STUDIA MATHEMATICA, T. XLII. (1972)

On the Fourier series of functions of bounded Φ-variation

by

ALBERT BAERNSTEIN II (Syracuse, N. Y.)

Abstract. Let Φ and Ψ be functions complementary in the sense of of W. H. Young. Salem proved that, when $\sum_{i=1}^{\infty} \Psi(i^{-1}) < \infty$, every continuous periodic function of bounded Φ-variation has a uniformly convergent Fourier series. We prove here that, when $\sum_{i=1}^{\infty} \Psi(i^{-1}) = \infty$, there exists a continuous periodic function of bounded Φ-variation whose Fourier series diverges at a point.

1. Let Φ and Ψ be functions complementary in the sense of W. H. Young. That is

$$\Phi(u) = \int_0^u \varphi(t)\,dt, \qquad \Psi(u) = \int_0^u \psi(t)\,dt, \qquad (u \geqslant 0),$$

where φ is a strictly increasing continuous function on $(0, \infty)$ with $\lim_{u\to 0} \varphi(u) = 0$, $\lim_{u\to\infty} \varphi(u) = \infty$, and ψ is the inverse of φ. For example, if $\Phi(u) = u^p$ $(1 < p < \infty)$ then $\Psi(u) = \text{const}\cdot u^q$, where $p^{-1}+q^{-1} = 1$. We say that a function F defined on $[-\pi, \pi]$ has bounded Φ-variation if

$$\sup \sum_{i=1}^{n} \Phi\big(|F(x_i) - F(x_{i-1})|\big) < \infty,$$

where the sup is taken over all partitions $\{x_i\}_{i=0}^{n}$ of $[-\pi, \pi]$. Note that functions of bounded variation have bounded Φ-variation. For $\Phi(u) = u^p$ $(1 < p < \infty)$ L. C. Young [4], generalizing Jordan's theorem, proved that every continuous periodic function of bounded Φ-variation has a uniformly convergent Fourier series. Young's result was extended by Salem ([3], p. 50), who showed the same conclusion holds as long as

$$\sum_{i=1}^{\infty} \Psi(i^{-1}) < \infty.$$

Salem's result is an easy consequence of a convergence criterion he devised.

This criterion may be stated as follows (see discussion in ([1], p. 305). Let

$$(1)\quad W_n(x) = \sum_{i=1}^{n-2}{}' i^{-1}\int_0^{\pi}\left[F\left(x+\frac{t+i\pi}{n}\right)-F\left(x+\frac{t+(i+1)\pi}{n}\right)\right]\sin t\,dt$$
$$+\sum_{i=1}^{n-2}{}' i^{-1}\int_0^{\pi}\left[F\left(x-\frac{t+i\pi}{n}\right)-F\left(x-\frac{t+(i+1)\pi}{n}\right)\right]\sin t\,dt,$$

where $\sum'$ means sum over odd indices only, and n is odd. Then, assuming F is continuous and periodic, we have, uniformly,

$$F(x)-S_n(x) = \pi^{-2}W_n(x)+o(1) \qquad (n\to\infty).$$

In this note we show that $\sum\Psi(i^{-1})<\infty$ is sharp by constructing, when $\sum\Psi(i^{-1}) = \infty$, a continuous periodic function of bounded Φ-variation whose Fourier series diverges at $x = 0$. This answers a question raised by Goffman and Waterman [2].

2. We will make use of the elementary theory of Orlicz sequence spaces. Denote by l_Φ^* the set of all real sequences $\alpha = \{\alpha_i\}_{i=1}^{\infty}$ such that

$$\varrho_\Phi(\alpha) = \sum \Phi(|\alpha_i|)<\infty,$$

and by l_Φ the set of all α such that

$$\|\alpha\|_\Phi = \sup\sum|\alpha_i|\beta_i<\infty,$$

where the sup is taken over all non-negative sequences $\beta = \{\beta_i\}_{i=1}^{\infty}$ with $\sum\Psi(\beta_i)\leqslant 1$. Then $\alpha\in l_\Phi$ iff $k\alpha\in l_\Phi^*$ for some constant k. The if part follows from Young's inequality $ab\leqslant\Phi(a)+\Psi(b)$, and the only if part from the fact that

$$(2)\qquad \|\alpha\|_\Phi\leqslant M\Rightarrow\varrho_\Phi(M^{-1}\alpha)\leqslant 1.$$

For proofs of these statements, see ([5], p. 76 ff.). We also need

PROPOSITION 1. *If* $\sum\Psi(i^{-1}) = \infty$, *then, given* $\varepsilon>0$, *there exists* $\alpha\in l_\Phi$ *with all* $\alpha_i\geqslant 0$ *and* $\alpha_i = 0$ *for all sufficiently large* i, *such that* $\|\alpha\|_\Phi<\varepsilon$ *and* $\sum i^{-1}\alpha_i>1$.

Proof. Define sequences $\beta^n\in l_\Psi$ by $\beta_i^n = i^{-1}$ $(1\leqslant i\leqslant n)$, $\beta_i^n = 0$ $(i>n)$. Then $\|\beta^n\|_\Psi\to\infty$ with n, for if not, we would have $\|\beta^n\|_\Psi\leqslant M$ (M some integer) for all n, hence, by (2), with Ψ in place of Φ,

$$\sum_{i=1}^{n}\Psi(M^{-1}i^{-1})\leqslant 1,$$

hence

$$\sum_{i=1}^{\infty}\Psi(M^{-1}i^{-1})\leqslant 1,$$

and this implies, since $\Psi\uparrow$, that

$$\sum_{i=1}^{\infty}\Psi(i^{-1})<\infty.$$

Take n so large that $\|\beta^n\|_\Psi>2\varepsilon^{-1}$. Then there exists $\gamma\in l_\Phi$ with $\gamma_i\geqslant 0$ for all i, $\varrho_\Phi(\gamma)\leqslant 1$, and

$$\sum_{i=1}^{n} i^{-1}\gamma_i>2\varepsilon^{-1}.$$

We may assume that $\gamma_i = 0$ for $i>n$. It follows from Young's inequality that $\|\gamma\|_\Phi\leqslant 1+\varrho_\Phi(\gamma)\leqslant 2$. Hence $\alpha = \frac{1}{2}\varepsilon\gamma$ has all the desired properties.

3. Now let F be a real function defined on $[-\pi,\pi]$. We define

$$\|F\|_\Phi = \sup\sum_{i=1}^{n}|F(x_i)-F(x_{i-1})|\beta_i$$

where the sup is taken over all partitions $\{x_i\}$ of $[-\pi,\pi]$ and over all non-negative sequences β with

$$\sum_{i=1}^{\infty}\Psi(\beta_i)\leqslant 1.$$

It follows from (2) that if $\|F\|_\Phi\leqslant 1$, then F has bounded Φ-variation.

PROPOSITION 2. *Let there be given* $\alpha\in l_\Phi$ *with* $\alpha_i\geqslant 0$ *for all* i *and* $\alpha_i = 0$ *for* $i>m$, *a positive integer* $n\geqslant 2m+1$, *and* $\varepsilon>0$. *Then there exists a continuous periodic function* F *with the following properties.*

(i) $F\geqslant 0$, $F = 0$ *on* $(0,\pi/n)\cup(2\pi m/n,\pi)$, *and* $\sup_x F(x) = \sup_i\alpha_i$,

(ii) $$\sum_{i=1}^{2m-1}{}' i^{-1}\int_0^{\pi}\left[F\left(\frac{t+i\pi}{n}\right)-F\left(\frac{t+(i+1)\pi}{n}\right)\right]\sin t\,dt>\sum\frac{\alpha_i}{i}-\varepsilon,$$

(iii) $\|F\|_\Phi\leqslant 2\|\alpha\|_\Phi$.

In the notation of (1), (ii) asserts that $W_n(0)>\sum i^{-1}\alpha_i-\varepsilon$.

Proof. Define $F = 0$ on $[-\pi,0]\cup[2\pi m/n,\pi]$ and on the intervals $[\pi i/n,\pi(i+1)/n]$ $(i = 0,2,\ldots,2m-2)$. For i odd, $1\leqslant i\leqslant 2m-1$, choose a proper subinterval $[s_i,t_i]$ of $[\pi i/n,\pi(i+1)/n]$, define $F = \alpha_{\frac{1}{2}(i+1)}$ on this subinterval, and let F be linear on $[\pi i/n,s_i]$ and $[t_i,\pi(i+1)/n]$. Choose $[s_i,t]$ so that

$$\int_0^{\pi}\left[F\left(\frac{t+i\pi}{n}\right)-F\left(\frac{t+(i+1)\pi}{n}\right)\right]\sin t\,dt>2\alpha_{\frac{1}{2}(i+1)}-2^{-i}\varepsilon,$$

and extend F by periodicity. Then (i) is clearly satisfied, and the sum in (ii) is

$$> 2\sum_{i=1}^{m} a_i(2i-1)^{-1} - \varepsilon > \sum_{i=1}^{m} i^{-1} a_i - \varepsilon .$$

To prove (iii), fix a non-negative sequence β with $\sum \Psi(\beta_i) \leqslant 1$ and a partition $\{x_j\}$ of $[-\pi, \pi]$. Assume first that $\{x_j\}$ is a refinement of Π, where Π is the partition whose points are $-\pi, \pi$, all the $\pi i/n$ $(i = 1, 2, \ldots, 2m)$, and all the s_i, t_i $(i = 1, 3, \ldots, 2m-1)$. For $i = 1, 2, \ldots, m$ let

$$A_i = \{j\colon (2i-1)\pi/n < x_j \leqslant s_{2i-1}\},$$

$$B_i = \{j\colon t_{2i-1} < x_j \leqslant 2i\pi/n\}$$

and write $\gamma_j = F(x_j) - F(x_{j-1})$. Then $\gamma_j = 0$ if

$$j \notin \bigcup_{i=1}^{m} (A_i \cup B_i)$$

and

$$\sum_{j\in A_i} |\gamma_j| = a_i, \qquad \sum_{j\in B_i} |\gamma_j| = a_i.$$

Hence

$$\sum_j |\gamma_j| \beta_j = \sum_{i=1}^{m} \Big(\sum_{j\in A_i} |\gamma_j| \beta_j + \sum_{j\in B_i} |\gamma_j| \beta_j \Big) \leqslant \sum_{i=1}^{m} a_i \beta_i^* + a_i \beta_i^{**}, \tag{3}$$

where $\beta_i^* = \max\limits_{j\in A_i} \beta_j$ and $\beta_i^{**} = \max\limits_{j\in B_i} \beta_j$. Since

$$\sum \Psi(\beta_i^*) \leqslant \sum \Psi(\beta_j) \leqslant 1$$

and similarly $\sum \Psi(\beta_i^{**}) \leqslant 1$, the right hand side in (3) is $\leqslant 2\|a\|_\Phi$.

Now assume that $\{x_j\}$ is an arbitrary partition of $[-\pi, \pi]$. Let $\{y_k\}$ be the least common refinement of $\{x_j\}$ and Π. With each x_j $(j \geqslant 1)$ we associate a point $y_{\sigma(j)}$ of $\{y_k\}$ according to the following scheme: If $F(x_j) = F(x_{j-1})$ take $y_{\sigma(j)} = x_j$. If $F(x_j) > F(x_{j-1})$ then $x_j \varepsilon (\pi i/n, \pi(i+1)/n)$ for some odd i. Take $y_{\sigma(j)} = x_j$ if $x_j \leqslant s_i$ and $y_{\sigma(j)} = s_i$ if $x_j > s_i$. If $F(x_j) < F(x_{j-1})$ then $x_{j-1} \varepsilon\ (\pi i/n, \pi(i+1)/n)$ for some odd i. Define $y_{\sigma(j)}$ by $y_{\sigma(j)-1} = t_i$ if $x_{j-1} \leqslant t_i$ and $y_{\sigma(j)-1} = x_{j-1}$ if $x_{j-1} > t_i$. In all cases we have $x_{j-1} < y_{\sigma(j)} \leqslant x_j$ and $|\theta_{\sigma(j)}| \geqslant |\gamma_j|$, where $\theta_k = F(x_k) - F(x_{k-1})$, $\gamma_j = F(x_j) - F(x_{j-1})$. Verification is left to the reader. In particular, $\sigma(j) < \sigma(j+1)$, so we can define (a new) β^* by $\beta_k^* = \beta_j$ when $k = \sigma(j)$, $\beta_k^* = 0$ for other k. Then

$$\sum |\gamma_i| \beta_i \leqslant \sum |\theta_{\sigma(j)}| \beta_j = \sum |\theta_k| \beta_k^* \leqslant 2\|a\|_\Phi,$$

where the last inequality holds because $\{y_k\}$ is a refinement of Π and $\sum \psi(\beta_k^*) = \sum \Phi(\beta_j) \leqslant 1$. This proves (iii).

4. We proceed with our construction. Assume that $\sum \Psi(i^{-1}) = \infty$. Then by Proposition 1, for $p = 1, 2, \ldots$ there exists $a^{(p)} \epsilon\, l_\Phi$ with all $a_i^{(p)} \geqslant 0$ and $a_i^{(p)} = 0$ for $i > m_p$, such that $\|a^{(p)}\|_\Phi < 2^{-(p+1)}$ and $\sum\limits_i i^{-1} a_i^{(p)} > 1$. Take an odd integer $n_1 \geqslant 2m_1 + 1$. Apply Proposition 2 with $n = n_1$ and $\varepsilon = 1$. Let F_1 be the function thus obtained. Choose now odd integers $n_2, n_3, \ldots$ and functions $F_2, F_3, \ldots$ inductively as follows: Assume that $n_1, \ldots n_{p-1}$ and $F_1, \ldots, F_{p-1}$ have already been chosen, where $p \geqslant 2$. Select an odd integer n_p which is so large that

$$n_p > (2m_p + 1) n_{p-1} \tag{4}$$

and, letting ω denote modulus of continuity,

$$\omega\Big(\sum_{i=1}^{p-1} F_i, \pi/n_p\Big) \log n_{p-1} < p^{-1}. \tag{5}$$

Let F_p be the function obtained from Proposition 2 with $n = n_p$ and $\varepsilon = p^{-1}$. Note that the support of F_p (in $[-\pi, \pi]$) lies strictly to the left of that of F_{p-1}. Define

$$F = \sum_{p=1}^{\infty} F_p .$$

This series converges uniformly, since $F_p(x) \neq 0$ for at most one p and

$$\sup_x F_p(x) = \sup_i a_i^{(p)} \leqslant \text{const} \cdot \|a^{(p)}\|_\Phi = o(1) \qquad (p \to \infty).$$

Thus F is continuous. Clearly $g \to \|g\|_\Phi$ is subadditive, and it is easy to prove that $g_n \to g$ pointwise implies

$$\|g\|_\Phi \leqslant \liminf_{p\to\infty} \|g_p\|_\Phi .$$

Thus

$$\|F\|_\Phi \leqslant \sum_{p=1}^{\infty} \|F_p\|_\Phi < 2\sum \|a^{(p)}\|_\Phi < 1$$

and hence F has bounded Φ-variation.

Consider now $W_{n_p}(0)$, where p is fixed. All the terms in the second sum of (1) are zero, since $F = 0$ on $[-\pi, 0]$. For $1 \leqslant i \leqslant 2m_p - 1$ (i odd) it follows from Proposition 2(i), applied to F_{p-1} and F_{p+1}, and from (4), that

$$F\Big(\frac{t+i\pi}{n_p}\Big) - F\Big(\frac{t+(i+1)\pi}{n_p}\Big) = F_p\Big(\frac{t+i\pi}{n_p}\Big) - F_p\Big(\frac{t+(i+1)\pi}{n_p}\Big) \qquad (0 \leqslant t \leqslant \pi). \tag{6}$$

For $2m_p+1 \leqslant i \leqslant k$, where k is the largest odd integer $\leqslant (n_p/n_{p-1})-2$, each term in the first sum in zero, where now Proposition 2 (i) has been applied to F_p and F_{p-1}. For $i \geqslant k$, the left hand side of (6) is $\leqslant$

$$\omega\Big(\sum_{j=1}^{p-1} F_j,\ \pi/n_p\Big),$$

which, by (5), is $< p^{-1}(\log n_{p-1})^{-1}$. We conclude from (6), and from Proposition 2 (ii) applied to F_p that

$$W_{n_p}(0) \geqslant \sum_i i^{-1}a_i^{(p)} - 2p^{-1}(\log n_{p=1})^{-1}\sum_{i=k+1}^{n_p-2} i^{-1}.$$

Since $k \geqslant (n_p/n_{p-1})-4$, the second sum on the right is $O(\log n_{p-1})$. The first sum on the right is >1. Hence $W_{n_p}(0) > 1-o(1)$ $(p\to\infty)$ and thus the Fourier series of F diverges at 0.

References

[1] N. K. Bary, *A Treatise on Trigonometric Series*, Vol. I, New York 1964.
[2] C. Goffman and D. Waterman, *Some aspects of Fourier series*, Amer. Math. Monthly 77 (1970), pp. 119–133.
[3] R. Salem, *Essais sur les séries trigonometrique*; Actualités Sci. Ind. No. 862, Paris 1940.
[4] L. C. Young, *Sur une généralization de la notion de variation de puissance p-ième bornée au sense de M. Wiener, et sur la convergence des séries de Fourier*, C. R. Acad. Sci. Paris 204 (1937), pp. 470–472.
[5] A. C. Zaanen, *Linear Analysis*, Amsterdam 1964.

SYRACUSE UNIVERSITY

Received April 8, 1971 (319)

STUDIA MATHEMATICA, T. XLII. (1972)

Convergence of convolution operators

by

CHARLES SWARTZ (Las Cruces, New Mexico)

Abstract. In this paper a locally convex topology is defined on the space of convolution operators over a general test space of functions. When the test space is the space $\mathscr{S}$ of rapidly decreasing functions, convergence in this topology coincides with the convergence introduced in $\mathscr{O}_c'$ by L. Schwartz. The topology is studied in some detail, and then the special case when the test space is a $K\{M_p\}$ space is considered.

In [8], L. Schwartz defined a class of convolution operators between certain spaces of distributions and introduced a topology on this space of operators. In this approach emphasis is placed on considering convolution by a fixed distribution as a linear operator between spaces of distributions. In Gelfand and Shilov [2], a somewhat different approach is taken. Gelfand and Shilov define a convolution operator on an arbitrary test space with continuous translation and then consider a few examples of such operators, some in very general test spaces. There is no topology defined on the space of convolution operators although one sequential limit theorem is proven ([2], III. 3.5).

In this paper we consider the approach of Gelfand and Shilov and introduce a locally convex topology on the space of convolution operators on a test space with continuous translation. In the first section some of the properties of this topology are studied and we compare this topology with the topology introduced by L. Schwartz in [8]. In the second section we consider this topology for a certain type of $K\{M_p\}$ space ([2], II. 2.1). Our results yield the characterization of sequential convergence in $\mathscr{O}_c'(K_1, K_1)$ as given in [12] and also a characterization of sequential convergence in the space $\mathscr{O}_c'$ of L. Schwartz ([7], VII. 5).

Our terminology and notation will basically be that of Gelfand and Shilov [2]. A *test space* is a vector space Φ of infinitely differentiable functions on R^k equipped with a locally convex Hausdorff topology such that