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Nonfactorization in group algebras

by

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Abstract. In this paper we study two large classes of group algebras — IFP group algebras and Segal algebras. Many examples of IFP group algebras and Segal algebras are given in Examples 2.4. The main results are as follows: I. (See Theorem 3.1 and 3.3) An IFP group algebra or a proper Segal algebra does not have any weak bounded approximate identity. II. (See Theorem 4.1 and Corollary 4.2). An IFP group algebra or a character Segal algebra with property F does not have the weak factorization property. III. If A is an IFP $*$-algebra such that A^2 is dense in A, then there exists a discontinuous positive linear functional on A.

1. Introduction. Factorization and nonfactorization (See 2.I for the definitions) in various group algebras have been studied by a number of mathematicians. W. Rudin in 1957 [12] proved $L^1(R)*L^1(R) = L^1(R)$, and in 1958 [13] proved $L^1(G)*L^1(G) = L^1(G)$, where G is any locally Euclidean abelian group. Using an interesting method, P. J. Cohen in 1959 [1] proved that a Banach algebra with a (left) bounded approximate identity (See 2.1 for the definition) has the factorization property. In particular, $L^1(G)*L^1(G) = L^1(G)$, where G is any locally compact group. In 1965, R. E. Edwards [3] proved that if $1 < p < \infty$ and G is any infinite compact abelian group, then the group algebra $L^p(G)$ does not have the weak factorization property (See 2.1 for the definition). L. Y. H. Yap in 1970 [17] proved that if $1 < p < \infty$ and G is any non-discrete locally compact abelian group, then the group algebra $L^1 \cap L^p(G)$ under the norm $\|f\| = \|f\|_1 + \|f\|_p$ has neither a weak bounded approximate identity nor the weak factorization property (See 2.1 for the definition). In 1970, J. C. Martin and L. Y. H. Yap [10] proved that if $1 \leqslant p < \infty$ and G is any non-discrete locally compact abelian group, then the group algebra $F^{1p}(G)$ of all f in $L^1(G)$ whose Fourier transforms are in L^p, under the norm $\|f\| = \|f\|_1 + \|\hat{f}\|_p$, has neither a weak bounded approximate identity nor the weak factorization property. There is, therefore, good reason to believe that an algebra which does not have a weak bounded approximate identity will not have the weak factorization property.

In this paper we prove that a large class of group algebras, the so-called IFP group algebras, lack both a weak bounded approximate identity and the weak factorization property. Many IFP algebras are listed in 2.4. Our main results are contained in Theorem 3.1 and Theorem 4.1. Edwards', Yap's, and Martin–Yap's results then are corollaries of our theorems.

Making use of Cohen's construction, N. Th. Varopoulos [14] in 1964 proved that for any locally compact group G, every positive functional on $L^1(G)$ is continuous. Using the results in Section 4, we proved in Section 5 that each known IFP algebra A is an $*$-algebra under some $*$ and there exists a discontinuous functional on A except the algebra $L^\infty(G)$.

These results are taken from the author's doctoral dissertation at the University of Iowa, written under the direction of Professor Richard R. Goldberg.

2. Definitions and examples. Throughout this paper unless the contrary is stated, G will denote a non-discrete locally compact abelian group with character group Γ. The identities in G and Γ will be denoted by e and d, respectively.

In this section we give definitions and examples of group algebras which form several chains. Afterwards, we shall define and study the IFP group algebras.

DEFINITION 2.1. Let $(A, \|\ \|_A)$ be a commutative Banach algebra. We say that A has the *factorization property* if for every $x \in A$, there exist $y, z \in A$ such that $x = yz$. We say that A has the *weak factorization property*, in symbols $A^2 = A$, if for every $x \in A$, there exist $y_1, \ldots, y_n, z_1, \ldots, z_n \in A$ such that $x = y_1z_1 + \ldots + y_nz_n$. We say that A has a *bounded approximate identity* if there is a constant D and a net $(x_\lambda)_{\lambda\in\Lambda}$ in A with $\|x_\lambda\|_A \leqslant D$ for every $\lambda \in \Lambda$, such that $\lim_\lambda x_\lambda x = x$ for every $x \in A$. We say that A has a *weak bounded approximate identity* if there exists a constant D such that, for every $x \in A$ and for every $\varepsilon > 0$, there exists $y \in A$ such that $\|y\|_A \leqslant D$ and $\|yx - x\|_A < \varepsilon$.

DEFINITION 2.2. A dense Banach algebra $(A, \|\ \|_A)$ in the group algebra $L^1(G)$ is a *L^1-dense subspace of* $L^1(G)$ and also a Banach algebra under $\|\ \|_A$ with convolution as multiplication.

DEFINITION 2.3. A dense Banach algebra $(S(G), \|\ \|_S)$ in $L^1(G)$ is a *Segal algebra on* G if the following properties are satisfied:

S-1. If $f \in S(G)$ and $a \in G$, then $L_af(x) = f(xa^{-1}) \in S(G)$ and $\|L_af\|_S = \|f\|_S$.

S-2. For each $f \in S(G)$, the mapping $x \to L_xf$ is a continuous mapping of G into $(S(G), \|\ \|_S)$.

By S-1, S-2 is equivalent to

S-2′. For each $f \in S(G)$, the mapping $x \to L_xf$ is continuous at e.

A Segal algebra $(S(G), \|\ \|_S)$ is said to be *character* if $f \in S(G)$, $\gamma \in \Gamma$ imply $\gamma f \in S(G)$ and $\|\gamma f\|_S = \|f\|_S$. $L^1(G)$ is a *character Segal algebra on* G. A Segal algebra on G is *proper* if it is not the whole of $L^1(G)$.

EXAMPLES 2.4. We list here many group algebras. All of them are Segal algebras, and some of them are character Segal algebras. We shall eventually prove that they are all IFP algebras (as defined in 2.9).

2.4.1. Let $1 \leqslant k < \infty$ and T be the circle group. The Banach algebra $C^k(T)$ of all functions with k continuous derivatives on T under the norm $\|f\|_{C^k} = \sup_{0\leqslant j\leqslant k} \max_{x\in T} |f^{(j)}(x)|$. $C^k(T)$ is a non-character Segal algebra.

2.4.2. Let G be an infinite compact abelian group. The Banach algebra $C(G)$ of all continuous functions on G under the norm $\|f\|_\infty = \max_{x\in G} |f(x)|$. $C(G)$ is a character Segal algebra.

2.4.3. Let $1 < p < \infty$ and G be an infinite compact abelian group with normalized Haar measure dx. The Banach algebra $L^p(G)$ of all measurable functions f with $\int_G |f(x)|^p dx < \infty$ under the norm $\|f\|_p = (\int_G |f(x)|^p dx)^{\frac{1}{p}}$. $L^p(G)$ is a character Segal algebra [7], [3], [5; p. 356].

2.4.4. Let R be the additive group of all real numbers. The Banach algebra $L^A(R)$ of all functions f in $L^1(R)$ which are absolutely continuous on R with $f' \in L^1(R)$, under the norm $\|f\|_{L^A} = \|f\|_1 + \|f'\|_1$. $L^A(R)$ is a non-character Segal algebra ([11]; p. 9).

2.4.5. Let G be a non-discrete locally compact abelian group having a discrete subgroup H such that G/H is compact. (There will then exist a compact set K of measure 1 in G such that $G = HK$). The Banach algebra $T(G)$ of all continuous functions of G for which $\sup_{u\in G} \sum_{h\in H} \max_{x\in K} |f(uhx)| < \infty$ under the norm $\|f\|_T = \sup_{u\in G} \sum_{h\in H} \max_{x\in K} |f(uhx)|$. $T(G)$ is a character Segal algebra [16], [2], [4], ([11]; p. 127). For example, $G = R$, $H =$ integers, $K = [0, 2\pi]$, Haar measure $= \frac{1}{2\pi} \cdot$ Lebesgue measure.

2.4.6. Let G be any non-discrete locally compact abelian group. The Banach algebra $L^1 \cap C_0(G)$ of all continuous functions in $L^1(G)$ which vanish at infinity under the norm $\|f\|_{10} = \|f\|_1 + \|f\|_\infty$ where $\|f\|_\infty = \sup_{x\in G} |f(x)|$. $L^1 \cap C_0(G)$ is a character Segal algebra ([11]; p. 9).

2.4.7. Let $1 < p < \infty$ and G be any non-discrete locally compact abelian group. The Banach algebra $L^1 \cap L^p(G)$ of all functions in $L^1(G)$ and $L^p(G)$, under the norm $\|f\|_{1p} = \|f\|_1 + \|f\|_p$. $L^1 \cap L^p(G)$ is a character Segal algebra [15], [17].

2.4.8. Let $1 \leqslant p < \infty$ and G be any non-discrete locally compact abelian group. The Banach algebra $F^{1p}(G)$ of all functions f in $L^1(G)$ whose Fourier transforms $\hat{f}$ belong to $L^p(\Gamma)$, under the norm $\|f\|_{F^{1p}} = \|f\|_1 + \|\hat{f}\|_p$. $F^{1p}(G)$ is a character Segal algebra [6], [8], [10].

Notations as in Examples 2.4, for the reference convenience we state without proving the following three theorems:

THEOREM 2.5. *The group algebras $C(G)$, $L^p(G)$, $T(G)$, $L^1 \cap C_0(G)$, $L^1 \cap L^p(G)$ and $F^{1p}(G)$ are character Segal algebras.*

THEOREM 2.6. *The algebras $C^k(T)$ and $L^A(R)$ are non-character Segal algebras.*

THEOREM 2.7. ([11]; p. 128) *If G is a discrete locally compact abelian group, then a Segal algebra on G is the whole of $L^1(G)$.*

THEOREM 2.8. *The group algebras in 2.4, together with the group algebra L^1, form a number of chains as follows:*

I. $\ldots \subset C^{k+1}(T) \subset C^k(T) \subset \ldots \subset C(T) \subset \ldots \subset L^r(T) \subset L^s(T) \subset \ldots \ldots \subset L^1(T)$, *where k is a positive integer and $r > s > 1$.*

II. $C(G) \subset \ldots \subset L^r(G) \subset L^s(G) \subset \ldots \subset L^1(G)$, *where G is an infinite compact abelian group, and $r > s > 1$.*

III. $L^A(R) \subset T(R) \subset L^1 \cap C_0(R) \subset L^1 \cap L^p(R) \subset F^{1t}(R) \subset L^1(R)$, *where $p > 1$, and $t = 2$ if $p > 2$, $t = p/p-1$ if $p \leqslant 2$.*

IV. $T(G) \subset L^1 \cap C_0(G) \subset L^1 \cap L^p(G) \subset F^{1t}(G) \subset L^1(G)$, *where G is a non-discrete locally compact abelian group with a discrete subgroup H such that G/H is compact, $p > 1$ and $t = 2$ if $p > 2$, $t = p/p-1$ if $p \leqslant 2$.*

V. $L^1 \cap C_0(G) \subset L^1 \cap L^p(G) \subset F^{1t}(G) \subset L^1(G)$, *where G is any non-discrete locally compact abelian group, $p > 1$ and $t = 2$ if $p > 2$, $t = p/p-1$ if $p \leqslant 2$.*

Proof. First we shall prove that $L^A(R) \subset T(R)$. Let $f \in L^A(R)$, and let $V_{-\infty}^{\infty} f = \lim_{n\to\infty} V_{-n}^{n} f$, where V_{-n}^{n} is the total variation on $[-n, n]$. By the Radon–Nikodym Theorem, $V_{-\infty}^{\infty} f = \|f'\|_1$. Moreover,

$$\sum_{k=-n}^{n-1} \max_{x\in[0,1]} |f(k+x)| \leqslant \sum_{k=-n}^{n-1} \min_{x\in[0,1]} |f(k+x)| + V_{-n}^{n} f \leqslant \|f\|_1 + \|f'\|_1 = \|f\|_{L^A}.$$

Therefore $\sum_{k=-\infty}^{\infty} \max_{x\in[0,1]} |f(k+x)| \leqslant \|f\|_{L^A}$. For any continuous function g on R, by routine computation, $\sup_{u\in R} \sum_{n=-\infty}^{\infty} \max_{x\in[0,1]} |g(u+n+x)| \leqslant 2 \sum_{n=-\infty}^{\infty} \max_{x\in[0,1]} |g(n+x)|$. Therefore $\|f\|_T \leqslant 2\, \|f\|_{L^A}$, so $f \in T(R)$.

Goldberg [4] proved $T(R) \subset L^1 \cap C_0(R)$. Similarly it may be shown that $T(G) \subset L^1 \cap C_0(G)$.

If $p \leqslant 2$ and $f \in L^1 \cap L^p(G)$, then, by the Hausdorff–Young theorem, $\hat{f} \in L^t(\Gamma)$ where $t = p/p-1$. This shows that $L^1 \cap L^p(G) \subset F^{1t}(G)$. If $p > 2$, and $f \in L^1 \cap L^p(G)$, then $f \in L^2(G), \hat{f} \in L^2(\Gamma)$. Hence $L^1 \cap L^p(G) \subset F^{1t}(G)$.

The other inclusions among the chains are clear. This completes the proof.

DEFINITION 2.9. Let G be a non-discrete locally compact abelian group with character group Γ. A dense Banach algebra $(A(G), \|\ \|_A)$ in $L^1(G)$ is an IFP group algebra if the following properties satisfied:

PROPERTY I. $\|f * g\|_A \leqslant \|f\|_1 \|g\|_A$, $(f \in L^1(G), g \in A)$

PROPERTY F. *$A^\wedge \subset L^{p_0}(\Gamma)$ for some p_0, $0 < p_0 < \infty$.*

PROPERTY P. *There exist sequence $(K_n)_{n=1}^{\infty}$, $(N_n)_{n=1}^{\infty}$ of subsets of Γ, a sequence $(f_n)_{n=1}^{\infty}$ in A and a sequence of positive numbers $(C_n \geqslant 1)_{n=1}^{\infty}$ such that*

P-1. $K_i \cap K_j = \emptyset$, *if* $i \neq j$. $N_n \subset \mathrm{Int}(K_n)$, $\nu(N_n) = \alpha > 0$, $\nu(K_n) = \beta < \infty$ $(n = 1, 2, \ldots)$, (*Here*, Int *denotes interior and ν denotes the Haar measure on Γ*).

P-2. $0 \leqslant \hat{f_n} \leqslant 1$, $\operatorname{supp} \hat{f_n} \subset K_n$, $\hat{f_n}(N_n) = 1$.

P-3. $\|f_n\|_A \leqslant C_n$ *and*

$$\sum_{n=1}^{\infty} \frac{1}{C_n^a} < \infty \qquad \textit{for some } a,\ 0 < a < \infty,$$

$$\sum_{n=1}^{\infty} \frac{1}{C_n^b} = \infty \qquad \textit{for some } b,\ 0 < b < \infty.$$

Every Segal algebra on G has the property I ([11]; p. 128) while the improper Segal algebra $L^1(G)$ does not have the property F. Moreover we have the following theorem.

THEOREM 2.10. *Every character Segal algebra $(S(G), \|\ \|_S)$ on G has the property* P.

Proof. Since G is non-discrete, Γ is non-compact. There exists a sequence $(\gamma_n)_{n=1}^{\infty}$ in Γ and a compact symmetric neighborhood K of d such that $\gamma_1 = d$, $\gamma_i K \cap \gamma_j K = \emptyset$ if $i \neq j$ ([9]; p. 116). Let N be a compact symmetric neighborhood of d with $N \subset \mathrm{Int} K$. There exists a generalized trapezium function $f \in L^1(G)$ such that ([11]; p. 111)

$$(*) \qquad \begin{aligned} &0 \leqslant \hat{f} \leqslant 1, \\ &\operatorname{supp} \hat{f} \subset K, \\ &\hat{f}(N) = 1. \end{aligned}$$

Then $f \in S(G)$ since f has compact support ([11]; p. 128). Let $K_n = \gamma_n K$, $N_n = \gamma_n N$, $f_n = \gamma_n f$ and $C_n = qn$, where $q \geqslant 1$ and $\|f\|_S \leqslant q$ $(n = 1, 2, \ldots)$. Then

P-1 holds by the construction of the K, N, and γ_n;

P-2 $0 \leqslant \hat{f_n} \leqslant 1$, $\operatorname{supp} \hat{f_n} \subset \gamma_n K = K_n$, $\hat{f_n}(N_n) = \hat{f_n}(\gamma_n N) = 1$ $(n = 1, 2, \ldots)$;

P-3 $\sum_{n=1}^{\infty} \frac{1}{(qn)^2} = \sum_{n=1}^{\infty} \frac{1}{q^2 n^2} < \infty, \qquad \sum_{n=1}^{\infty} \frac{1}{qn} = \infty$, and

$$\begin{aligned} \|f_n\|_S &= \|\gamma_n f\|_S \\ &= \|f\|_S \quad \text{since } (S(G), \|\ \|_S) \text{ is character} \\ &\leqslant q \leqslant qn = C_n \quad (n = 1, 2, \ldots). \end{aligned}$$

Thus $(S(G), \|\ \|_S)$ has the Property P. This completes the proof.

THEOREM 2.11. *Every Banach algebra $(A(G), \|\ \|_A)$ in 2.4. is an IFP group algebra.*

Proof. We divide the proof into three parts:

Property I: Each algebra in 2.4. is a Segal algebra, and hence ([11]; p. 128) has Property I.

Property F: By Theorem 2.8.

Property P: Except for $C^k(T)$ $(k \geqslant 1)$, and $L^A(R)$, the Banach algebras in 2.4 are character Segal algebras and so, by Theorem 2.10, all of them have the Property P. Moreover,

I. $C^k(T)$ $(k \geqslant 1)$ has the Property P: The character group of T is Z, the integers. Let $K_n = N_n = \{n\}$, $f_n(t) = e^{int}$, and $C_n = n^k$ $(n = 1, 2, \ldots)$. By a simple calculation, $C^k(T)$ has the Property P.

II. $L^A(R)$ has the Property P: Let $K_n = [n - \frac{1}{4}, n + \frac{1}{4}]$, $N_n = [n - \frac{1}{8}, n + \frac{1}{8}]$ in R. There is a function f in $L^1(G)$ such that $0 \leqslant \hat{f} \leqslant 1$, $\operatorname{supp} \hat{f} \subset K_1$, $\hat{f}(N_1) = 1$. Since $L^A(R)$ is a Segal algebra, $f \in L^A(R)$. Let $f_n(t) = e^{i(n-1)t} f(t)$, and $C_n = n\|f\|_{L^A}$ $(n = 1, 2, \ldots)$. We have

$$\begin{aligned} \|f_n\|_{L^A} &= \|f_n\|_1 + \|f_n'\|_1 \\ &= \|f\|_1 + \|e^{i(n-1)t} f'(t) + i(n-1) e^{i(n-1)t} f(t)\|_1 \\ &\leqslant \|f\|_1 + \|f'\|_1 + (n-1)\|f\|_1 \\ &\leqslant n\|f\|_1 + n\|f'\|_1 = n\|f\|_{L^A} = C_n. \end{aligned}$$

Now it is easy to see that $L^A(R)$ has the property P.

These complete the proof.

3. Bounded approximate identity.

THEOREM 3.1. *A proper dense Banach algebra $(A(G), \|\ \|_A)$ in $L^1(G)$ with property* I *does not have any weak bounded approximate identity* (see Definition 2.1). *In particular, an IFP group algebra does not have any weak bounded approximate identity.*

Proof. The identity map of $A(G)$ into $L^1(G)$ is a continuous map as a homomorphism of a semisimple Banach algebra into another. So there is a constant $D > 0$ such that $\|f\|_1 \leqslant D\|f\|_A$ for all $f \in A(G)$. By our assumptions, the norms $\|f\|_1$ and $\|f\|_A$ are non-equivalent and so there is no constant $D_1 > 0$ such that $\|f\|_A \leqslant D_1\|f\|_1$ for all $f \in A(G)$. Thus, for every integer $k > 1$ there is a $g \in A(G)$ with $\|g\|_A > k$ and $\|g\|_1 = 1$. Let $h \in A(G)$ with $\|h * g - g\|_A < 1$. Then

$$1 < k < \|g\|_A < \|h * g\|_A + 1 \leqslant \|h\|_A \|g\|_1 + 1 < \|h\|_A + 1.$$

We have $\|h\|_A > k - 1$. So for every $k > 1$ there exists $g \in A(G)$ such that $\|h * g - g\|_A < 1$ implies $\|h\|_A > k - 1$. This proves that $(A(G), \|\ \|_A)$ does not have any weak bounded approximate identity.

COROLLARY 3.2. *Each Banach algebra in 2.4 has no weak bounded approximate identity.*

THEOREM 3.3. *Every proper Segal algebra does not have any weak bounded approximate identity.*

Proof. A Segal algebra has the property I ([11]; p. 128).

4. Factorization property.

In this section we shall show that none of the algebras $A(G)$ in 2.4 has the weak factorization property.

The main ideas are as follows:

(i) *If $A(G) \subset L^1(G)$ has the weak factorization property (that is, $A(G)^2 = A(G)$) and if $\widehat{A(G)} \subset L^p(\Gamma)$ for some p, $0 < p < \infty$, then $\widehat{A(G)} \subset L^p(\Gamma)$ for all p, $0 < p \leqslant \infty$.*

(ii) *But if $A(G)$ is in 2.4 then $\widehat{A(G)} \subset L^p(\Gamma)$ for some p, but $\widehat{A(G)} \not\subset L^r(\Gamma)$ for some r.*

(iii) *Hence $A(G)^2 \neq A(G)$.*

THEOREM 4.1. *A Banach algebra $(A(G), \|\ \|_A)$ in $L^1(G)$ with properties* F *and* P *does not have the weak factorization property. In particular, any IFP group algebra does not have the weak factorization property.*

Proof. I. Let $(A(G), \|\ \|_A)$ be a subalgebra of $L^1(G)$ such that $A(G) = A(G)^2$ where $A(G)^2 = \{\sum_{i=1}^{n} f_i * g_i : f_i, g_i \in A(G),\ n \text{ is any positive integer}\}$. Then $\widehat{A(G)} \subset L^{p_0}(\Gamma)$ for some p_0, $0 < p_0 < \infty$ (that is, $A(G)$ has property F) iff $\widehat{A(G)} \subset L^p(\Gamma)$ for *all* p, $0 < p \leqslant \infty$.

The "If" part is clear.

"Only-if" part. Suppose $\widehat{A(G)} \subset L^{p_0}(\Gamma)$ for some p_0, $0 < p_0 < \infty$. Since $\widehat{A(G)} \subset L^\infty(\Gamma)$ we have $\widehat{A(G)} \subset L^p(\Gamma)$ for all p, $p_0 \leqslant p \leqslant \infty$. If

$f \in A(G)$, there exists g_i, $h_i \in A(G)$, $i = 1, 2, \ldots, n$ with $f = \sum_{i=1}^{n} g_i * h_i$, or $f\hat{\ } = \sum_{i=1}^{n} g_i\hat{\ } h_i\hat{\ }$. By the Schwarz inequality, for $i = 1, 2, \ldots, n$, we have

$$\int_\Gamma |g_i\hat{\ }(\gamma) h_i\hat{\ }(\gamma)|^{p_0/2} d\gamma = \int_\Gamma |g_i\hat{\ }(\gamma)|^{p_0/2} |h_i\hat{\ }(\gamma)|^{p_0/2} d\gamma$$
$$\leqslant \Big(\int_\Gamma |g_i\hat{\ }(\gamma)|^{p_0} d\gamma\Big)^{1/2} \Big(\int_\Gamma |h_i\hat{\ }(\gamma)|^{p_0} d\gamma\Big)^{1/2}$$
$$< \infty \quad \text{since } g_i\hat{\ },\ h_i\hat{\ } \in L^{p_0}(\Gamma).$$

Thus $g_i\hat{\ }\ h_i\hat{\ } \in L^{p_0/2}(\Gamma)$ $(i = 1, 2, \ldots, n)$. So $f\hat{\ } \in L^{p_0/2}(\Gamma)$. Hence $\widehat{A(G)} \subset L^{p_0/2}(\Gamma)$. Continuing this process, we have $\widehat{A(G)} \subset L^{p_0/2^n}$ for $n = 1, 2, \ldots$ For any p, $0 < p \leqslant p_0$, there is a positive integer n such that $\frac{p_0}{2^n} \leqslant p \leqslant p_0$. Therefore $\widehat{A(G)} \subset L^p(\Gamma)$ for all $0 < p \leqslant p_0$. This completes the proof.

II. If the Banach algebra $(A(G), \|\ \|_A)$ in $L^1(G)$ has the Property P, then there is $f \in A(G)$ with $f\hat{\ } \in L^1(\Gamma)$ but $f\hat{\ } \notin L^r(\Gamma)$ for some r, $0 < r < 1$.

Notations as in 2.9. Since $\sum_{n=1}^{\infty} \frac{\|f_n\|_A}{C_n^{a+1}} \leqslant \sum_{n=1}^{\infty} \frac{C_n}{C_n^{a+1}} = \sum_{n=1}^{n} \frac{1}{C_n^a} < \infty$, there exists $f \in A(G)$ with $f = \sum_{n=1}^{\infty} \frac{f_n}{C_n^{a+1}}$, the series converging with respect to $\|\ \|_A$. Then $f\hat{\ } = \sum_{n=1}^{\infty} \frac{f_n\hat{\ }}{C_n^{a+1}}$ since $\|g\hat{\ }\|_\infty \leqslant \lim_{n\to\infty} \|g^n\|_A^{\frac{1}{n}} \leqslant \|g\|_A$ for each $g \in A(G)$.

Since, $\operatorname{supp} f_n\hat{\ } \subset K_n$ $(n = 1, 2, \ldots)$ and $K_i \cap K_j = \emptyset$ if $i \neq j$, we have

$$|f\hat{\ }| = \sum_{n=1}^{\infty} \frac{|f_n\hat{\ }|}{C_n^{a+1}} \quad \text{and} \quad |f\hat{\ }|^{\frac{b}{a+1}} = \sum_{n=1}^{\infty} \frac{|f_n\hat{\ }|^{\frac{b}{a+1}}}{C_n^b}.$$

Then

$$\int_\Gamma |f\hat{\ }(\gamma)| d\gamma = \sum_{n=1}^{\infty} \int_\Gamma \frac{|f_n\hat{\ }(\gamma)|}{C_n^{a+1}} d\gamma = \sum_{n=1}^{\infty} \int_{K_n} \frac{|f_n\hat{\ }(\gamma)|}{C_n^{a+1}} d\gamma$$
$$\leqslant \sum_{n=1}^{\infty} \frac{1}{C_n^{a+1}} \nu(K_n) \quad \text{since } 0 \leqslant f_n\hat{\ } \leqslant 1$$
$$= \sum_{n=1}^{\infty} \frac{1}{C_n^{a+1}} \beta \leqslant \sum_{n=1}^{\infty} \frac{1}{C_n^a} \beta \quad \text{since } C_n \geqslant 1 \text{ for every } n < \infty.$$
$$< \infty.$$

Also

$$\int_\Gamma |f\hat{\ }(\gamma)|^{\frac{b}{a+1}} d\gamma = \sum_{n=1}^{\infty} \int_\Gamma \frac{|f_n\hat{\ }(\gamma)|^{\frac{b}{a+1}}}{C_n^b} d\gamma$$
$$\geqslant \sum_{n=1}^{\infty} \int_{N_n} \frac{1}{C_n^b} d\gamma \quad \text{since } f_n\hat{\ }(N_n) = 1$$
$$= \sum_{n=1}^{\infty} \frac{1}{C_n^b} \nu(N_n) = \sum_{n=1}^{\infty} \frac{1}{C_n^b} a = \infty. \quad \text{Note } a > 0.$$

Thus $f\hat{\ } \in L^1(\Gamma)$ but $f\hat{\ } \notin L^r(\Gamma)$, where $r = \frac{b}{a+1}$, and $0 < r < 1$ since $f\hat{\ } \in L^s(\Gamma)$ for every s, $1 \leqslant s \leqslant \infty$.

III. Suppose $A(G)$ has the weak factorization property, that is, $A(G) = A(G)^2$. Then, by I, $\widehat{A(G)} \subset L^p(\Gamma)$ for all, $0 < p \leqslant \infty$. But, by II., there is a function f in $A(G)$ such that $f\hat{\ } \in L^1(\Gamma) \setminus L^r(\Gamma)$, for some r, $0 < r < 1$. This is a contradiction. Therefore $A(G)$ does not have the weak factorization property.

COROLLARY 4.2. *A character Segal algebra $S(G)$ with $\widehat{S(G)} \subset L^{p_0}(\Gamma)$, for some p_0, $0 < p_0 < \infty$, does not have the weak factorization property.*

Proof. Theorem 2.10 and Theorem 4.1.

COROLLARY 4.3. *Each Banach algebra in 2.4 does not have the weak factorization property.*

Proof. Theorem 2.11 and Theorem 4.1.

Remark 4.4. Let G be an infinite compact abelian group. Let $L^\infty(G)$ be the convolution algebra of all essentially bounded functions on G under the norm $\|f\|_\infty = \operatorname{ess\,sup}_{x \in G} |f(x)|$. Now $L^\infty(G)$ is a Banach algebra but is not a Segal algebra, since S-2 of Definition 2.3 does not hold. However it may be shown that $L^\infty(G)$ is an IFP group algebra and hence lacks both a weak bounded approximate identity and the weak factorization property. We omit the proof.

5. Discontinuous positive functionals. By Corollary 3.2 and Corollary 4.3, every Banach algebra in 2.4 lacks both a weak bounded approximate identity and the weak factorization property. Each one of them is commutative, so they are $*$-algebras under the trivial involution $*$. That is, $f^* = f$ for every f. Evidently they are $*$-algebras under some non-trivial involution $*$. In this section we shall apply previous results to prove that there exists a discontinuous positive functional on each of them.

THEOREM 5.1. *Let $(A, \|\ \|_A)$ be a Banach algebra such that $A^2 \subsetneqq A$ and A^2 is dense in $(A, \|\ \|_A)$. If A is also a $*$-algebra, then there exists a discontinuous positive functional on A.*

Proof. Since A^2 is a subspace of A, by Zorn's lemma there exist a Hamel basis Λ for A^2 and a Hamel basis Δ for A such that $\Lambda \subsetneqq \Delta$. Take $\varphi \in \Delta \setminus \Lambda$. Let M be the linear space spanned by $\Delta \setminus \{\varphi\}$. We have $A = M \oplus C\varphi$, where C denotes the complex numbers. Clearly $A^2 \subset M$. For every $f \in A$, f can be uniquely represented by $g + \alpha\varphi$ for some $g \in M$, $\alpha \in C$. Define $p(f) = \alpha$. Then p is a non-zero linear functional on A such that $p(h) = 0$ for any $h \in M$. For $f \in A$, we have $f^* f \in A^2 \subset M$, so $p(f^* f) = 0$. Thus p is a non-zero *positive* functional on A. Since A^2 is dense in $(A, \|\ \|_A)$ and $p(A^2) = 0$, p is discontinuous.

THEOREM 5.2. *Let $(A(G), \|\ \|_A)$ be any Banach algebra in 2.4. Then there exists a discontinuous positive functional on $A(G)$.*

Proof. By Corollary 4.3, $A(G)^2 \subsetneqq A(G)$. Since $A(G)$ is a Segal algebra, $A(G)^2$ is dense in $(A(G), \|\ \|_A)$ [11; p. 128]. By Theorem 5.1, there is a discontinuous positive functional on $A(G)$.

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