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Flat spaces of continuous functions*

by

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Abstract. A Banach space is flat if there exists on the surface of its unit ball a curve of length 2 with antipodal endpoints. We characterize the flat spaces among the spaces $C(K)$ and $C_\sigma(K)$, K compact, σ an involutory automorphism of K, as well as some other spaces of continuous functions, by topological properties of the domain. We note that every infinite-dimensional space $L^\infty(\mu)$ is flat.

1. Introduction. In [7], Harrell and Karlovitz call a Banach space *flat* if there exists on the surface of its unit ball a curve of length 2 with antipodal endpoints. They observe that $L^1(\mu)$, where μ is Lebesgue measure on the unit interval, is flat, but that $l^1(\aleph_0)$ is not. They had shown earlier [6] that a flat space is not reflexive, and that $C([0,1])$ is flat. In [12], Schäffer showed that $L^1(\mu)$ for a general measure space is flat if and only if μ is not purely atomic.

Continuing the investigation of the flatness of "classical" spaces, we are led to consider the space $C(K)$ for a compact Hausdorff space K, and, more generally, the subspace $C_\sigma(K)$ of those functions that are skew with respect to an involutory automorphism σ of K. The purpose of this paper is to give a complete account of which $C_\sigma(K)$ are flat: in terms of the topology of K, they are exactly those for which there exists a non--empty dense-in-itself set in K not containing fixed points of σ. The flatness of $C_\sigma(K)$ can also be characterized in terms of the geometry of its dual: in particular, $C_\sigma(K)$ is flat if and only if its dual is flat.

These various characterizations yield a similar account for $C(K)$ itself and for $C_0(T)$, the space of continuous functions vanishing at infinity on the locally compact Hausdorff space T. Among other results concerning spaces congruent to some $C_\sigma(K)$ we note the fact that every infinite-dimensional space $L^\infty(\mu)$ is flat.

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The spaces $C_\sigma(K)$ are discussed and characterized by their metric properties in ([2]; pp. 87–96), an account of work due in the main to Jerison. Lindenstrauss [8] proposes an interesting definition of "classical Banach spaces in the isometric sense"; he points out that they turn out to be exactly the Banach spaces congruent to $L^p(\mu)$ for $1 \leqslant p < \infty$, together with those whose dual is congruent to some $L^1(\mu)$. Now the $L^p(\mu)$ are reflexive, and therefore not flat, for $1 < p < \infty$; and the $L^1(\mu)$ were classified as to their flatness in [12]. The spaces $C_\sigma(K)$ are important instances of spaces with duals congruent to L^1-spaces, but do not exhaust this class by far (see [9] and references given there for a complete description). It would be interesting to decide which of the remaining such spaces are flat — thus completing the survey of all "classical" spaces — or at least which M-spaces or G-spaces are flat (terminology as in [9]). The fragmentary results available on this point are not included here.

The question of flatness of Banach spaces belongs to an area of investigation begun in [11] and continued in other papers, dealing with certain metric parameters of the unit spheres of normed spaces. In another paper [13] one of us shall discuss the values of these, viz., the *inner diameter*, the *perimeter*, and the *girth*, for all the spaces treated here.

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2. Preliminaries. If X is a normed space, a *subspace* of X is a linear manifold of X (not necessarily closed), provided with the norm of X. A *congruence* is an isometric isomorphism of one normed space onto another.

A curve in X is a "rectifiable geometric curve" as defined in ([1]; pp. 23–26); for terminological details see ([11]; p. 61). The length of a curve $\mathfrak{c}$ is $l(\mathfrak{c})$, and its standard representation in terms of arc-length is $g_{\mathfrak{c}}\colon [0, l(\mathfrak{c})] \to X$.

X is *flat* if there is a curve of length 2 in the boundary of the unit ball of X such that its endpoints are antipodal; i.e., a curve $\mathfrak{c}$ with $l(\mathfrak{c}) = 2$, $\|g_{\mathfrak{c}}(s)\| = 1$ for $s \in [0, 2]$, and $g_{\mathfrak{c}}(0) + g_{\mathfrak{c}}(2) = 0$. If a subspace is flat, it obviously follows that X itself is flat.

Let T be a Hausdorff space; then $C(T)$ is the Banach space of all bounded real-valued continuous functions on T with the supremum norm. Let σ be an involutory automorphism of T, i.e., a homeomorphism of T onto T with $\sigma \circ \sigma = \mathrm{id}$. Then $C_\sigma(T)$ denotes the closed subspace $\{f \in C(T)\colon f(t) + f(\sigma t) = 0,\ t \in T\}$ of $C(T)$; it is also a Banach space. We set $T^\sigma = \{t \in T\colon \sigma t \neq t\}$ the open set of points not fixed by σ, and observe once and for all that

$$(1) \qquad f \in C_\sigma(T) \quad \text{implies } f(T \setminus T^\sigma) \subset \{0\}.$$

If T is locally compact, $C_0(T)$ denotes the closed subspace of $C(T)$ consisting of the real-valued continuous functions on T that vanish at infinity. If σ is as before, we set $C_{0\sigma}(T) = C_0(T) \cap C_\sigma(T)$.

We summarize a useful remark for the study of $C_\sigma(K)$, K compact, in the following lemma.

LEMMA 1. *Let K be a compact Hausdorff space and σ an involutory automorphism of K. Let $K' = K^\sigma \cup \{\infty\}$ be the one-point compactification of the locally compact space K^σ, and $\sigma'\colon K' \to K'$ defined by $\sigma' t = \sigma t$, $t \in K^\sigma$ and $\sigma'\infty = \infty$. Then σ' is an involutory automorphism of the compact Hausdorff space K', $K'^{\sigma'} = K^\sigma$, and the mapping $f \mapsto f'\colon C_\sigma(K) \to C_{\sigma'}(K')$ defined by $f'(t) = f(t)$, $t \in K^\sigma$, and $f'(\infty) = 0$ is a congruence.*

Proof. Immediate from the definitions and (1).

A Hausdorff space T contains a largest dense-in-itself subset; this set is closed and is called the *perfect core* of T. A space is *scattered* if its perfect core is empty. Pełczyński and Semadeni [10] have given a great number of equivalent conditions for a compact space K to be scattered, and especially some involving $C(K)$ and $(C(K))^*$. We reformulate for our use three of these conditions. If K is a compact Hausdorff space and $t \in K$, the *evaluation functional* $e_t \in (C(K))^*$ is defined by $\langle f, e_t \rangle = f(t)$, $f \in C(K)$.

THEOREM 2. (PEŁCZYŃSKI AND SEMADENI). *Let K be a compact Hausdorff space. The following statements are equivalent:*

(a) *K is not scattered;*

(b) *there exists $h \in C(K)$ such that $h(K) = [0, 1]$;*

(c) *the linear mapping $\Gamma\colon l^1(K) \to (C(K))^*$ defined by $\Gamma y = \sum_{t \in K} y(t) e_t$, $y \in l^1(K)$, is not surjective.*

Proof. ([10]; Main Theorem, (0), (3), (11)).

3. The main result. We examine the following properties that a normed space X may have:

(F1): *X is flat;*

(F2): *X^* is not the closed linear span of the extreme points of its unit ball;*

(F3): *X^* is not congruent to $l^1(A)$ for any set A;*

(F4): *X^* is flat.*

We observe that $l^1(A)$ is the closed linear span of the extreme points of its unit ball, so that (F2) always implies (F3).

Before we discuss these conditions as applicable to a space $C_\sigma(K)$, we look at a special case. We define $\pi\colon [-1, 1] \to [-1, 1]$ by $\pi t = -t$, an involutory automorphism of $[-1, 1]$. The proof of the following lemma is an adaptation of a construction in [6].

LEMMA 3. *The space $C_\pi([-1,1])$ is flat.*

Proof. We define $g\colon [0,2]\to C_\pi([-1,1])$ by

$$\big(g(s)\big)(t) = -\big(g(s)\big)(-t) = \begin{cases} 2(1-s)t, & 0\leqslant t\leqslant \tfrac12,\\ |4-4t-s|-1, & \tfrac12\leqslant t\leqslant 1,\end{cases}\qquad 0\leqslant s\leqslant 2.$$

Then $\|g(s)\| = \big|\big(g(s)\big)(1-\tfrac14 s)\big| = 1$, and $\|g(s')-g(s)\| = |s'-s|$, as is easily verified directly. Therefore g is Lipschitzian, and is the standard representation in terms of arc-length of a curve of length 2 in the boundary of the unit ball of $C_\pi([-1,1])$. But $g(2) = -g(0)$, so the endpoints of the curve are antipodal, and the space is flat.

In the rest of this section, we shall be dealing with a given compact Hausdorff space K and an involutory automorphism σ of K. The following construction is useful. Let V be a closed set in K with $V\cap\sigma V = \emptyset$, and let $f_0\in C(V)$ be given. By the Tietze Extension Theorem there exists $f_1\in C(K)$ with $\|f_1\| = \|f_0\|$ and $f_1(t) = -f_1(\sigma t) = f_0(t)$, $t\in V$. We define $f\colon K\to R$ by $f(t) = \tfrac12\big(f_1(t)-f_1(\sigma t)\big)$, $t\in K$, and find $f\in C_\sigma(K)$, $\|f\| = \|f_0\|$, and $f(t) = f_0(t)$, $t\in V$. Such a function f shall be called a *skew Tietze extension of* f_0.

For every $t\in K$, we consider the evaluation functional $e_t^\sigma\in\big(C_\sigma(K)\big)^*$, (the restriction of e_t to $C_\sigma(K)$) defined by $\langle f, e_t^\sigma\rangle = f(t)$, $f\in C_\sigma(K)$. *The set* $\{e_t^\sigma\colon t\in K^\sigma\}$ *is exactly the set of extreme points of the unit ball of* $\big(C_\sigma(K)\big)^*$ ([2]; p. 89).

LEMMA 4. *With K, σ as specified, let a non-empty set $P\subset K$ satisfy $P\cap\sigma P = \emptyset$. Then the linear mapping $\Gamma_P\colon l^1(P)\to\big(C_\sigma(K)\big)^*$ defined by $\Gamma_P y = \sum_{t\in P} y(t)e_t^\sigma$, $y\in l^1(P)$, is isometric.*

Proof. Obviously, $P\subset K^\sigma$. Now $\|e_t^\sigma\| = 1$, $t\in P$, so Γ_P is well defined, linear, and bounded, and $\|\Gamma_P\|\leqslant 1$. It remains to prove that $\|\Gamma_P y\|\geqslant\|y\|$ for all $y\in l^1(P)$, or at least for all those with finite support. If $Q\subset P$ is finite and $y(t) = 0$, $t\in P\setminus Q$, we can find, by means of a skew Tietze extension, $f\in C_\sigma(K)$ with $\|f\| = 1$ and $f(t) = \operatorname{sgn} y(t)$, $t\in Q$. Then

$$\|\Gamma_P y\|\geqslant\|f\|\,\|\Gamma_P y\|\geqslant\Big|\Big\langle f, \sum_{t\in Q} y(t)e_t^\sigma\Big\rangle\Big| = \sum_{t\in Q} y(t)\operatorname{sgn} y(t) = \sum_{t\in Q}|y(t)| = \|y\|.$$

We are now ready to characterize those K and σ for which $C_\sigma(K)$ satisfies (F1)–(F4).

THEOREM 5. *Let K be a compact Hausdorff space and σ an involutory automorphism of K. Then* (F1), (F2), (F3), (F4) *are equivalent for $X = C_\sigma(K)$, and also equivalent to each of the following statements:*

(a): *K^σ is not scattered;*

(b): *there exists $h\in C_\sigma(K)$ with $h(K^\sigma) = [-1,1]$;*

(c): *there exists $h\in C_\sigma(K)$ with $h(K) = [-1,1]$.*

Proof. We add one more statement to the list:

(d): *if $P\subset K$ satisfies $P\cap\sigma P = \emptyset$, $P\cup\sigma P = K^\sigma$, the isometric linear mapping $\Gamma_P\colon l^1(P)\to\big(C_\sigma(K)\big)^*$ defined in Lemma 4 is not surjective;*

and prove the implications

$$\begin{tikzcd}
 & & (a) \arrow[dl] & & & & \\
 & (b) \arrow[dr] & & (d) \arrow[ul] \arrow[rr] & & (F2) \arrow[dl] & \\
(F1) \arrow[ur] & & (c) \arrow[ll] \arrow[ur] & & (F3) \arrow[ul] \arrow[rr, leftrightarrow] & & (F4)
\end{tikzcd}$$

In view of the formulation of statements (a), (b), (c), (d) it is possible to apply Lemma 1 (observing (1)) and assume without loss, as we shall in this proof, that $K\setminus K^\sigma$ is a singleton, say $\{\infty\}$. If $K^\sigma = \emptyset$, the theorem is trivial. We therefore assume without loss that $K^\sigma\neq\emptyset$.

The implication (F2) $\to$ (F3) was noted above, and the implications (b) $\to$ (c) and (F3) $\to$ (d) are trivial.

(a) implies (b). The perfect core S of K^σ is not empty; choose $t_0\in S$. Since $t_0\neq\sigma t_0$, there exists an open neighborhood U of t_0 such that $\operatorname{cl}U\cap\operatorname{cl}(\sigma U) = \emptyset$; in particular, $\operatorname{cl}U\subset K^\sigma$. Then $U\cap S$ is non-empty and dense-in-itself, hence $\operatorname{cl}U$ is compact and not scattered. By Theorem 2 there exists $h_0\in C(\operatorname{cl}U)$ with $h_0(\operatorname{cl}U) = [0,1]$. A skew Tietze extension h of h_0 satisfies $h\in C_\sigma(K)$, $[-1,1]\supset h(K^\sigma)\supset h(\operatorname{cl}U)\cup h\big(\operatorname{cl}(\sigma U)\big) = [0,1]\cup\cup[-1,0] = [-1,1]$, as required by (b).

(c) implies (F1). With h as in (c), the mapping $\varphi\mapsto\varphi\circ h$ is a congruence of $C_\pi([-1,1])$ onto a closed subspace of $C_\sigma(K)$. By Lemma 3, this subspace is flat; hence $C_\sigma(K)$ itself is flat.

(F1) implies (b). Let $\mathfrak{c}$ be a curve of length 2 in the boundary of the unit ball of $C_\sigma(K)$, with antipodal endpoints. Let $r\in[-1,1]$ be given. Since $g_{\mathfrak{c}}(1-r)\in C_\sigma(K)$, $\|g_{\mathfrak{c}}(1-r)\| = 1$, there exists $t_r\in K^\sigma$ such that $\big(g_{\mathfrak{c}}(1-r)\big)(t_r) = 1$. Then

$$\begin{aligned} r = 1-(1-r) &\leqslant 1-\|g_{\mathfrak{c}}(1-r)-g_{\mathfrak{c}}(0)\|\leqslant 1-\Big(1-\big(g_{\mathfrak{c}}(0)\big)(t_r)\Big)\\ &= \big(g_{\mathfrak{c}}(0)\big)(t_r) = \Big(1+\big(g_{\mathfrak{c}}(0)\big)(t_r)\Big)-1\leqslant\|g_{\mathfrak{c}}(1-r)+g_{\mathfrak{c}}(0)\|-1\\ &= \|g_{\mathfrak{c}}(2)-g_{\mathfrak{c}}(1-r)\|-1\leqslant 2-(1-r)-1 = r.\end{aligned}$$

Therefore $r = \big(g_{\mathfrak{c}}(0)\big)(t_r)\in\big(g_{\mathfrak{c}}(0)\big)(K^\sigma)$; since $r\in[-1,1]$ is arbitrary and $\|g_{\mathfrak{c}}(0)\| = 1$, we conclude that (b) is satisfied with $h = g_{\mathfrak{c}}(0)$.

(c) implies (d). With h as in (c), consider once more the congruence $\varphi \mapsto \varphi \circ h$ of $C_\pi([-1,1])$ onto a closed subspace of $C_\sigma(K)$. If, contrary to (d), Γ_P were surjective for some $P \subset K$, $P \cap \sigma P = \emptyset$, then every element of $\big(C_\pi([-1,1])\big)^*$ would, by the Hahn–Banach Theorem, be of the form $\sum_{t\in P} y(t) e^{\pi}_{h(t)}$, $y \in l^1(P)$; however the linear functional $\varphi \mapsto \int_0^1 \varphi(r)\,dr$ on $C_\pi([-1,1])$ is bounded, but not of this form.

(d) implies (a). Assume, contrary to (a), that K^σ is scattered. Since $K \setminus K^\sigma$ is a singleton, K itself is scattered. Let $x^* \in \big(C_\sigma(K)\big)^*$ be given. By the Hahn–Banach Theorem, x^* can be extended to an element of $\big(C(K)\big)^*$. By Theorem 2, there exists $y_0 \in l^1(K)$ such that

$$\langle f, x^*\rangle = \langle f, \Gamma y_0\rangle = \Big\langle f, \sum_{t\in K} y_0(t) e_t\Big\rangle = \Big\langle f, \sum_{t\in K^\sigma} y_0(t) e^\sigma_t\Big\rangle, \qquad f \in C_\sigma(K),$$

since $\langle f, e_\infty\rangle = f(\infty) = 0$. Thus

$$x^* = \sum_{t\in K^\sigma} y_0(t) e^\sigma_t. \tag{2}$$

Let P be any set in K that is maximal with respect to the condition $P \cap \sigma P = \emptyset$ (such exist, by Zorn's Lemma); then $P \cup \sigma P = K^\sigma$. We define $y \in l^1(P)$ by $y(t) = y_0(t) - y_0(\sigma t)$, $t \in P$ (so that $\|y\| \leqslant 2\,\|y_0\|$). Then (2) implies — since $e^\sigma_{\sigma t} = -e^\sigma_t$, $t \in P$ —

$$\begin{aligned} x^* &= \sum_{t\in P} y_0(t) e^\sigma_t + \sum_{t\in\sigma P} y_0(t) e^\sigma_t = \sum_{t\in P} \big(y_0(t) e^\sigma_t + y_0(\sigma t) e^\sigma_{\sigma t}\big) \\ &= \sum_{t\in P} y(t) e^\sigma_t = \Gamma_P y. \end{aligned}$$

Since $x^* \in \big(C_\sigma(K)\big)^*$ was arbitrary, Γ_P is surjective, in contradiction to (d).

(d) implies (F2). Let $P \subset K$ satisfy $P \cap \sigma P = \emptyset$, $P \cup \sigma P = K^\sigma$; we have just shown that such a set exists. As noted earlier in this section, the set of extreme points of the unit ball of $\big(C_\sigma(K)\big)^*$ is $\{e^\sigma_t\colon t \in K^\sigma\} = \{\pm e^\sigma_t\colon t \in P\}$. But the image of Γ_P contains this set, and hence also contains (actually, coincides with) the closed linear span of this set of extreme points. The required implication follows.

(F3) is equivalent to (F4). Since $\big(C_\sigma(K)\big)^*$ is an abstract L-space (cf. [9]), it is flat if and only if it is not congruent to $l^1(A)$ for any set A [12].

Remark 1. Using statement (a), it is possible to apply the equivalences of Pełczyński and Semadeni [10] to derive many other conditions equivalent to (F1)–(F4) for $X = C_\sigma(K)$; e.g., *there exists a non-atomic regular finite Borel measure ν on K such that $\nu(K^\sigma) > 0$.*

Remark 2. In [13] we shall give further conditions on the metric structure of the unit balls of X, X^* that are equivalent to (F1)–(F4) for $X = C_\sigma(K)$.

4. Applications to other spaces. Theorem 5 provides criteria for the flatness of Banach spaces congruent to $C_\sigma(K)$. The following theorems summarize some of these criteria.

The proofs will merely be sketched. We recall that a topological space is *basically disconnected* if the closure of every co-zero set is open; extremally disconnected spaces are basically disconnected.

THEOREM 6. *Let T be a completely regular Hausdorff space. Then (F1), (F2), (F3), (F4) are equivalent for*

the space $X =$	*where* T *is*	*and hold if and only if*
$C_0(T)$	*locally compact*	T *is not scattered; or equivalently, there exists* $h \in X$ *with* $h(T) = [0,1]$;
$C(T)$	*compact*	
	pseudocompact	*there exists* $h \in X$ *with* $h(T) = [0,1]$;
	not pseudocompact	(*always*)
	paracompact (*in particular, metrizable*)	T *is not both compact and scattered*;
	basically disconnected	T *is infinite.*

Proof. 1. If T is locally compact, there is a natural congruence between $C_0(T)$ and $C_\sigma(K)$, where the compact space K is $T \times \{-1, 1\}$ if T is compact, and the one-point compactification of this product otherwise, and $\sigma\colon K \to K$ interchanges the second components -1 and 1 in the product. The conclusion for $C_0(T)$ — and for $C(T)$ if T is compact — then follows from Theorem 5.

2. In general, there is a natural congruence between $C(T)$ and $C(\beta T)$ where βT is the Stone–Čech compactification of T. Application of the preceding result to $C(\beta T)$ shows that $C(T)$ satisfies (F1), (F2), (F3), (F4) if and only if

(*) *there exists $h \in C(T)$ such that $h(T)$ is a dense subset of* $[0,1]$.

If T is pseudocompact, every continuous image of T in R is pseudocompact, hence compact. In this case, (*) is equivalent to the existence of $h \in C(T)$ with $h(T) = [0,1]$. If T is not pseudocompact, we prove (*) by an argument adapted from [4]. Let $f\colon T \to R$ be an unbounded continuous function; there exists a countably infinite set $S \subset f(T)$ that is closed and discrete in R. Since R is normal, there exists a continuous $\varphi\colon R \to [0,1]$ such that $\varphi(S)$ is dense in $[0,1]$. Then $h = \varphi \circ f$ satisfies (*).

The conclusion for paracompact T follows from these results, since such a space is pseudocompact if and only if it is compact.

3. If T is basically disconnected and infinite, βT contains a subset homeomorphic to βN ([5]; 9H); but βN is not scattered, hence βT is not scattered. The conclusion follows from the preceding results applied to $C(\beta T)$.

Remark. The condition for non-compact pseudocompact spaces in Theorem 6 is unsatisfactory: pseudocompactness itself has a simple *intrinsic* characterization for completely regular Hausdorff spaces ([3]; p. 232), but we lack such a characterization of those pseudocompact spaces that can be mapped continuously onto $[0, 1]$. Such spaces may well be scattered: it is easy to construct a suitable instance of the scattered locally compact pseudocompact space Ψ described in ([5]; 5I) so that it has a continuous mapping onto $[0, 1]$; the construction is suggested by ([5]; 6Q).

THEOREM 7. *If* $(S, \mathbf{S}, \mu)$ *is any measure space,* (F1), (F2), (F3), (F4) *are equivalent for* $X = L^\infty(\mu)$ *and hold unless this space is finite-dimensional.*

Proof. $L^\infty(\mu)$ is congruent to $C(T)$, where T is the Stone space of the σ-complete Boolean measure algebra of μ ([14]; pp. 206–207). T is compact and basically disconnected; a proof might use ([14]; pp. 85–86) and ([5]; Theorem 16, 17). The conclusion follows from Theorem 6.

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