathrm{Hs}(L^2(G), Z)$ according to Theorem 1.5. Futhermore, also according to this theorem,

$$\|T_z\|_2^2 = \int_G \|xz\|^2 d\mu(x).$$

Since Z is assumed to be a unitary G-module and the Haar measure μ on G is assumed to be normalized, we obtain the fact that $\|T_z\|_2 = \|z\|$. (This is the only place where we use the assumption that Z is unitary.)

Now T_z is an $L(G)$-module homomorphism since

$$T_z(g*f) = (g*f)z = g(fz) = g(T_z f)$$

for any $g\in L(G)$ and $f\in L^2(G)$, and so $T_z\in\mathrm{Hs}_G(L^2(G), Z)$. Furthermore it is easily verified that the map $z\to T_z$ from Z to $\mathrm{Hs}_G(L^2(G), Z)$ is an S-G-module homomorphism, that it is injective (by using the facts that Z is an essential $L(G)$-module and that $L^2(G)$ is dense in $L(G)$), and that it is natural. Thus what needs to be shown is that this map is surjective.

Let $T\in\mathrm{Hs}_G(L^2(G), Z)$ be given. We wish to find $z\in Z$ such that $T = T_z$. Now according to Theorem 1.5 there is an $F\in L^2(G, Z)$ such that for all $f\in L^2(G)$

$$T(f) = \int_G f(x)F(x)\,d\mu(x).$$

The fact that T is a G-module homomorphism means that

$$\int_G f(y^{-1}x)F(x)\,d\mu(x) = y\int_G f(x)F(x)\,d\mu(x)$$

or

$$\int_G f(x)F(yx)\,d\mu(x) = \int_G f(x)y\big(F(x)\big)\,d\mu(x)$$

or all $f\in L^2(G)$ and $y\in G$. It follows that

$$F(yx) = y\big(F(x)\big) \quad \text{a. e. } x$$

for each $y\in G$. Now it is easily verified that $F(yx) - y\big(F(x)\big)$ is a jointly measurable function of x and y. Then by Fubini's theorem

$$Q = \{(x, y)\colon F(yx) \neq y\big(F(x)\big)\}$$

is a null set of $G\times G$, and so, again by Fubini's theorem, for all x outside of some fixed null set of G we have

$$F(yx) = y\big(F(x)\big) \quad \text{a. e. } y.$$

If we let x_0 be such a point outside the given null set and if we let $z = x_0^{-1}\big(F(x_0)\big)$, then we see that $F(x) = xz$ a. e., so that $T = T_z$ as desired.

COROLLARY. 4.4. *If as a G-module Z is a unitary right G-module, then*

$$Z\otimes_G L^2(G) \cong Z.$$

Under this isomorphism the element of Z which corresponds to an elementary tensor $z \otimes f$ is zf.

Proof. The dual, Z^*, of Z is an S-G-bimodule which as a G-module is a left unitary G-module. Applying Lemma 4.3 we obtain

$$Z^* \cong \mathrm{Hs}_G(L^2(G), Z^*) = \mathrm{Hs}_G(L^2(G), \mathrm{Hs}(Z, C)),$$

where C denotes the complex number field. Applying the adjointness relation, Theorem 2.10, to the right hand side, we obtain

$$Z^* \cong \mathrm{Hs}(Z \otimes_G L^2(G), C).$$

Passing to the dual Hilbert space we obtain the desired isomorphism. If $h \in Z^*$, $z \in Z$ and $f \in L^2(G)$, then, tracing back the isomorphisms in this proof, we obtain

$$h(z \otimes f) = T_h(f)(z) = (fh)(z) = h(zf),$$

so that $z \otimes f$ corresponds to zf as desired.

Corollary. 4.5. *If as a G-module Z is a unitary right G-module, then*

$$\mathrm{Hs}_G(\tilde{L}^2(G), Z) \cong Z.$$

Under this isomorphism the operator, T_z, which corresponds to $z \in Z$ is defined by $T_z(f) = z\tilde{f}$.

Proof. Let $\tilde{G}$ denote the group opposite to that of G. Then Z becomes a left $\tilde{G}$-module, $\tilde{L}^2(G)$ becomes a left-right G-bimodule, and it is easily seen that

$$\mathrm{Hs}_G(\tilde{L}^2(G), Z) \cong \mathrm{Hs}_{\tilde{G}}(\tilde{L}^2(G), Z).$$

Furthermore, as a $\tilde{G}$-module it is easily seen that $\tilde{L}^2(G)$ is naturally isomorphic to $L^2(\tilde{G})$, the isomorphism being given by $f \mapsto \tilde{f}$. Thus

$$\mathrm{Hs}_G(\tilde{L}^2(G), Z) \cong \mathrm{Hs}_{\tilde{G}}(L^2(G), Z).$$

We can now apply Lemma 4.3. to obtain the desired result.

4.6. Corollary. *If as a G-module Z is a unitary left G-module, then*

$$\tilde{L}^2(G) \otimes_G Z \cong Z.$$

Under this isomorphism the element of Z which corresponds to an elementary tensor $f \otimes z$ is $\tilde{f}z$.

Proof. This follows from Corollary 4.5 in the same way that Corollary 4.4 followed from Lemma 4.3.

The proof of Theorem 4.2 is now complete.

One reason that the results of this paper do not apply to representations induced from non-compact subgroups is that Theorem 4.2 is false for non-compact groups. In fact it fails so badly that even our definition of induced representations is useless in this case.

Proposition. 4.7. *If H is a non-compact subgroup of the locally compact group G, and if V is any left Hilbert H-module, then*

$$L^2(G) \otimes_H V = 0 \quad \text{and} \quad \mathrm{Hs}_H(L^2(G), V) = 0,$$

whether H is viewed as acting on $L^2(G)$ by left translation or right translation.

Proof. If $T \in \mathrm{Hs}_H(L^2(G), V)$, then $T^*T \in \mathrm{Hs}_H(L^2(G), L^2(G))$. If $T \neq 0$, then T^*T will have non-trivial finite dimensional eigensubspaces since it is a compact self-adjoint operator, and these eigensubspaces will be invariant under H. Thus what we need to know in order to show that $\mathrm{Hs}_H(L^2(G), V) = 0$ is the following lemma which is somewhat related to a result of Weil (see page 70 of [27]). The proof that $L^2(G) \otimes_H V = 0$ then follows from these results in the same way that Corollary 4.4 followed from Lemma 4.3.

Lemma. 4.8. *If H is a subgroup of G which is not compact, then $L^2(G)$ has no finite dimensional subspaces which are invariant under either left or right translation by H.*

Proof. The case of right invariant subspaces is complicated slightly by the fact that the modular function, Δ, of G comes into play, and so we will treat this case. Let V be a subspace of $L^2(G)$ which is invariant under right translation by H. Choose $f \in V$ with $\|f\| = 1$. Let $\varepsilon > 0$ be given. Since $f \in L^2(G)$, we can find a compact subset, E, of G such that

$$\int_{E'} |f(x)|^2 d\mu(x) \leqslant (\varepsilon/2)^2$$

(where E' denotes the complement of E). Since H is not compact, we can find a sequence, $\{s_i\}$, of points of H such that the sequence $\{Es_i\}$ of subsets of G is disjoint. Then by considering integrals over E and E' it is easily calculated that

$$|\langle fs_i, fs_j\rangle| \leqslant \varepsilon(\Delta(s_i)\Delta(s_j))^{1/2},$$

so that $|\langle e_i, e_j\rangle| \leqslant \varepsilon$, where $e_i = fs_i/\|fs_i\|$. But one can then easily verify that $\|e_i - e_j\| \geqslant (2-2\varepsilon)^{1/2}$ if $i \neq j$. Thus the unit ball of V can not be norm compact, and so V is not finite dimensional.

An alternate proof of Proposition 4.7 can be given as follows if the action of H is unitary. Let $T \in \mathrm{Hs}_H(L^2(G), V)$. Arguing as we did at the end of the proof of Lemma 4.3, we find that there is an $F \in L^2(G, V)$ such that $T = T_F$ and, if, say, H acts by left translation, then for all x outside some null set

$$F(sx) = (sF(x)) \quad \text{a. e. } s.$$

It follows that off a null set the norm of F is essentially constant on cosets of H. Then by using Proposition 10.1 of [22] it is not difficult to convince oneself that if H is not compact then the only way such an F can be square-integrable is for F to be a null function. Thus $T = 0$.

We now derive some of the basic properties of the induced representations which we have defined. The derivations are very similar to those for induced representations of finite groups. We begin with the theorem on induction in stages.

THEOREM. 4.9. *Let H be a compact subgroup of the locally compact group G, and let K be a closed subgroup of H. If V is any Hilbert K-module, then*

$${}^{G}({}^{H}V) \cong {}^{G}V \quad \textit{and} \quad (V^{H})^{G} \cong V^{G},$$

the isomorphisms being unitary and natural.

Proof. Applying Theorems 2.11 and 4.2 we obtain

$$\begin{aligned} {}^{G}({}^{H}V) &= L^2(G)\otimes_H\big(L^2(H)\otimes_K V\big) \cong \big(L^2(G)\otimes_H L^2(H)\big)\otimes_K V \\ &\cong L^2(G)\otimes_K V = {}^{G}V. \end{aligned}$$

Applying in addition Theorem 2.10, we obtain

$$\begin{aligned} (V^{H})^{G} &= \mathrm{Hs}_H\big(\tilde{L}^2(G), \mathrm{Hs}_K(\tilde{L}^2(H), V)\big) \cong \mathrm{Hs}_K\big(\tilde{L}^2(H)\otimes_H\tilde{L}^2(G), V\big) \\ &\cong \mathrm{Hs}_K\big(\tilde{L}^2(G), V\big) = V^{G}. \end{aligned}$$

COROLLARY. 4.10. *If V is the left regular representation of H, then ${}^{G}V$ and V^{G} are both equivalent to the left regular representation of G.*

Proof. Let $K = \{e\}$ where e is the identity element of G, and let W be the one-dimensional representation of K. It is easily seen that ${}^{H}W$ and W^{H} are equivalent to the left regular representation of H and that ${}^{G}W$ and W^{G} are equivalent to the left regular representation of G. We can now apply Theorem 4.9 to obtain the desired result.

Another basic property of induced representations follows immediately from the fact that tensor products and Hs commute with direct sums (Theorem 2.12).

THEOREM. 4.11. *Let H be a compact subgroup of the locally compact group G. If $\{V_i\}$ is a collection of unitary H-modules, then*

$${}^{G}(\oplus V_i) \cong \oplus {}^{G}V_i \quad \textit{and} \quad (\oplus V_i)^{G} \cong \oplus V_i^{G}.$$

COROLLARY. 4.12. *If V is reducible then so are ${}^{G}V$ and V^{G}.*

COROLLARY. 4.13. *If V is equivalent to a subrepresentation of the left regular representation of H (in particular if V is an irreducible unitary H-module) then ${}^{G}V$ and V^{G} are both equivalent to subrepresentations of the left regular representation of G.*

Proof. This follows immediately from Theorem 4.11 and Corollary 4.10.

Next, we consider the Frobenius reciprocity theorem, which should say that induced representations are the adjoints of the restriction functor from the category of unitary G-modules to the category of unitary H-modules (or, more generally, of Hilbert H-modules). If V is any Hilbert H-module and W is any unitary G-module, then, applying the adjointness relation Theorem 2.10, we obtain

$$(*)\qquad \mathrm{Hs}_G({}^{G}V, W) = \mathrm{Hs}_G(L^2(G)\otimes_H V, W) \cong \mathrm{Hs}_H\big(V, \mathrm{Hs}_G(L^2(G), W)\big)$$

and

$$(**)\quad \mathrm{Hs}_G(W, V^{G}) = \mathrm{Hs}_G\big(W, \mathrm{Hs}_H(\tilde{L}^2(G), V)\big) \cong \mathrm{Hs}_H\big(\tilde{L}^2(G)\otimes_G W, V\big).$$

If G is not compact, then from Proposition 4.7 we unfortunately conclude that

$$\mathrm{Hs}_G({}^{G}V, W) = 0 = \mathrm{Hs}_G(W, V^{G}).$$

The only useful piece of information which we can obtain from this is the following special case of Theorem 8.2 of [15].

PROPOSITION. 4.14. *If G is not compact, then ${}^{G}V$ and V^{G} contain no finite dimensional subrepresentations.*

However, if G itself is compact, then Theorem 4.2 is applicable to (*) and (**) and we obtain:

THEOREM. 4.15. (THE FROBENIUS RECIPROCITY THEOREM FOR COMPACT GROUPS). *Let G be a compact group and let H be a closed subgroup of G. Then the functor ${}^{G}V$ and V^{G} are respectively the left and right adjoints of the restriction functor from the category of unitary G-modules to the category of Hilbert H-modules. That is, for any Hilbert H-module, V, and any unitary G-module, W, there are natural unitary isomorphisms*

$$\mathrm{Hs}_G({}^{G}V, W) \cong \mathrm{Hs}_H(V, W_H)$$

and

$$\mathrm{Hs}_G(W, V^{G}) \cong \mathrm{Hs}_H(W_H, V)$$

where W_H denotes W viewed as an H-module.

If V and W are taken to be irreducible modules then one can immediately deduce from this theorem the more classical form of the Frobenius reciprocity theorem for compact groups as formulated in [29], or on pages 82–83 of [27], or on pages 12 and 27 of [30]. This is done by noting, for example, that in this case the dimension of $\mathrm{Hs}_H(V, W_H)$ is equal to the multiplicity of V in W_H.

Joseph Wolf has pointed out to us that a part of one version of the Peter–Weyl theorem is a consequence of the Frobenius reciprocity theorem. Specifically, if G is compact, if $H = \{e\}$, and if V is the one-dimensional representation of H, then, as mentioned before, ${}^{G}V$ and V^{G} are both equivalent to the left regular representation of G. If we let W be any irreducible representation of G, then by applying the Frobenius reciprocity theorem we find that there are non-zero Hilbert–Schmidt operators which intertwine W with the left regular representation of G so that W must be finite dimensional, and furthermore that the multiplicity of W in the left regular representation of G is equal to the dimension of W.

Of course, the rest of the Peter–Weyl theorem can be obtained in the usual way. It suffices to show that any invariant subspace of the left regular representation contains a finite dimensional invariant subspace. But to do this it suffices to check that the operator R_f of right convolution by any function f in $L(G)$ is a Hilbert–Schmidt operator which is in the commutant of the left regular representation, and that given any $g \in L^2(G)$ there is a self-adjoint $f \in L(G)$ such that $R_f g \neq 0$. For then g will have a non-zero component in one of the finite dimensional eigensubspaces of R_f, and the eigensubspaces of R_f are invariant subspaces of the left regular representation.

Although the comments just preceding Proposition 4.14 show that if G is not compact then ${}^G V$ and V^G are not adjoints of the restriction functor, one can still imagine that some other construction might provide such adjoints. We show now that this is not the case (at least if the morphisms are taken to be Hilbert–Schmidt operators), again by applying Proposition 4.7.

THEOREM. 4.16. *If H is a compact subgroup of a locally compact group G, and if G is not compact, then the restriction functor from unitary G-modules to unitary H-modules has neither a left nor a right adjoint.*

Proof. Suppose that the restriction functor had a right adjoint, $V \mapsto R(V)$, so that

$$\mathrm{Hs}_G\big(W, R(V)\big) \cong \mathrm{Hs}_H(W_H, V)$$

for every unitary G-module W and unitary H-module V. Then this relation must hold in particular if we set $W = L^2(G)$. But for this choice of W the left-hand side will always be 0 according to Proposition 4.7, whereas there will be many choices of V for which the right-hand side is not 0. Thus no such adjointness relation can exist. The argument showing the non-existence of a left adjoint is similar.

The final property of induced representations which we will derive in this section concerns their contragradient (or adjoint, in Mackey's terminology). If V is a unitary G-module, then its dual is by definition the G-module $V^* = \mathrm{Hs}(V, C)$. If V is a left module then V^* is a right module and conversely. In order to be able to work always with left modules it is customary to apply the involution $x \mapsto x^{-1}$ on G to a right module, V, to obtain a corresponding left module, which we will denote by $\tilde{V}$(this is consistent with our usage in $\tilde{L}^2(G)$). The contragradient of a Hilbert left G-module V is then defined to be the left module $(\tilde{V})^*$ (or equivalently $(V^*)^{\sim}$), and we will denote it by $V^{\check{}}$.

PROPOSITION. 4.17. *If H is a compact subgroup of a locally compact group G, and if V is a Hilbert left H-module, then*

$$(V^{\check{}})^G \cong ({}^G V)^{\check{}}.$$

Proof. $(V^{\check{}})^G = \mathrm{Hs}_H\big(\tilde{L}^2(G), \mathrm{Hs}(\tilde{V}, C)\big) = \mathrm{Hs}\big(\tilde{L}^2(G) \otimes_H \tilde{V}, C\big) = \big(\tilde{L}^2(G) \otimes_H \tilde{V}\big)^*$. But it is easily seen that $\tilde{L}^2(G) \otimes_H \tilde{V} = \big(L^2(G) \otimes_H V\big)^{\sim}$, and so the result follows.

In the next section we will show that ${}^G V$ and V^G are naturally equivalent if V is unitary, and it will then follow that

$${}^G(V^{\check{}}) \cong (V^G)^{\check{}} \cong ({}^G V)^{\check{}} \cong (V^{\check{}})^G.$$

But we do not know how to derive these facts in a more direct manner.

John L. Kelley has suggested that we mention that all of the results of this section, suitably interpreted, remain true when instead of assuming that H is a subgroup of G we assume only that we are given a continuous homomorphism of H into G.

5. The realization of ${}^G V$ and V^G as function spaces. In this section we will show that the definitions of induced representations which we gave in the previous section are equivalent to the definition in terms of function spaces which was given by Mautner [18, 19] and to the special case of Mackey's definition [13, 14, 15] in which the subgroup is compact. In the process we will see that ${}^G V$ and V^G are naturally equivalent.

The first theorem is an analogue of Theorem 10.4 of [22].

THEOREM. 5.1 *Let H be a compact subgroup of the locally compact group G, and let V be a unitary left H-module. Then ${}^G V$ is naturally isomorphic to the unitary G-module consisting of all Bochner square-integrable functions, F, from G to V such that*

$$F(xs) = s^{-1}\big(F(x)\big)$$

for all $x \in G$ and all $s \in H$ (with functions which are equal a. e. identified) on which the action of G is defined by

$$(yF)(x) = F(y^{-1}x).$$

Proof. Let K be the subspace of $L^2(G) \otimes V$ which is spanned by elements of the form $fs \otimes v - f \otimes sv$, where $f \in L^2(G)$, $s \in H$, and $v \in V$, so that ${}^G V = \big(L^2(G) \otimes V\big)/K$. But this quotient is naturally isomorphic to the orthogonal complement of K, that is, ${}^G V \cong K^{\perp}$. Now applying Theorem 1.6 we identify $L^2(G) \otimes V$ with $L^2(G, V)$, and we identify K and $K^{\perp}$ with the corresponding subspaces of $L^2(G, V)$.

Suppose now that $F \in K^{\perp}$. Then for every $f \in L^2(G)$, $s \in H$, and $v \in V$, we have, since V is unitary,

$$\begin{aligned} 0 &= \langle F, fs \otimes v - f \otimes sv \rangle \\ &= \int_G \langle F(x), v \rangle \bar{f}(xs^{-1})\, d\mu(x) - \int_G \langle F(x), sv \rangle \bar{f}(x)\, d\mu(x) \\ &= \int_G \langle F(xs) - s^{-1}(F(x)), v \rangle \bar{f}(x)\, d\mu(x). \end{aligned}$$

Thus for each $s \in H$ and $v \in V$ we have

$$\langle F(xs) - s^{-1}(F(x)), v \rangle = 0 \quad \text{a. e. } x.$$

Since F is Bochner measurable, its range is contained in a separable subspace of V. If we let v range over a countable dense subset of this subspace and if we take the union of the corresponding null sets, we find that for each $s \in H$

$$F(xs) = s^{-1}(F(x)) \quad \text{a. e. } x.$$

Conversely it is easily seen that any function satisfying this condition is an element of $K^{\perp}$. But this condition on the function F is not as strong as that in the statement of the theorem. To verify that this stronger condition holds we prove the following lemma which is also of independent interest. Remarks concerning the map π introduced in this lemma can be found immediately following Theorem 10.4 of [22]. This map is also used at the bottom of page 27 of [30].

LEMMA. 5.2. *The projection, π, of $L^2(G, V)$ onto $K^{\perp}$ is given by*

$$\pi g(x) = \int_H s(g(xs))\, d\beta(s)$$

for all $g \in L^2(G, V)$. The range of π is the space, $L^2_H(G, V)$, consisting of all those functions, F, in $L^2(G, V)$ which for all x outside of some null set of G satisfy

$$F(xs) = s^{-1}(F(x))$$

for all $s \in H$ (where two such functions are identified if they are equal a. e.).

Proof. Arguments analogous to those in the proof of Lemma 10.5 of [22] show that $s(g(xs))$ is a Bochner square-integrable function on $(G \times H, \mu \times \beta)$, that πg is defined a. e., that $\pi g \in L^2(G, V)$, that $\|\pi g\|_2 \leqslant \|g\|_2$, that $\pi(yg) = y(\pi g)$ for all $y \in G$, and, finally, that $\pi g \in L^2_H(G, V)$.

On the other hand, if $F \in L^2_H(G, V)$, then for all x outside of the null set referred to in the definition of $L^2_H(G, V)$

$$\pi F(x) = \int_H s(F(xs))\, d\beta(s) = \int_H F(x)\, d\beta(s) = F(x),$$

so that π is a projection of $L^2(G, V)$ onto $L^2_H(G, V)$. A routine calculation shows that π is self-adjoint.

It remains to show that the kernel of π is K. Now if $f \in L^2(G)$, $v \in V$ and $r \in H$, then

$$\pi(f \otimes rv)(x) = \int_H f(xs) srv\, d\beta(s) = \int_H f(xsr^{-1}) sv\, d\beta(s) = \pi(fr \otimes v)(x),$$

so that K is contained in the kernel of π. On the other hand, since $K^{\perp}$ is invariant under G, it is closed under convolution by continuous functions of compact support. Since such convolutions are continuous (Theorem 20.16 of [9]) it follows that $K^{\perp}$ contains a dense submanifold consisting of continuous functions. But if F is such a continuous function in $K^{\perp}$, then it is clear from the argument just preceding this lemma that $F(xs) = s^{-1}(F(x))$ for all $x \in G$ and $s \in H$, that is, that F is in the space of functions described in the theorem, and in particular is preserved by π. Thus π acts as the identity operator on a dense submanifold of $K^{\perp}$, and so $K^{\perp}$ is in the range of π. It follows that K contains the kernel of π.

To conclude the proof of Theorem 5.1 it suffices to remark that any function in $L^2_H(G, V)$ can be modified on a null set so as to satisfy the conditions of the theorem (for example by defining its value to be 0 on this null set).

We note that the space of functions described in Theorem 5.1 is the same as that used by Mautner ([18], [19]) to define induced representations, which in turn is easily seen to be the same as the special case of the space of functions used by Mackey ([13], [14], [15]) in which the subgroup is compact (as was remarked by Mautner himself).

THEOREM. 5.3. *Let H be a compact subgroup of the locally compact group G and let V be a Hilbert H-module. Then V^G is unitarily equivalent to ${}^G V$, and so to the G-module consisting of the space of functions described in Theorem 5.1.*

Proof. Given $T \in \mathrm{Hs}_H(\tilde{L}^2(G), V)$, one can find, according to Theorem 1.6, an $F \in L^2(G, V)$ which represents T, that is, such that $Tf = \int_G f(x) F(x)\, d\mu(x)$ for all $f \in \tilde{L}^2(G)$. The fact that T is a module homomorphism means that $s(Tf) = T(sf)$ for all $s \in H$ and $f \in \tilde{L}^2(G)$, that is, that

$$s \int_G f(x) F(x)\, d\mu(x) = \int_G f(xs) F(x)\, d\mu(x),$$

or

$$\int_G f(x) s(F(x))\, d\mu(x) = \int_G f(x) F(xs^{-1})\, d\mu(x).$$

It follows that for each $s \in H$ we have $F(xs) = s^{-1}(F(x))$ a. e. x. Conversely it is easily seen that any such function does represent an element of $\mathrm{Hs}_H(L^2(G), V)$. The rest of the proof follows by arguments similar to those in the proof of Lemma 5.2 and in the end of the proof of Lemma 4.3.

We remark that in this theorem we did not need to assume that V is unitary, contrary to the case in Theorem 5.1. If in Theorem 5.1 it is only assumed that V is a Hilbert module then one can only conclude that the functions F must satisfy $F(xs) = s^*(F(x))$ where s^* denotes the adjoint of the action of s.

If G is compact then the Frobenius reciprocity theorem (Theorem 4.13) says that ${}^G V$ and V^G are the left and right adjoints of the restriction functor, and so in this case we see that the restriction functor exhibits the interesting phenomenon that its left and right adjoints are naturally equivalent. In a purely algebraic setting this phenomenon has been studied first by Higman [28], and then by Morita [20], who showed that it is very closely related to the concept of Frobenius extensions. In the terminology of Morita, the restriction functor and the functor ${}^G V$ which we have defined above form a strongly adjoint pair.

In a subsequent paper in which we will discuss unitary representations induced from subalgebras of the complexified Lie algebra of a compact Lie group, we will see that the left and right adjoints of the restriction functor are no longer equivalent in this more general setting.

6. Blattner's construction of induced representations. In [1] Blattner showed how the induced representations of Mackey could be defined in terms of positive definite measures. In this section we will show how Blattner's construction is related to the definition of induced representations which we have given in terms of tensor products and spaces of Hilbert–Schmidt operators.

For various reasons, including the fact that we have been using left rather than right Haar measures, we will find it convenient to use conventions which are slightly different from those used by Blattner. The conventions which we will use, together with a reformulation of Blattner's theorem in terms of these conventions, can be found in unpublished notes by J. M. G. Fell. We are indebted to Fell for having provided us with a copy of these notes. We will view $C_c(G)$ as a dense subalgebra of $L(G)$ with its usual convolution multiplication and involution. We will say that a complex Radon measure, m, is positive definite if $m(f^* * f) \geqslant 0$ for all $f \in C_c(G)$. Given a positive definite Radon measure, m, one can then equip $C_c(G)$ with the positive Hermitian form

$$(6.1) \qquad \langle f, g\rangle_m = m(g^* * f), \quad f, g \in C_c(G),$$

and from this obtain in the usual way a unitary representation of G on the corresponding Hilbert space (for details see [5]). We will denote the resulting G-module by V_m.

Let H be a compact subgroup of G and let m be a positive definite measure on H. Let $\hat{m}$ denote m viewed as a measure on G in the obvious way. Then by imitating part of the proof of Blattner's theorem using our slightly different conventions, one can verify that $\hat{m}$ is in fact positive definite on G (this verification is carried out in the notes of Fell mentioned above). Blattner's theorem states then (though in the more general case in which H need not be compact) that

$${}^G(V_m) \cong V_{\hat{m}},$$

where the induced representation ${}^G(V_m)$ which Blattner uses is essentially that defined by Mackey in terms of function spaces. In this section we will show how to prove this theorem of Blattner (for the special case in which H is compact) in terms of the definition of induced representation which we have given using tensor products.

Our approach to Blattner's theorem is based in part on the observation that the construction of the representation V_m can be reformulated in terms of positive intertwining operators. Specifically, if m is any Radon measure on G then an easy calculation shows that

$$(6.2) \qquad m(g^* * f) = \langle P_m f, g\rangle$$

where P_m is the operator defined by

$$(P_m f)(x) = \int f(xy)\, dm(y),$$

although this operator need not have all its values even in $L^2(G)$. If, however, m is a finite measure and if the modular function of G has value 1 on the support of m (in particular, if m is supported on a compact subgroup of G, as will be the case in our applications), then P_m is a bounded operator on $L^2(G)$. In fact it is just convolution on the right by the measure $\tilde{m}$ which is the image of m under the homeomorphism $x \to x^{-1}$ of G. From this last remark it is clear that P_m commutes with the left regular representation of G on $L^2(G)$. Furthermore, it is clear from 6.2 that if m is positive definite then P_m is a positive (not necessarily definite) operator. In this case one can see by comparing 6.1 with 6.2 that the construction of V_m is a particular case of the following construction, whose details are easily verified.

PROPOSITION. 6.3. *Let A be a normed *-algebra and let V be an essential unitary A-module. Let P be a positive operator on V such that $P \in \mathrm{Hom}_A(V, V)$. Define a positive Hermitian form on V by*

$$\langle v, v'\rangle_P = \langle Pv, v'\rangle.$$

Then the action of A on V induces on the Hilbert space corresponding to this positive Hermitian form the structure of an essential unitary A-module.

We remark that a similar construction can be made for unbounded P, although in this case it seems to be necessary to assume that A has an approximate identity in order to conclude that the module is essential.

We will denote the A-module constructed in the way described above by V/P. In particular, we note that if m is a positive definite finite measure supported so that P_m is bounded, then it is clear from the discussion above that

$$V_m \cong L^2(G)/P_m.$$

Further information about the construction described in Proposition 6.3 is given by the following:

PROPOSITION. 6.4. *Let A, V and P be as in Proposition 6.3, and let E be the orthogonal projection of V onto the closure of the range of P (so $E \in \mathrm{Hom}_A(V, V)$). Then $V/P \cong V/E$. Furthermore, V/E is isomorphic to the A-submodule of V consisting of the range of E.*

Proof. Let Q be the positive square-root of P (so $Q \in \mathrm{Hom}_A(V, V)$). Then for $v, v' \in V$ we have

$$\langle Qv, Qv'\rangle_E = \langle EQv, Qv'\rangle = \langle EPv, v'\rangle = \langle v, v'\rangle_P.$$

Thus Q extends to an isometric intertwining operator of V/P into V/E. We show that $Q(V)$ is dense in V/E so that this isometry is unitary. Let $\{E(r)\}$ be a resolution of the identity for P (continuous from the right), and let T_t be defined for $t \geqslant 0$ by

$$T_t = \int_t^\infty r^{-(1/2)} dE(r).$$

Then a simple calculation shows that $\|v - QT_t v\|_E$ approaches 0 for any $v \in V$ as t approaches 0. Thus $Q(V)$ is dense in V/E. Finally, it is a trivial matter to verify the last statement of the proposition.

As a corollary of this proposition we obtain the following result which we have not seen mentioned before:

COROLLARY. 6.5. *Let m be a finite positive definite Radon measure on G and suppose that the modular function of G has value 1 on the support of m. Then V_m is unitarily equivalent to a subrepresentation of the left regular representation of G.*

Proof. We see from Proposition 6.4 that V_m is in fact unitarily equivalent to the submodule of the left regular representation consisting of the closure of the range of P_m.

Suppose now that H is a compact subgroup of G and that m is a positive definite measure on H. Then in terms of the notation and results described above, Blattner's theorem becomes the statement that

$$L^2(G) \otimes_H \big(L^2(H)/P_m\big) \cong L^2(G)/P_{\hat{m}}.$$

We will now give a direct proof of this relation. To do this we need to know how P_m and $P_{\hat{m}}$ are related.

LEMMA. 6.6. *If we identify $L^2(G)$ with $L^2(G) \otimes_H L^2(H)$ according to Theorem 4.2, then*

$$P_{\hat{m}} = I \otimes_H P_m,$$

where I denotes the identity operator on $L^2(G)$.

Proof. Under the identification of $L^2(G) \otimes_H L^2(H)$ with $L^2(G)$ an elementary tensor $f \otimes \varphi$ is carried, according to Theorem 4.2, to the element $f * \varphi$ of $L^2(G)$, where for the purposes of defining this convolution the elements of $L^2(H)$ are viewed as measures on H and so on G. Then

$$(I \otimes P_m)(f \otimes \varphi) = f \otimes P_m \varphi = f * (\varphi * \tilde{m}) = (f * \varphi) * \tilde{\tilde{m}} = P_{\hat{m}}(f * \varphi).$$

Since the span of the elementary tensors is dense, it follows that $I \otimes P_m = P_{\hat{m}}$, as desired.

The proof of Blattner's theorem is thus reduced to proving the following theorem:

THEOREM. 6.7 *Let Z be a unitary left-G-right-H-bimodule, let V be a unitary left H-module, and let P be a positive operator in $\mathrm{Hom}_H(V, V)$. Then*

$$Z \otimes_H (V/P) \cong (Z \otimes_H V)/I \otimes_H P,$$

where I denotes the identity operator on Z.

Proof. We remark first that $I \otimes_H P$ is a positive operator since it clearly has a self-adjoint square root. Let E be the projection onto the closure of the range of P. We show first that the projection onto the closure of the range of $I \otimes_H P$ is $I \otimes_H E$. Now, $E = \lim P^{1/n}$ in the strong operator topology, as n approaches infinity. Similarly the projection onto the range of $I \otimes_H P$ is $\lim (I \otimes_H P)^{1/n}$. But it is clear that $(I \otimes_H P)^{1/n} = I \otimes_H (P)^{1/n}$, and it is easily seen that $\lim I \otimes_H (P)^{1/n} = I \otimes_H E$, so that we obtain the desired result.

Then, according to Proposition 6.4,

$$Z \otimes_H (V/P) \cong Z \otimes_H (V/E) \quad \text{and} \quad (Z \otimes_H V)/I \otimes_H P \cong (Z \otimes_H V)/I \otimes_H E,$$

so that we need only show that

$$Z \otimes_H (V/E) \cong (Z \otimes_H V)/I \otimes_H E,$$

that is, we need only prove the theorem when P is a projection, E. But let $V_1 = EV$ and $V_2 = (I - E)V$ so that $V = V_1 \oplus V_2$ as H-modules. Now $V/E \cong V_1$ as mentioned in Proposition 6.4, so $Z \otimes_H (V/E) \cong Z \otimes_H V_1$. But $I \otimes_H E$ is clearly the projection of $Z \otimes_H V$ onto $Z \otimes_H V_1$ so that we also have $(Z\otimes_H V)/I\otimes_H E \cong Z\otimes_H V_1$. Thus $Z\otimes_H (V/E) \cong (Z\otimes_H V)/I\otimes_H E$ as desired.

References

[1] R. J. Blattner, *Positive Definite Measures*, Proc. Amer. Math. Soc. 14 (1963), pp. 423–428.
[2] C. W. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, New York 1962.
[3] J. Dieudonné, *Representaciones de Groupos Compactos y Functiones Esfericas*, Cursos y Seminarios de Matemática 14, Universidad de Buenos Aires, 1964.
[4] J. Dixmier, *Les Algèbres d'Operateurs dans l'Espace Hilbertien (Algèbres de von Neumann)*, Paris 1957.
[5] — *Les C*-algèbres et leurs Représentations*, Paris 1964.
[6] N. Dunford and J. Schwartz, *Linear Operators*, Vol. II, New York, 1963.
[7] L. C. Grove, *A generalized group algebra for compact groups*, Studia Math. 26 (1965), pp. 73–90.
[8] — *Tensor products over H*-algebras*, Pacific J. Math. 15 (1965), pp. 857–863.
[9] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis*, Vol. I, Berlin, 1963.
[10] S. Lang, *Algebra*, Reading 1965.
[11] F. E. J. Linton, *Autonomous categories and duality of functors*, J. Algebra 2 (1965), pp. 315–349.
[12] L. H. Loomis, *An Introduction to Abstract Harmonic Analysis*, New York 1953.
[13] G. W. Mackey, *Imprimitivity for representations of locally compact groups*, I. Proc. Nat. Acad. Sci. U. S. 35 (1949), pp. 537–545.
[14] — *On induced representations of groups*. Am. J. Math. 73 (1951). pp. 576–592.
[15] — *Induced representations of locally compact groups*, I. Ann. Math. 55 (1952), pp. 101–139.
[16] — *The Theory of Group Representations*, Mimeographed notes, University of Chicago, 1955.
[17] S. MacLane, *Categorial algebra*, Bull. Am. Math. Soc. 71 (1965), pp. 40–106.
[18] F. I. Mautner, *A generalization of the Frobenius reciprocity theorem*, Proc. Nat. Acad. Sci. U. S., 37 (1951), pp. 431–435.
[19] — *Induced representations*, Amer. J. Math. 74 (1952), 737–758.
[20] K. Morita, *Adjoint pairs of functors and Frobenius extensions*, Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A, 9 (1965), pp. 40–71.
[21] F. J. Murray and J. von Neumann, *On rings of operators*, Ann. Math. 37 (1936), pp. 116–229.
[22] M. A. Rieffel, *Induced Banach representations of Banach algebras and locally compact groups*, J. Func. Anal. 1 (1967), pp. 443–491.
[23] — *Square-integrable representations of Hilbert algebras*, J. Func. Anal. 3 (1969), pp. 265–300.
[24] — *Multipliers and tensor products of L^p-spaces of locally compact groups*, Studia Math. 33 (1969), pp. 71–82.
[25] R. Schatten, *A Theory of Cross-spaces*, Princeton, New Jersey, 1950.
[26] K. Vala, *Sur le produit tensoriel des espaces hilbertiens*, Suomalarsen Tiedeakatemia 267 (1959), pp. 3–17.
[27] A. Weil, *L'intégration dans les Groupes Topologiques et ses Applications*, Actual. Sci. et Ind. 869, Paris, 1940.
[28] D. G. Higman, *Induced and produced modules*, Canadian J. Math. 7 (1955), pp. 490–508.
[29] T. Nakayama, *A remark on representations of groups*, Bull. Amer. Math. Soc. 44 (1938), pp. 233–235.
[30] L. Pukanszky, *Leçons sur les Représentations des Groupes*, Paris 1967.

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