

## Reciprocity theorems in the theory of representations of groups

by

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**Abstract.** A new and general version of the duality theorem in the theory of induced representations of groups is given. The theorem gives the description of the discrete part of the decomposition of the induced representation into irreducibles in terms of spaces of continuous invariant sesquilinear forms on the product of the space of inducing representation of the subgroup with some dense invariant subspace of the given irreducible representation.

Induced representations are meant in the sense close to that of Bruhat and groups are assumed to be of Yamabe type and possessing a large subgroup.

**Introduction.** This paper deals with duality theorems in the theory of induced representations of groups, which are also called Frobenius reciprocity theorems.

We recall here the classical formulation:

For a compact  $G$  the intertwining number (i.e. the dimension of space of intertwining operators) for a given irreducible unitary representation  $U$  of  $G$  and the induced representation  $U^V$  of  $G$  equals the intertwining number for the inducing representation  $V$  of a subgroup  $\Gamma$  and the restriction of  $U$  to  $\Gamma$ .

There were numerous attempts to generalize the duality theorem to noncompact groups — the interested reader may consult the Chicago lecture notes of G. W. Mackey for the history and several formulations — other papers of interest in that connection are Mackey [6], Fell [3] and Bruhat [1]. The main difficulty in the noncompact case is the presence of continuous spectrum of decomposition into irreducibles.

In this paper we deal with discrete part of decomposition of the induced representation into irreducibles and we find the dimension of space of intertwining operators for this and the given irreducible one.

Several versions of theorems of that type are already known — the first, and the most famous is the Gelfand, Piateckii–Šapiro duality theorem for automorphic forms [4]. It was proved for  $G$  semisimple, discrete subgroup  $\Gamma$  such that  $G/\Gamma$  is compact and  $V = \text{id}$  and asserts that for a given irreducible representation  $U$  the intertwining number for  $U$  and

$U^\vee$  is equal the dimension of space of the generalized invariant vectors for the restriction of  $U$  to  $\Gamma$  which just correspond to automorphic forms.

In the latter years several generalizations were made by L. Maurin [10], Y. Olsanskiĭ [11] and finally a very general version was proved in the joint work of K. Maurin & L. Maurin [9], however all those theorems formulated for unitary inducing representations.

In this paper, by adopting Bruhat's definition of the differentiable induced representation we are able to carry over the Gelfand & Piateckiĭ-Šapiro theorem to a large class of groups — the so called Yamabe groups or projective limits of Lie groups, which we assume to have large compact subgroups, and inducing representations — arbitrary in the Fréchet space. We found however that certain growth condition replacing the original condition of compactness of  $G/\Gamma$  is necessary.

The organization of the paper is following. Section 1 contains preliminaries and notations — there is also explained what we mean by unitary induced representation when the inducing representation is not unitary. Section 2 contains the construction of the dual object to the space of intertwining operators, corresponding to the generalized invariant vectors of Gelfand & Piateckiĭ-Šapiro, which is the space of "appropriately continuous" sesquilinear, invariant forms on  $\Phi \times E$ , where  $\Phi$  is a certain subspace of the representation space. There is also proved the general version of the duality theorems — Theorem 2.8.

Possible applications of the scalar version of the Theorem 2.8 were discussed at length in [12] and [13]. However, the first of those contains a version of duality theorem which is untrue, the other is not easy to deal with, so the present results may be thought of as an improvement and generalization of previous results.

The last section contains the duality theorem for unitary induced representation and also discusses connections between our results and the previously known formulations of duality theorems.

**1. Notations and fundamental concepts.** The aim of this introductory paragraph is to establish notations which will be used in the sequel and recall some basic definitions and theorems which we shall freely use afterwards without mentioning them explicitly.

a) Regular functions and distributions on Yamabe groups ([2], [8]). By a *Yamabe group* we mean a countable projective limit of Lie groups. We shall use standard notations for spaces of functions e.g.  $\mathcal{D}$  for *infinitely differentiable scalar functions with compact support*,  $\mathcal{D}(E)$  for the same class of functions valued in a locally convex Hausdorff topological vector space  $E$ . Those spaces are equipped with usual topology of double inductive limit. We shall call an *E-distribution* a continuous linear functional on the space  $\mathcal{D}(E)$ . It is known that in the case when

$E$  is a Fréchet space there is a canonical isomorphism between the space of  $E$ -distributions and the space of bilinear separately continuous forms on  $\mathcal{D} \times E$ . In the following we shall not distinguish between those spaces. In agreement with the formation of tensor product of spaces  $\mathcal{D}$  and  $E$  we shall denote by  $f \otimes e$  the function on  $G$ :

$$G \ni g \rightarrow f(g)e \in E \quad \text{where } f \in \mathcal{D}, e \in E.$$

b) Various notations concerning representations [1]. Let  $G$  denote a locally compact unimodular group of the Yamabe type. By a *representation of  $G$*  in a Hausdorff topological vector space  $X$  we mean a strongly continuous homomorphism  $g \rightarrow U_g$  of  $G$  into the group of linear topological isomorphisms of  $X$  with an additional property:

(\*) for every compact  $K \in G$  the set of operators  $\{U_g\}_{g \in K}$  is equicontinuous in  $X$ .

When  $X$  is barreled, (\*) is automatically satisfied. It is well known that left translations define in all reasonable function spaces like  $\mathcal{D}$ ,  $\mathcal{D}(E)$ , etc. representations of  $G$ . The notation used for a representation of  $G$  is  $(U, X)$  where  $U$  is the homomorphism and  $X$  the underlying space.

Let  $(U, X)$  be a representation of  $G$  in the complete space  $X$  and  $\mathfrak{M}$  the convolution algebra of measures on  $G$  with compact support. The group homomorphism  $U$  can be extended to the homomorphism of  $\mathfrak{M}$  into the algebra  $\mathcal{L}(X)$ . If, in addition  $(U, X)$  is a unitary representation in Hilbert space this homomorphism is involution preserving. We shall be interested in the representation of a subalgebra of  $\mathfrak{M}$ , namely the convolution algebra  $\mathcal{D} \subset \mathfrak{M}$ . We shall denote it by  $\mathcal{D} \ni f \rightarrow U_f \in \mathcal{L}(X)$ .

Every character of a compact subgroup  $K \subset G$  defines a projection  $P_K$  in  $X$ , where

$$P_K x = \frac{1}{n_K} \int_K \chi(k) U_k x dk$$

$dk$  is the Haar measure on  $K$ ,  $n_K$  denotes the dimension of the irreducible representation with character  $\chi$ .

The range of this projection is the maximal subspace of  $X$  on which the restriction of  $U$  to  $K$  is equivalent to the multiple of the representation of  $K$  associated with character  $\chi$ .

c) Induced representations [1], [6]. We shall employ the Bruhat's definition, with minor changes, of the *induced representations*. It goes as follows.

Let  $\Gamma \subset G$  be a closed subgroup of  $G$  and  $(V, E)$  its representation in a space  $E$ . Denote by  $\mathcal{D}^\Gamma$  the space of regular functions on  $G$  valued in  $E$ , which satisfy the following conditions:

(a)  $\text{supp } f/\Gamma$  is compact in  $G/\Gamma$ ,

(b)  $f(g\gamma) = \Delta(\gamma) V_\gamma^{-1} f(g)$ ,  $\Delta$  denotes the modular function on  $\Gamma$ .  
The map  $\beta: \mathcal{D}(E) \rightarrow \mathcal{D}^V$  defined by

$$\beta f(g) = \int_{\Gamma} \Delta^{-1}(\gamma) V_\gamma f(g\gamma) d\gamma, \quad d\gamma \text{ the left Haar measure on } \Gamma$$

is continuous, surjective and commutes with left translations, hence we can form the quotient representation of the left regular representation in  $\mathcal{D}(E)$ . This quotient representation in the space  $\mathcal{D}^V \simeq \mathcal{D}(E)/\text{Ker}\beta$  equipped with quotient topology is called the differentiable representation induced by the representation  $(V, E)$  of  $\Gamma$  and denoted by  $(U^V, D^V)$ . By a *left-invariant prehilbert seminorm* on  $\mathcal{D}^V$  we mean a prehilbert seminorm  $q$  such that  $q(L_g f, L_g h) = q(f, h)$  for every  $g \in G$  and all  $f, h \in \mathcal{D}(E)$ .  $L_g$  as usual denotes the left regular representation of  $G$ .

Examples are furnished by seminorms of the type

$$\mathcal{D}(E) \times \mathcal{D}(E) \ni (f, h) \rightarrow \int_G (Df(g) | Dh(g)) dg,$$

where  $D$  is a left invariant differential operator on  $\mathcal{D}(E)$  and  $(\cdot | \cdot)$  a prehilbert seminorm on  $E$ .

Having defined a hermitian positive, invariant form  $q$  on  $\mathcal{D}^V$  we can form a unitary representation of  $G$  by taking the Hilbert space  $H^q$ , the completion of  $\mathcal{D}^V / (\text{null space of } q)$  normed by  $\|f\|^2 = q(f, f)$  and extending operators of the induced representation to the whole  $H^q$ . This unitary representation, that is the pair  $(U^V, H^q)$  we shall call the *unitary induced representation* of  $G$ . This notion, as explained in Section 3 extends the notion of the induced representation of Mackey.

d) Groups with large subgroups [6]. A compact subgroup  $K \subset G$  is called *large* if every irreducible representation of  $G$  in a Banach space contains any irreducible representation of  $K$  with finite multiplicity. This being the case projections  $P_\chi$ , defined above, have finite dimensional ranges for any character of the large subgroup.

Convention. All groups considered in the sequel are separable locally compact unimodular Yamabe groups with a large subgroup  $K$ .

Such groups are called simply groups without any further reference.

**2. A duality theorem.** We begin with considering an irreducible unitary representation  $(U, H)$  of the group  $G$  acting in Hilbert space  $H$  and a certain nuclear space  $\Phi$  connected with it.

Let us define a mapping from  $\mathcal{D}$  into  $H$  as follows:

$$\alpha_x: \mathcal{D} \rightarrow H; \quad \alpha_x(f) := U_\gamma x,$$

where  $x$  belongs to the subspace  $P_\chi H$  for some character  $\chi$ . We shall show that the image of  $\alpha_x$  which we denote by  $\Phi_x$  does not depend on

$x$  and equip it with the nuclear topology transported by  $\alpha_x$ . To this end let us note that the following holds

$$P_\chi \circ \alpha_x(f) = \alpha_x(P_\chi f), \quad U_h \circ \alpha_x(f) = \alpha_x(h*f), \quad h, f \in \mathcal{D}.$$

At the right hand side of the first equality  $P_\chi f$  is formed with respect to the left regular representation of  $G$  in  $\mathcal{D}$ .

PROPOSITION 2.1.

1°  $\Phi_x = \Phi_{x'} =: \Phi$  for every  $x, x' \in \bigoplus_\chi P_\chi H$ .

2° Natural injections  $\Phi_x \rightarrow \Phi_{x'}$  are linear isomorphisms for topologies transported by  $\alpha_x$ .

3° The representation of convolution algebra  $\mathcal{D}$  in  $\Phi$  is algebraically irreducible.

Proof. 1°. For every character  $\chi$   $P_\chi \Phi_x$  is a dense subspace of finite dimensional space  $P_\chi H$ , hence is identical with it. This shows that for  $x' \in P_{\chi'} H = P_\chi \Phi_x \subset \Phi_x$  we have  $\Phi_{x'} = \text{Im } \alpha_{x'} \subset \Phi_x$ . Reversing the roles of  $x$  and  $x'$  in this argument we obtain opposite inclusion. This implies  $\Phi_x = \Phi_{x'}$ .

3° follows immediately from the argument in 1°.

2° The basis of neighbourhoods in the topology transported by  $\alpha_x$  is obtained by taking sets of the form  $\alpha_x(V)$ , where  $V$  ranges through neighbourhoods in  $\mathcal{D}$ . From this and continuity of convolution in  $\mathcal{D}$  we infer that

$$\begin{aligned} (*) \quad \alpha_x(V) &= \{\varphi \in \Phi: \varphi = U_\psi U_\gamma x'; \psi \in V\} \\ &= \{\varphi \in \Phi: \varphi = U_h x'; h \in V*f\} \subset \alpha_{x'}(W), \end{aligned}$$

where  $W$  is a neighbourhood of 0 in  $\mathcal{D}$  and  $U_\gamma x' = x$ .

Since we also have  $\alpha_{x'}(V') \subset \alpha_x(W')$ , that means that the topologies transported by  $\alpha_x$  and by  $\alpha_{x'}$  are equivalent.

We shall need the following slight extension of the proposition above, which we state separately:

PROPOSITION 2.1'. For every  $x \in \Phi$  define  $\alpha_x(f) = U_\gamma x$ . Then

$$\alpha_x: \mathcal{D} \rightarrow \Phi$$

is 1° surjective 2° open.

Proof. 1° follows from 3° above. Since  $x = U_\gamma x'$  where  $x' \in P_\chi H$  in virtue of 3° applies and the proof follows.

The representation of  $\mathcal{D}$  in  $\Phi$  characterises completely the representation  $(U, H)$ .

PROPOSITION 2.2. Let  $(U^1, H^1)$  and  $(U^2, H^2)$  be irreducible unitary representations of  $G$  and let  $\Phi^1, \Phi^2$  be corresponding spaces constructed as above. The representations  $(U^1, H^1)$  and  $(U^2, H^2)$  are equivalent if and only

if there exists a linear operator  $A: \Phi^1 \rightarrow \Phi^2$  intertwining for representations of  $\mathcal{D}$  in  $\Phi^1$  and  $\Phi^2$ . This being the case  $A$  turns out to be the restriction to  $\Phi^1$  of a scalar multiple of a unitary operator on  $H^1$ . (We do not suppose the continuity of  $A$ ).

**Proof.** The necessity of this condition is evident. We prove the sufficiency. Since the representation of  $\mathcal{D}$  in  $\Phi^1$  is irreducible it follows that  $A$  is an algebraical isomorphism onto.

$A$  is automatically continuous, because the preimage of a neighbourhood in  $\Phi^2$  of the form  $\{U_\varphi^2 x: \varphi \in V\}$  is the set  $\{U_\varphi^1 A^{-1}x: \varphi \in V\}$  which is in turn a neighbourhood in  $\Phi^1$ .

Because of this we can define a hermitian,  $G$  - invariant continuous form on  $\Phi^1$ :

$$(\varphi|\psi) := (\varphi|\psi)_1 + (A\varphi|A\psi)_2,$$

where  $(\varphi|\psi)_i$  stands for the scalar product in  $H^i$ . Let  $\mathcal{H}$  denote the Hilbert space obtained by completion of  $\Phi^1$  normed by  $(\cdot|\cdot)$ , and  $U$  unitary representation of  $G$  in  $\mathcal{H}$  obtained as extension of  $U^1$ .

We shall show that the extension of the identity of  $\Phi^1$  to the continuous operator  $I: \mathcal{H} \rightarrow H^1$  is still injective. The argument runs as follows. Suppose that  $\text{Ker } I \neq \{0\}$ , hence by the intertwining property of  $I$  the subspace  $\text{Ker } I$  is  $K$ -invariant subspace of  $\mathcal{H}$ , hence decomposes into sum of subspaces  $P_x(\text{Ker } I) \subset \text{Ker } I$  ( $P_x$  is formed with respect to  $(U, H)$ ).

But  $P_x \mathcal{H} = \text{closure of } P_x \Phi^1$ . The restriction of  $P_x$  to  $\Phi^1$  equals the restriction of  $P_x^1$  to  $\Phi^1$  ( $P_x^1$  being formed with respect to  $(U^1, H^1)$ ), hence we have  $P_x \mathcal{H} = \text{closure of } P_x^1 \Phi^1 = P_x^1 \Phi^1$  because the latter is finite dimensional. Moreover  $P_x \mathcal{H} = P_x \Phi^1 \subset \Phi^1$ , hence  $P_x(\text{Ker } I) \subset P_x \Phi^1 \subset \Phi^1$  and this is a contradiction since  $I$  on  $\Phi^1$  is the identity. Being so we shall identify  $\mathcal{H}$  with  $I(\mathcal{H})$ .

Now we show that  $(U, \mathcal{H})$  is irreducible. Assuming the opposite let  $\{0\} \neq \mathcal{H}' \subset \mathcal{H}$  be closed and invariant. For some character  $\chi$  we have  $\{0\} \neq P_\chi \mathcal{H}' \subset P_\chi \mathcal{H} \subset \Phi^1$ . But then

$$\Phi^1 = \{x \in \mathcal{H}: x = U_\chi y, y \in P_\chi \mathcal{H}'\} \subset \mathcal{H}'.$$

Density of  $\Phi^1$  in  $\mathcal{H}$  implies  $\mathcal{H}' = \mathcal{H}$  and Schurs lemma in turn shows that  $(\varphi|\psi)_i = \lambda_i(\varphi|\psi)$  for appropriate  $\lambda_i$ .

**DEFINITION 2.3.** For a Fréchet space  $E$  let  $Q_E$  denote the family of left-invariant prehilbert seminorms on  $\mathcal{D}(E)$  and put  $\mathcal{X}_E = (\mathcal{D}(E), Q_E)$  i.e. the space  $\mathcal{D}(E)$  topologized by  $Q_E$ .

We note that this family of seminorms is a basis of continuous seminorms on  $\mathcal{X}_E$  in the sense that the sets  $\lambda V_q$ , where  $\lambda > 0$  and  $V_q$  is the unit semiball of  $q \in Q_E$ , form a basis of neighbourhoods for the topology of  $\mathcal{X}_E$ .

Let  $H^q$  denote the Hilbert space obtained by completion of  $\mathcal{D}(E)/N_q$  where  $N_q = \{f: q(f, f) = 0\}$  for  $q \in Q_E$ .

The space of sesquilinear separately continuous forms on  $\Phi \times E$ , linear with respect to the second variable can be imbedded into the space  $B(\mathcal{D}, E)$  of separately continuous bilinear forms on  $\mathcal{D} \times E$  which in turn is canonically isomorphic to the dual of  $\mathcal{D}(E)$ . This imbedding is given by

$$j_x b(f, e) := b(a_x(f^*), e), \quad b: \Phi \times E \rightarrow \mathbb{C}.$$

We shall be interested here in such sesquilinear forms which under this identification give continuous form on  $\mathcal{X}_E$ . We shall also regard  $H^q$  as the subspace of  $\mathcal{X}'_E$ .

**LEMMA 2.4.** If  $j_x b \in H^q \subset \mathcal{X}'_E$  then for every  $x' \in \Phi$ ,

$$j_x b \in H^q.$$

**Proof.** Let  $\varphi \in \mathcal{D}$  be such that  $x' = U_\varphi x$ . By hypothesis

$$|j_x b(f, e)| \leq C q(f \otimes e), \quad C > 0.$$

Then

$$\begin{aligned} |j_x b(f, e)| &= |b(a_x(f^*), e)| = |b(U_\varphi x', e)| \\ &= |b(U_\varphi U_\varphi x, e)| = |j_x b(\psi^* f, e)| \leq C \int_G |\psi^*| dg q(f \otimes e) \end{aligned}$$

in view of the left-invariance of  $q$ .

**DEFINITION 2.5.** Let us denote:  $I(\Phi, E) := j_x^{-1}(\mathcal{X}'_E)$  — the space of sesquilinear forms on  $\Phi \times E$  continuous relatively to some  $q \in Q_E$ . Let  $\Gamma$  be a subgroup of  $G$  and let  $(V, E)$  be a representation of  $\Gamma$ . By  $V_\Gamma(\Phi, E)$  we denote the subspace of  $I(\Phi, E)$  consisting of forms invariant relatively to the action of a subgroup  $\Gamma \subset G$  on  $\Phi$  and  $E$ . Explicitly it means  $\{I(\Phi, E) \ni b \text{ belongs to } V_\Gamma(\Phi, E)\} \Leftrightarrow \bigwedge_{\gamma \in \Gamma} b(U_\gamma \varphi, V_\gamma e) = b(\varphi, e)$ .

We are now in the position to formulate one of the main results.

**PROPOSITION 2.6.** There exists an injection of  $I(\Phi, E)$  into the space of operators intertwining for the left regular representation of  $G$  in  $\mathcal{D}(E)$  and the representation  $(U, H)$ .

**Proof.** Let  $b \in I(\Phi, E)$ . We define  $\sigma: \mathcal{D}(E) \rightarrow \bar{\Phi}'$  where  $\bar{\Phi}'$  is the space of antilinear continuous functionals on  $\Phi$ .

$$\langle \varphi, \sigma(f) \rangle = \int_G b(U_\sigma^{-1} \varphi, f(g)) dg.$$

Observe that for fixed  $\varphi$  this defines a bilinear form

$$\mathcal{D} \times E \ni (h, e) \rightarrow \int_G b(U_\sigma^{-1} \varphi, (h \otimes e)(g)) dg = b(U_\sigma \varphi, e) = j_\sigma b(h, e).$$

In view of Lemma 2.4.  $j_\sigma b$  and  $j_x b$  are equicontinuous i.e.  $j_x b \in H^q$  implies  $j_\sigma b \in H^q$ , hence  $j_\sigma b \in \mathcal{X}'_E$ .

This shows that  $\sigma: \mathcal{X}_{\bar{\Phi}} \rightarrow \bar{\Phi}'$  is continuous when  $\bar{\Phi}'$  is equipped with its weak topology, and can be extended to a continuous mapping from  $H^q$  into  $\bar{\Phi}'$ . This extension is justified, because  $\bar{\Phi}'$  is weakly closed subspace of the space of distributions on  $G$ , hence complete.

Let  $\eta$  be canonical injection of  $H$  into  $\bar{\Phi}'$

$$\langle \varphi, \eta x \rangle := (\varphi | x).$$

Next step in the proof consists in constructing a commutative diagram of spaces and mappings:

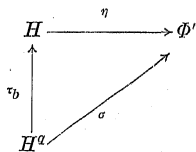


Diagram 1.

For this we shall need the following lemma

LEMMA 2.7. a) The subspace of functionals from  $\bar{\Phi}'$  vanishing identically on  $P_{\chi'}\Phi$  for all  $\chi \neq \chi'$  is equal to  $\eta(P_{\chi}H)$ .

b) Let  $X\mathcal{D}(E)$  denote the algebraic direct sum of all subspaces of the form  $P_{\chi}\mathcal{D}(E)$ , where  $\chi$  is a character of  $K$ . Then we have  $\sigma(X\mathcal{D}(E)) \subset \eta(\Phi)$ .

Proof. Consider a character  $\chi$  on  $K$  for which  $P_{\chi}\Phi \neq 0$ . We denote by  $\Phi_0$  the closure in  $\Phi$  of the algebraic direct sum of the subspaces  $P_{\chi'}\Phi$ , where  $\chi' \neq \chi$ , i.e.

$$\overline{\bigoplus_{\chi' \neq \chi} P_{\chi'}\Phi} = \Phi_0.$$

Since  $P_{\chi}\Phi = \Phi_{\chi}$  is finite dimensional  $\Phi_0 \oplus \Phi_{\chi}$  is closed in  $\Phi$  and contains  $\Phi_{\chi}$  for all  $\chi$ . We know that  $X\mathcal{D}$  is dense in  $\mathcal{D}$  [13] what implies  $\bigoplus_{\chi} \Phi_{\chi}$  dense in  $\Phi$ . Hence we have

$$\Phi = \Phi_{\chi} \oplus \Phi_0 \quad \text{and} \quad \bar{\Phi}' = \bar{\Phi}'_{\chi} \oplus \bar{\Phi}'_0.$$

By elementary algebra  $\dim\{w \in \bar{\Phi}': w(\Phi_0) = 0\} = \dim \bar{\Phi}'_{\chi} = \dim \bar{\Phi}_{\chi}$ . On the other hand  $\dim \eta(\Phi_{\chi}) = \dim \Phi_{\chi}$  and every form from  $\eta(\Phi_{\chi})$  vanishes identically on  $\Phi_0$ . This together with the equality  $P_{\chi}\Phi = P_{\chi}H$  mentioned before implies part a) of the lemma.

Property b) follows from the observation that for  $f = P_{\chi}f$  we have  $\langle \Phi_0, \sigma f \rangle = 0$ . In fact, for  $\chi \neq \chi'$  and  $\varphi \in \Phi_{\chi'}$

$$\langle P_{\chi'}\varphi, \sigma f \rangle = \langle P_{\chi'}\varphi, \sigma(P_{\chi}f) \rangle = \langle P_{\chi'}P_{\chi}\varphi, \sigma(f) \rangle = 0.$$

We proceed with the proof of the Proposition 2.6. As remarked above  $\sigma$  can be extended to the operator on Hilbert space  $H^q$ . Taking quotient space modulo  $\text{Ker } \sigma$  and the operator associated with  $\sigma$  in quotient space (which we denote also  $\sigma$ ) we obtain the continuous algebraic isomorphism intertwining for the representation of  $G$  in  $H_1 = H^q/\text{Ker } \sigma$  and the transpose of representation  $U$  in  $\bar{\Phi}'$ .

Let us suppose that  $H_1 = H' \oplus H''$  where both subspaces are closed and invariant. We show that one of those must be empty. For otherwise there are  $\chi'$  and  $\chi''$  such that  $P_{\chi'}H' \neq \{0\}$  and  $P_{\chi''}H'' \neq \{0\}$ . But then  $\sigma(P_{\chi'}H') \subset \eta(\Phi)$  and similarly  $\sigma(P_{\chi''}H'') \subset \eta(\Phi)$  by Lemma 2.7. b). That implies

$$\sigma(H') \supset \{U_f x: x \in P_{\chi'}H', f \in \mathcal{D}\} = \eta(\Phi)$$

and  $\sigma(H'') \supset \eta(\Phi)$  by the same argument.

This is impossible since  $\sigma$  is an isomorphism (algebraic) on  $H_1$  and  $H'$  and  $H''$  are orthogonal.

Lemma 2.7 b) shows that the following definition makes sense

$$\tau(f) = \eta^{-1} \circ \sigma(f)$$

at least for  $f \in X\mathcal{D}(E)$ .

We easily observe that both  $\sigma$  and  $\eta$  are intertwining operators for the representations of  $\mathcal{D}$  in corresponding spaces, so this equality defines  $\tau$  as an operator from

$$\Phi_1 = \{U_f x: f \in \mathcal{D}, x \in \bigoplus_{\chi} P_{\chi}H_1\}$$

into  $\Phi \subset H$  intertwining for the representation of the convolution algebra  $\mathcal{D}$  in  $\Phi_1$  and  $\Phi$  correspondingly. Hence the hypotheses of the Proposition 2.2 are satisfied and we see that  $\tau$  can be extended to the continuous intertwining operator  $\tau: H_1 \rightarrow H$ .

So we came to the end of the construction of Diagram 1.

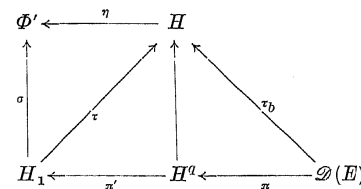


Diagram 2.

Here  $\pi$  denotes canonical projection of  $\mathcal{D}(E)$  into  $H^q$ , and  $\pi'$  projection onto. To end the proof it suffices put  $\tau_b = \tau \circ \pi' \circ \pi$ .

The main results of this section is formulated as follows.



**THEOREM 2.8. (THE DUALITY THEOREM)** Let  $(U, H)$  be a unitary irreducible representation of  $G$  and  $\Phi$  the space constructed above for the representation  $(U, H)$ . Let  $\Gamma$  be an arbitrary closed subgroup of  $G$ . We denote  $(V, E)$  a continuous representation of  $\Gamma$  in a Fréchet space  $E$ . Let  $\mathcal{D}^V$  be the space of the differentiable representation of  $G$  induced by  $V$  in the sense explained in Section 1.

Then there exists a bijection of the space  $V_\Gamma(\Phi, E)$  onto the space  $\mathcal{L}_G(\mathcal{D}^V, H)$  of intertwining operators for the representations of  $G$  in  $\mathcal{D}^V$  and  $H$ .

Proof. After the Proposition 2.6 we have an assignment

$$I(\Phi, E) \ni b \rightarrow \tau_b \in \mathcal{L}(\mathcal{D}(E), H).$$

We are going to show that the condition of  $\Gamma$ -invariance imposed on  $b$  results in the possibility of completing the diagram

$$\begin{array}{ccc} \mathcal{D}(E) & \xrightarrow{\tau_b} & H \\ \beta \downarrow & \nearrow T_b & \\ \mathcal{D}^V & & \end{array}$$

Diagram 3.

It suffices to prove that  $\beta(f) = 0$  implies  $\tau_b(f) = 0$ . We know (cf G. Mackey [6]) that for an arbitrary quasi-invariant measure  $\mu$  on  $G/\Gamma$  there exists a continuous function on  $G$ , say  $\varrho$ , with following properties

$$\varrho(x\gamma) = \Delta(\gamma)\varrho(x)$$

and

$$(2.1) \quad \int_G f(g)\varrho(g)dg = \int_{G/\Gamma} d\mu(x) \int_\Gamma f(g\gamma)d\gamma.$$

So we have

$$\begin{aligned} (\varphi | \tau_b(f)) &= \int_G b(U_\sigma^{-1}\varphi, f(g))dg \\ &= \int_{G/\Gamma} d\mu(x) \int_\Gamma \varrho^{-1}(g\gamma) b(U_\sigma^{-1}\varphi, f(g\gamma))d\gamma \\ &= \int_{G/\Gamma} d\mu(x) \varrho^{-1}(g) \int_\Gamma \Delta(\gamma^{-1}) b(U_\sigma^{-1}\varphi, V_\gamma f(g\gamma))d\gamma \\ &= \int_{G/\Gamma} d\mu(x) \varrho^{-1}(g) b(U_\sigma^{-1}\varphi, \beta f(g)) = 0 \end{aligned}$$

for  $f \in \text{Ker } \beta$  and an arbitrary  $\varphi$ . That implies  $\tau_b(f) = 0$ .

The proof will be completed once we have shown that this assignment is onto  $\mathcal{L}_G(\mathcal{D}^V, H)$ . To this end let us define for  $T \in \mathcal{L}_G(\mathcal{D}^V, H)$  a form  $b$  on  $\Phi \times E$  as follows

$$(2.2) \quad b(U_\sigma x, e) = (x | T\beta(f^* \otimes e)).$$

We observe that  $T = T_b$  for this definition of  $b$ . In fact we have

$$\begin{aligned} (U_h x | T_b(\beta f)) &= \int_G b(U_\sigma^{-1} U_h x, f(g))dg \\ &= \int_G (x | T(\beta(L_\sigma^{-1} h)^* \otimes f(g)))dg = \int_G (x | T\beta(\bar{h}(g) \otimes L_\sigma^{-1} f))dg \\ &= \int_G (x | U_\sigma^{-1} \bar{h}(g) T(\beta f))dg = (U_h x | T\beta(f)) \end{aligned}$$

for any  $h \in \mathcal{D}$  and  $f \in \mathcal{D}(E)$ .

Moreover we have

$$\begin{aligned} b(U_\gamma U_\varphi x, V_\gamma e) &= (x | T(\beta L_\gamma^{-1} \varphi)^* \otimes V_\gamma e) \\ &= (x | T\beta(R_\gamma(L_\gamma^{-1} \varphi)^* \otimes e)) = (x | T(\beta(\varphi^* \otimes e))) = b(U_\varphi x, e) \end{aligned}$$

what shows  $\Gamma$  invariance of  $b$ .

It remains to show the continuity of  $j_x b$  relatively to some seminorm from  $Q_E$ . Applying Schwarz inequality we get

$$|j_x b(f, e)| = |b(\alpha_x(f^*), e)| = |(x | T\beta(f \otimes e))| \leq \|T\beta(f \otimes e)\| \|x\|.$$

As the required seminorm we can take  $q(f) = \|T\beta(f)\|$ . This completes the proof.

**3. Applications.** The duality theorem proved in Section 2 allows us to describe the discrete irreducible components of an arbitrary unitary induced representation. By discrete component we understand such representation for which there exists an operator intertwining for this and the given representation.

Given an unitary induced representation  $(U^V, H^p)$  let us consider the sesquilinear form on  $\mathcal{D}(E)$  defined by

$$q(f, h) := p(\beta f, \beta h), \quad f, h \in \mathcal{D}(E).$$

**THEOREM 3.1.** The intertwining number for the representations  $(U^V, H^p)$  and a given unitary irreducible representation  $(U, H)$  is equal to the dimension in  $V_\Gamma(\Phi, E)$  of the subspace of forms for which

$$j_x b \in H^q$$

i.e.  $j_x b$  is continuous relatively to the seminorm  $q$ .

Proof. Let  $b \in V_r(\Phi, E)$  be continuous relatively to the seminorm  $p$  i.e.  $j_x b \in H^q$ . According to diagram 1 the operator  $\tau \circ \pi$  maps continuously  $H^q$  into  $H$ . To see that  $H^p$  is isometric to  $H^q$  recall that:  $H^q$  is the completion of  $\mathcal{D}(E)/\text{Ker } p \circ \beta$  normed by  $p$  and  $H^p$  is the completion of  $\beta \mathcal{D}(E)/\text{Ker } p$  normed by  $p$ .

The very definition of  $q$  shows that the operator  $T_b$  intertwines representation  $U^V$  with  $U$  for every form satisfying the hypothesis.

To prove the opposite let  $T$  be such an intertwining operator and let  $b \in V_r(\Phi, E)$  be corresponding form (cf 2.2).

We know that  $j_x b$  is continuous respectively to the seminorm  $q(f) = \|\tau \circ \pi_1(f)\|$  which in turn is continuous relatively to  $q$ .

In the following we shall explain the connection between the notion of the unitary induced representation and the induced representation in the sense of Mackey.

Let  $E$  be a Hilbert space with the scalar product  $(\cdot | \cdot)$  and  $V$  be a representation of the subgroup in  $E$  of the form  $\Gamma \ni \gamma \rightarrow V_\gamma = \Delta^{1/2}(\gamma) U(\gamma)$  where  $U$  denotes an arbitrary unitary representation of  $G$  in  $E$ .

In this case we can define a positive definite left invariant hermitian form on  $\mathcal{D}^V$ :

$$p(f, h) = \int_{G/\Gamma} \varrho^{-1}(x) (f(x) | h(x)) d\mu(x),$$

where  $\mu$  is a quasi-invariant measure on  $G/\Gamma$  and  $\varrho$  corresponding to it via formula (2.1).

The condition imposed on  $b$  in the hypothesis of the Theorem 3.1 i.e.  $b \in V_r(\Phi, E)$  and  $j_x b \in H^q$  is equivalent to the continuity of the functional

$$\mathcal{D}(E) \ni f \xrightarrow{l} \int_G b(U_\sigma^{-1} x, f(g)) dg = \int_{G/\Gamma} b(U_\sigma^{-1} x, \beta f(g)) \varrho^{-1}(g) d\mu(g)$$

relatively to the seminorm  $q = p \circ \beta$ .

This means that the functional  $l$  vanishes on  $\text{Ker } \beta$  and can be regarded as a continuous functional on  $H^p$ . Applying the Fréchet–Riesz theorem we see that there exists an element  $h \in H^p$  for which the following equality holds

$$l(f) = p(h, \beta f).$$

Comparing the above facts we obtain the following

$$(h(g) | e) = b(U_\sigma^{-1} x, e)$$

which holds for an arbitrary  $e \in E$  and  $\mu$  almost all  $g$ .

This gives the following

COROLLARY 3.2. *The intertwining number of the induced representation  $(U^V, H^p)$  and  $(U, H)$  is equal to the dimension of the subspace of forms from  $V_r(\Phi, E)$  such that the function*

$$F: G \ni g \rightarrow \varrho^{-1/2}(g) b(U_\sigma x, \cdot) \in E$$

*is square integrable.*

Explicitly it means that the function

$$G \ni g \rightarrow (F(g) | F(g)) \text{ belongs to } L^1(G/\Gamma, d\mu).$$

This corollary corresponds to the following well known theorems and generalize them in case of groups with large subgroups.

1. Let  $G$  be a semisimple group with finite center,  $E$ -finite dimensional,  $\Gamma$  discrete and  $G/\Gamma$  compact; it is the Gelfand and Piateckii–Šapiro theorem in the theory of automorphic functions [4].

2. Under the assumption that  $G/\Gamma$  is compact and possesses an invariant measure and  $E$  is finite-dimensional cf. Olšanskii [11].

Recently K. Maurin and L. Maurin have proved a general version of the duality theorem for Mackey's induced representations and arbitrary locally compact groups assuming however a different continuity condition for the form  $b$  [9].

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## Über die Struktur der rationalen Operatoren in der zweidimensionalen diskreten Operatorenrechnung

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**Zusammenfassung.** In der Mikusińskischen Operatorenrechnung sind die in  $p$  (Differentiationsoperator) echt gebrochenen rationalen Operatoren stetige Funktionen aus dem Grundring. Im Zweidimensionalen gilt ein entsprechender Satz nicht.

In der diskreten zweidimensionalen Operatorenrechnung ist der Grundring die Menge der für alle nichtnegativen ganzzahligen  $m, n$  definierten Funktionen. Man betrachtet hier Operatoren, die in  $p, q$  (Verschiebungsoperatoren) rational sind. Es werden Kriterien für die Zugehörigkeit solcher Operatoren zum Grundring angegeben. Die Ergebnisse lassen sich bei der Lösung linearer partieller Differenzengleichungen verwerten.

In der von J. Mikusiński begründeten Operatorenrechnung gehört bekanntlich der Integrationsoperator zum Integritätsbereich (der für  $t \geq 0$  definierten und dort stetigen Funktionen), während das inverse Element, der Differentiationsoperator  $p$ , ein „eigentlicher“ Operator ist. Die in  $p$  echt gebrochenen rationalen Operatoren sind aber stetige Funktionen aus dem Integritätsbereich.

Eine analoge Rolle spielen in der diskreten Operatorenrechnung der Verschiebungsoperator  $v$  und sein inverses Element  $q$ . Der Operator  $v$  gehört zum Integritätsbereich (der für  $n = 0, 1, \dots$  definierten Zahlenfolgen), während  $q$  ein (im zugehörigen Quotientenkörper existierender) eigentlicher Operator ist. Wie im kontinuierlichen Fall gilt dann analog, daß jeder in  $q$  echt gebrochene rationale Operator eine Zahlenfolge aus dem Integritätsbereich darstellt (s. etwa [1]).

In der zweidimensionalen Operatorenrechnung gelten die entsprechenden Sätze nicht, wie im kontinuierlichen Fall aus [9] und im diskreten aus [4] zu ersehen ist. Im letzteren Fall ist z. B.  $1/(p-q)$  kein Element des Integritätsbereichs der für nichtnegative ganze  $m, n$  definierten Doppelfolgen. Die Operatoren  $p$  bzw.  $q$  sind dabei die inversen Elemente der Verschiebungsoperatoren  $u$  bzw.  $v^{(1)}$ , die ihrerseits im Integritätsbereich

$$^{(1)} u = \begin{cases} 1 & \text{für } m = 1, n = 0, \\ 0 & \text{sonst;} \end{cases} \quad v = \begin{cases} 1 & \text{für } m = 0, n = 1, \\ 0 & \text{sonst.} \end{cases}$$