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Invariant norms for $C(T)$

by

STEPHEN SCHEINBERG (*) (Stanford, Ca.)

Abstract. The space of continuous functions with supremum norm has a huge group of isometries. Given a subgroup of this group one can ask whether there is another algebra norm for the continuous functions having isometry group containing the given subgroup. This paper presents various constructions of algebra norms designed to accommodate several natural groups of isometries and gives conditions under which certain groups of isometries characterize the sup norm among all algebra norms.

In many calculations on function algebras an important property of the sup norm, in addition to completeness and the indispensible inequalities defining "norm", is that a particular collection of mappings of the algebra are isometric, or perhaps norm-decreasing. It is often evident that the sup norm could be replaced by any other "invariant" norm. This gives rise to a natural question: are there any other norms besides $\|\ \|_\infty$ which have a given invariance behavior, and how much invariance must be imposed in order to characterize $\|\ \|_\infty$ among all norms? The purpose of this note is to exhibit several distinct norms which are invariant under large collections of mappings and to give conditions sufficient to ensure that a norm must be identical with the sup norm. For simplicity let us consider $C(T)$, where T is the circle. Generalizations to $C(G)$, G a compact abelian group, and in some cases to $C(X)$, X a compact Hausdorff space, will be apparent.

If $\|\ \|$ is an algebra norm for $C(T)$, then $\|f\| \geqslant \|f\|_\infty$, by a theorem of Kaplansky ([1], Theorem 6.2). An algebra norm is complete if and only if $\|\ \|_\infty \leqslant \|\ \| \leqslant K\|\ \|_\infty$, for some $K < \infty$. A theorem of Bade and Curtis ([2], Theorem 4.1) asserts that $\|f\| \leqslant K\|f\|_\infty$ for all f vanishing on a neighborhood of a certain finite set, which may be empty. If $\|\ \|$ is translation-invariant $\big(\|f(t+s)\| = \|f(t)\|\big)$, then it immediately follows

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that $\|f\| \leqslant 2K\|f\|_\infty$ for all $f \in C(T)$. Hence, a translation-invariant algebra norm for $C(T)$ is complete and satisfies $\| \ \|_\infty \leqslant \| \ \| \leqslant$ constant $\| \ \|_\infty$.

It is easy to see directly that multiplication by f is an isometry on $(C(T), \| \ \|)$ if and only if f never vanishes and $\|f\| = \|1/f\| = 1$. Of course, multiplication by f is norm decreasing if and only if $\|f\| \leqslant 1$. I thank K. De Leeuw for suggesting that the norm in Theorem 2 below might be a norm for convolution product. In fact, the following proposition holds.

PROPOSITION. *If $\| \ \|$ is a translation-invariant algebra norm for $C(T)$, then $\|f*g\| \leqslant \|f\| \, \|g\|$, where $*$ is convolution on T.*

Proof. From the Riemann sums for $f*g$ it is evident that

$$\|f*g\| \leqslant \Big(\int |f|\Big)\|g\| \leqslant \|f\|_\infty \|g\| \leqslant \|f\| \, \|g\|.$$

THEOREM 1. *If $\| \ \|$ is any algebra norm on $C(T)$, then $\{n\colon \|e^{int}\| = 1\}$ is a semigroup containing 0. Conversely, if $S \subseteq \mathbf{Z}$ is a semigroup containing 0, there is a translation invariant Banach algebra norm on $C(T)$ such that $S = \{n\colon \|e^{int}\| = 1\}$.*

Proof. By definition $\|1\| = 1$. $1 = \|e^{i(n+m)t}\|_\infty \leqslant \|e^{i(n+m)t}\| \leqslant \|e^{int}\| \, \|e^{imt}\|$, proving the first statement. For any subset $S \subseteq \mathbf{Z}$, define

$$A_S = \Big\{f \in C(T)\colon \int_0^{2\pi} f(t)e^{-int}dt = 0 \text{ for all } n \notin S\Big\}$$

= the closure of those trigonometric polynomials whose exponents lie in S. A_S is a closed translation-invariant subspace of $C(T)$. If S is a semi-group containing 0, then A_S is a subalgebra containing 1.

Define $p_S(f) = \inf\{\|f-a\|_\infty\colon a \in A_S\}$; p_S is a semi-norm which vanishes exactly on A_S. The desired norm will be $\|f\| = \|f\|_\infty + p_S(f)$. To prove that $\|fg\| \leqslant \|f\| \, \|g\|$, it is sufficient to show that $p_S(fg) \leqslant \|f\|_\infty p_S(g) + \|g\|_\infty p_S(f)$. Let $\|f-a_n\|_\infty \to p_S(f)$ and $\|g-b_n\|_\infty \to p_S(g)$; we may assume $\|b_n\|_\infty \leqslant \|g\|_\infty$. Then $fg - a_n b_n = f(g-b_n) + b_n(f-a_n)$, implying $p_S(fg) \leqslant \|fg - a_n b_n\|_\infty \leqslant \|f\|_\infty \|g-b_n\|_\infty + \|g\|_\infty \|f-a_n\|_\infty \to \|f\|_\infty p_S(g) + \|g\|_\infty p_S(f)$. The other properties of $\| \ \|$ are immediate.

Remark 1. Every closed translation-invariant subspace of $C(T)$ has the form A_S for some set S. It is a subalgebra exactly when S is a semi-group.

Remark 2. If S_j are semigroups and $a_j > 0$ with $\sum a_j < \infty$ then $\| \ \|_\infty + \sum a_j p_{S_j}$ defines a norm. If $S_0 = \{0\}$, $S_+ = \{n \geqslant 0\}$, and $S_- = \{n \leqslant 0\}$, then $p_{S_0} \neq p_{S_+} + p_{S_-}$. However, they agree on all exponentials.

Remark 3. The set of norms is closed under convex combinations and under the taking of the sup of a bounded collection.

THEOREM 2. *There is a Banach algebra norm for $C(T)$ which is unequal to $\| \ \|_\infty$ but for which these mappings are isometric:*

(1) $f(t) \to e^{int}f(t)$ *(all n)*,

(2) $f(t) \to f(t+t_0)$ *(all t_0)*,

(3) $f(t) \to f(-t)$,

(4) $f(t) \to f(nt)$ *(all n)*,

(5) $f \to \bar{f}$.

Furthermore $\|f*g\| \leqslant \|f\| \, \|g\|$.

Proof. Let $U = \{f \in C(T)\colon \|f\|_\infty \leqslant 1\}$ and $V = \{f = \sum_{-\infty}^{\infty} a_n e^{int}\colon \sum |a_n| \leqslant 1\}$. V is a closed subset of U, since $f^{(j)} \to f \Rightarrow a_n^{(j)} \to a_n$. It is well known that V is a proper subset of U. This means that for some $\varepsilon_0 > 0$, $V + 2\varepsilon_0 U$ does not contain U. Put W = the convex hull of $V \cup \varepsilon_0 U$. Then $V, \ \varepsilon_0 U \subseteq W \subseteq \overline{W} \subsetneq U$.

W is convex, circled, and translation invariant. Furthermore, $WW = \{w_1 w_2\colon w_i \in W\} \subseteq W$. Indeed, $VV \subseteq V$, $V\varepsilon_0 U \subseteq U\varepsilon_0 U = \varepsilon_0 U$, and $\varepsilon_0 U \varepsilon_0 U = \varepsilon_0^2 U \subseteq \varepsilon_0 U$ since $\varepsilon_0 \leqslant 1$. And the product of convex combinations is a convex combination of products. A direct proof that $W*W \subseteq W$ is similar. The Minkowski functional $p(f) = \inf\{r\colon f \in rW\} = 1/\sup\{r\colon rf \in W\}$ is then a translation-invariant algebra norm for $C(T)$ and $p(f*g) \leqslant p(f)p(g)$. Since $\varepsilon_0 U \subseteq W \subseteq U$, p is equivalent to $\| \ \|_\infty$ and is complete. Since $1 = \|e^{int}\|_\infty \leqslant p(e^{int}) \leqslant 1$, (1) follows. It is evident that $f(t) \in W \Leftrightarrow f(-t) \in W$ and the same for f and $\bar{f}$.

It remains to show that $f(t) \in W \Leftrightarrow f(nt) \in W$. The direction "$\Rightarrow$" is clear. For the converse define $\tilde{g}(t) = 1/n \cdot \sum_{j=1}^{n} g(t + 2\pi j/n)$ for any g. Observe that $\tilde{U} \subseteq U$ and $\tilde{V} \subseteq V$, hence $\tilde{W} \subseteq W$. Further, if $g(t) = f(nt)$, then $\tilde{g} \equiv g$. So if $f(nt) = \lambda v(t) + (1-\lambda)\varepsilon_0 u(t)$, then $f(nt) = \lambda\tilde{v}(t) + (1-\lambda)\varepsilon_0\tilde{u}(t)$. Now it is clear that $\tilde{v}(t) = v_1(nt)$ and $\tilde{u}(t) = u_1(nt)$ for some $v_1 \in V$ and $u_1 \in U$. Thus, $f = \lambda v_1 + (1-\lambda)\varepsilon_0 u_1 \in W$. This completes the proof.

THEOREM 3. *If $\| \ \|$ is an algebra norm on $C(T)$, then these are equivalent:*

(1) $\|f\| = \|f\|_\infty$ *for all f.*

(2) $\|f^2\| = \|f\|^2$ *for all f.*

(3) $\|f\| = 1$ *for all f such that $|f| \equiv 1$.*

Proof. The equivalence $1 \Leftrightarrow 2$ is well known. The equivalence $1 \Leftrightarrow 3$ follows from the known fact (easily established for T) that convex combinations of such functions are dense in U.

Theorem 3 shows that the collection of isometries of Theorem 2 cannot be enlarged to include the mapping $f \to |f|$.

Let $\mathscr{G}$ be the group of all $g \in C(T)$ such that $|g| \equiv 1$. Given an algebra norm $\| \ \|$ for $C(T)$, define G = the subset of $C(T)$ consisting of all g for which $f \to gf$ is an isometry of $(C(T), \| \ \|)$. G is a uniformly closed subgroup of $\mathscr{G}$ containing the constants, if $\| \ \|$ is equivalent to $\| \ \|_\infty$. Theorem 3 says that $G \subsetneq \mathscr{G}$ unless $\| \; \| \equiv \| \; \|_\infty$. The next theorem shows that this is the only restriction to enlarging the isometry group of Theorem 2, part (1).

THEOREM 4. *If G is any proper uniformly closed subgroup of $\mathscr{G} = \{g \in C(T)\colon |g| \equiv 1\}$ containing the constants, then there is an algebra norm $\| \; \|$ for $C(T)$ such that $f \to gf$ is an isometry of $(C(T), \| \; \|) \Leftrightarrow g \in G$. (Since G is proper, $\| \; \| \neq \| \; \|_\infty$.)*

Proof. We must produce a norm such that for $g \in \mathscr{G}$, $\|g\| = 1 \Leftrightarrow g \in G$. Choose ε_0, $0 < \varepsilon_0 < 1$. Let W = the uniform closure of the circled convex hull of $G \cup \varepsilon_0 U$, where U is the sup-norm unit ball of $C(T)$. Since $\varepsilon_0 U \subseteq W \subseteq U$ and $WW \subseteq W$, it follows that the Minkowski functional p of W is an algebra norm equivalent to the sup-norm. Since W is uniformly closed, it is identical with the unit ball of the new norm p. $W \subseteq U$ implies $p(f) \geqslant \|f\|_\infty$, hence $p(g) \geqslant 1$ for $g \in \mathscr{G}$. However, $g \in G$ implies $g \in W$; hence $p(g) \leqslant 1$ for $g \in G$. Thus $p(g) = 1$ for all $g \in G$. Finally we show that $g \in \mathscr{G}$ and $p(g) = 1$ imply $g \in G$.

Since W is closed, $p(g) = 1 \Rightarrow g \in W$. Thus, $\lambda_n g_n + (1-\lambda_n) e_n \to g$, where $\|e_n\|_\infty \leqslant \varepsilon_0$ and $g_n \in G$. Since $|g| \equiv |g_n| \equiv 1$ we see that $\lambda_n \to 1$ and hence $g_n \to g$. Therefore $g \in \bar{G} = G$.

We can generalize Theorem 2 in a different direction as follows.

THEOREM 5. *Let S be a proper closed convex subset of $\{|z| \leqslant 1\}$ which is a semigroup under multiplication and which contains 1. Then there is an algebra norm $\| \; \|$ for $C(T)$, distinct from $\| \; \|_\infty$, such that*

(1) *$\|f\| = \|f\|_\infty$ if the range of $f \subseteq S$.*

(2) *If $\varphi\colon T \to T$ is continuous, then $\|f \circ \varphi\| \leqslant \|f\|$ for all f. In particular, $\|f \circ \varphi\| = \|f\|$ if φ is a homeomorphism. Hence $\|f * g\| \leqslant \|f\| \, \|g\|$.*

EXAMPLES. (i) If S contains $[-1, 1]$, then $\|f\| = \|f\|_\infty$ for any real valued f.

(ii) The convex hull of any proper closed sub-semigroup containing 1 will be appropriate for S. For example, the regular N-gon is the convex hull of the Nth roots of 1. Using this for S we have $\|f\| = \|f\|_\infty$ for any f with range contained in a regular N-gon centered at 0 and having radius $\|f\|_\infty$.

Proof. Without loss of generality we may assume $[0, 1] \subseteq S$, for the set $\{rs\colon s \in S, 0 \leqslant r \leqslant 1\}$ will be a proper closed subset containing $S \cup [0, 1]$. Put U' = the convex circled hull of the set of continuous functions having range $\subseteq S$. Because S is a convex semigroup, it is clear that $U'U' \subseteq U'$, $U' * U' \subseteq U'$, and $U' \circ \varphi \subseteq U'$. Since $S \supseteq [0, 1]$, $\frac{1}{4} f \in U'$ if $\|f\|_\infty \leqslant 1$. The support functional p' of U' is the desired norm. Since $U' \subseteq U$ we have $p'(f) \geqslant \|f\|_\infty$ for all f. The reverse inequality holds if the range of $f \subseteq S$. This proves (1). All other properties of p' are evident and we have only see that $p' \neq \| \; \|_\infty$.

Let a, b be any two distinct points of the circle and α any complex number such that $|\alpha| = 1$ and $\alpha \notin S$. Define $Lf = f(a) + \bar{\alpha} f(b)$. Evidently, f can be chosen in $C(T)$ so that $\|f\|_\infty = 1$ and $L(f) = 2$. However, if range of $f \subseteq S$, it is clear that $|f(a) + \bar{\alpha} f(b)| < 2$ and indeed $|f(a) + \bar{\alpha} f(b)| \leqslant 2 - \delta < 2$ by compactness of $S \times S$. The inequality $|Lf| \leqslant 2 - \delta$ persists to U' and this shows that $p' \neq \| \; \|_\infty$.

The following theorem shows that if an algebra norm is invariant under homeomorphism, then it cannot also be invariant under multiplication by unimodular functions (unless it is the sup norm). This fact justifies the restrictions of Theorem 5.

THEOREM 6. *Suppose $\| \; \|$ is an algebra norm for $C(T)$ which is invariant under all homeomorphisms (as in Theorem 5, part (2)). Suppose there is $f_0 \in C(T)$ such that $\|f_0\| = 1$ and the range of f_0 contains λ_1, λ_2 with $|\lambda_1| = |\lambda_2| = 1$ and λ_1/λ_2 not a root of unity. Then $\| \; \| \equiv \| \; \|_\infty$.*

Proof. Put $B = \{f\colon \|f\| \leqslant 1\}$. We know that B is a closed convex circled subset of $U = \{f\colon \|f\|_\infty \leqslant 1\}$. We need to see that $B = U$. By Theorem 3 if $B \subsetneq U$, then $u \in U$ exists with $|u| = 1$ and $u \notin B$. Then a linear functional exists separating u and B; that is, there exist $\delta > 0$ and a finite complex Borel measure μ such that $|\int b\,d\mu| \leqslant 1 - \delta < 1 = \int u\,d\mu$ for all $b \in B$. This inequality persists to all $b \in \tilde{B}$ = the smallest class of functions containing B and closed under the taking of everywhere pointwise sequential limits. Since B is convex, circled, multiplicatively closed, and closed under composition with homeomorphisms of T, the same is true for $\tilde{B}$. The proof will be completed by showing that $f_0 \in B \subseteq \tilde{B} \Rightarrow u \in \tilde{B}$, which will be a contradiction.

Multiplying f by a scalar and composing with a homeomorphism we may assume $f(0) = 1$, $f(\pi) = \lambda$, with λ not a root of unity, where f is defined and continuous on $[0, 2\pi)$ with $f(2\pi^-) = f(0)$. Let $\varphi_n(t) = \pi(t/\pi)^n$ for $0 \leqslant t < \pi$ and $\pi(-1 + t/\pi)^n + \pi$ for $\pi \leqslant t < 2\pi$. Then φ_n is a homeomorphism of $[0, 2\pi)$ and tends pointwise to 0 for $0 \leqslant t < \pi$ and π for $\pi \leqslant t < 2\pi$. Hence $f \circ \varphi_n$ tends to 1 for $0 \leqslant t < \pi$ and λ for $\pi \leqslant t < 2\pi$. By moving the arcs and multiplying the functions we can get this function in $\tilde{B}$: $g(t) = \lambda^{n_j}$ on $t_j \leqslant t < t_{j+1}$, where n_j are integers $\geqslant 0$ and

$$0 = t_0 < t_1 < \ldots < t_k = 2\pi.$$

Now $\lambda^{n_1}, \ldots, \lambda^{n_k}$ can simultaneously approximate any k numbers of modulus 1, since λ is not a root of unity. Hence the members of $\tilde{B}$ approximate any given u, $|u| \equiv 1$. Thus, $u \in \tilde{B}$, and this contradiction establishes the theorem.

In regard to the two kinds of isometries $f \to gf$ and $f \to f \circ \varphi$, Theorem 2 shows that one can have some of each kind with a norm unequal to the sup norm. Theorem 4 shows that one can greatly enlarge the class $f \to gf$ subject to restrictions imposed by Theorem 3. On the other hand, the second class may be greatly enlarged (Theorem 5); however, this will enormously restrict the first class (Theorem 6).

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STANFORD UNIVERSITY

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An example of infinite dimensional reflexive Banach space non-isomorphic to its Cartesian square(*)

by

T. FIGIEL (Warszawa)

Abstract. The paper gives the first example of an infinite dimensional reflexive Banach space X non-isomorphic to X^2. The sense of this non-isomorphism depends on a difference between the structure of finite dimensional subspaces of X and those of X^2. The proof involves certain properties of subspaces of l_p^n, some of them seeming to be new.

Introduction. The problem whether every infinite dimensional Banach space X is isomorphic to its square X^2 was raised in Banach's monograph and remained unsolved until 1959. The first counterexamples were the space J of R. C. James and some related spaces (cf. [2]) and the space $C(\Gamma_{\omega_1})$ (cf. [12]).

The proofs of these non-isomorphisms were based on certain additive, isomorphic invariants $\delta(X)$ characterizing the natural embedding $\varkappa\colon X \to X^{**}$ (additive in the sense that $\delta(X_1 \oplus X_2) = \delta(X_1) + \delta(X_2)$ for any Banach spaces X_1, X_2).

Putting e.g. $\delta(X) = \dim\big(X^{**}/\varkappa(X)\big)$ we obtain such an invariant and since $\delta(J) = 1 \neq 2\delta(J) = \delta(J^2)$ we get the non-isomorphism $J \not\approx J^2$.

The question stated nowadays by several authors (cf. [1], [2]), whether X can, in addition, be reflexive required other methods and remained unsolved until today.

Below we show that spaces with such properties can be constructed from familiar spaces l_p^n, i.e. spaces of all n-tuples $a = (a_1, \ldots, a_n)$ of numbers with $\|a\| = \big(\sum_{i=1}^{n} |a_i|^p\big)^{1/p}$ if $1 \leqslant p < \infty$ and $\|a\| = \max_{1\leqslant i\leqslant n} |a_i|$ if $p = \infty$. (Letters i, j, k, m, n will always denote in the sequel positive integers.)

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