Further, for arbitrarily fixed positive a and $b, b > a$, by a similar reasoning as in Section 4, (S_1) implies the following statement

$$(S_1') \quad \sup_{s\in[a,b]} \left| \mathrm{V.P.} \int_a^b \frac{x(u)}{s-u}\,du \right| \leqslant \mathrm{const}\cdot \sup_{u\in[a,b]} |x(u)| \quad \text{for every} \quad x \in \mathscr{D}(a,b).$$

Now the whole indirect proof is completed by showing that (S_1') is not true. Indeed, if $\varepsilon \in \left(0, \frac{b-a}{2}\right)$ and $x_\varepsilon \in \mathscr{D}(a,b)$ is such that $0 \leqslant |x_\varepsilon(s)| \leqslant 1$ for $s \in (a,b)$, and that $x_\varepsilon(s) = 1$ for $s \in [a+\varepsilon, b-\varepsilon]$, then

$$\int_a^b \frac{x(u)}{b-u}\,du \geqslant \int_{a+\varepsilon}^{b-\varepsilon} \frac{du}{b-u} = \log\left(\frac{b-a}{\varepsilon} - 1\right).$$

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On uniform symmetrization of analytic matrix functions

by

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Abstract. Let A be real-analytic function of ξ on open set $M \subset R^n$, which values $A(\xi)$ are $m \times m$ matrices with purely diagonal and real canonical Jordan form. If the characteristic roots of $A(\xi)$ are restricted to change their multiplicities only in a suitable, very simple manner, then for every $\xi \in M$ we construct a hermitean positive $m \times m$ matrix $H(\xi)$, such that $H(\xi)A(\xi)$ is hermitean and that $\|H(\xi)\|$ and $\|H^{-1}(\xi)\|$ are locally bounded functions of ξ.

1. The result. Let A be a function defined on a set M, which values are $m\times m$ complex matrices. We shall say that A is *uniformly symmetrizable on M* if the following condition is satisfied:

(S) there is a constant $c \geqslant 1$, such that for every $\xi \in M$ there is a hermitean $m\times m$ matrix $H(\xi)$, such that $c^{-1} \leqslant H(\xi) \leqslant c$ and that $H(\xi)A(\xi)$ is hermitean.

According to Kreiss [2], [3], the uniform symmetrizability of A on M is equivalent to either of the following conditions:

(D) there is a constant $d \geqslant 1$, such that for every $\xi \in M$ there is an on singular $m\times m$ matrix $T(\xi)$, such that $\|T(\xi)\| \leqslant d$, $\|T^{-1}(\xi)\| \leqslant d$ and that $T^{-1}(\xi)A(\xi)T(\xi)$ is purely diagonal and real;

(E) $\sup\{\|\exp(itA(\xi))\| : t \in (-\infty, \infty),\ \xi \in M\} < \infty$;

(R) $\sup\{\|(E - isE - itA(\xi))^{-1}\| : s, t \in (-\infty, \infty),\ \xi \in M\} < \infty$, where E denotes the unit $m\times m$ matrix.

The theorem, which we state below may be treated as a contribution to the following problem. Let A be a matrix-valued function on a set M and suppose that $A(\xi)$ is symmetrizable for every fixed $\xi \in M$. Under which additional conditions A is uniformly symmetrizable on M? Our additional conditions have the form of restrictions on the behaviour of characteristic roots of $A(\xi)$ near the points of branching. We consider only the simplest case, when two roots come together.

THEOREM. *Let $M \subset R^n$ be open and let A be an analytic function on M, which values are $m\times m$ complex matrices. Suppose that for every $\xi \in M$ the matrix $A(\xi)$ has purely diagonal and real canonical Jordan form. Moreover,* *suppose that the characteristic polynomial* $\det(\lambda E - A(\xi))$ *has a factorization*

$$\det(\lambda E - A(\xi)) = (\lambda^2 + 2b(\xi)\lambda + c(\xi))^{q_0} \prod_{k=1}^{m'} (\lambda - \lambda_k(\xi))^{q_k}, \quad \lambda \in C, \ \xi \in M,$$

where $q_0, \ldots, q_m$, *are positive integers, such that*

1° *the characteristic roots* $\lambda_k(\xi)$, $k = 1, \ldots, m'$, *have multiplicities* q_k *independent of* ξ *in whole* M *and*

2° *if we put*

$$N = \{\xi\colon \ \xi \in M, \ b^2(\xi) = c(\xi)\},$$

then

$$(*) \qquad b^2(\xi) - c(\xi) \geqslant kd^2(\xi, N)$$

for every $\xi \in M \setminus N$, *where* $k > 0$ *is independent of* ξ *and* $d(\xi, N)$ *denotes the distance from* ξ *to* N.

Under these assumptions A *is uniformly symmetrizable on every compact subset of* M.

2. Remarks.

2.1.

Under assumptions of the theorem from Section 1 *the functions* $\xi \to b(\xi)$ *and* $\xi \to c(\xi)$ *are analytic on* M.

Proof. Because any of the roots $\lambda_k(\xi)$, $k = 1, \ldots, m'$, has constant multiplicity, these roots are analytic functions of ξ in M and therefore the coefficients $a_j(\xi)$ of the polynomial

$$\lambda^{m'} + a_{m'-1}(\xi)\lambda^{m'-1} + \ldots + a_1(\xi)\lambda + a_0(\xi) = \prod_{k=1}^{m'} (\lambda - \lambda_k(\xi))^{q_k}$$

are also analytic functions of ξ. Because

$$\begin{aligned}\det(\lambda E - A(\xi)) = \lambda^m &+ (a_{m'-1}(\xi) + 2q_0 b(\xi))\lambda^{m-1} \\ &+ (q_0 c(\xi) + q_0(q_0 - 1)b(\xi) + 2q_0 b(\xi) a_{m'-1}(\xi) + \\ &+ a_{m'-2}(\xi))\lambda^{m-2} + \ldots,\end{aligned}$$

it follows that $b(\xi)$ and $c(\xi)$ are analytic functions of ξ in M.

2.2.

Under the assumptions of the theorem from Section 1 *any maximal connected part of the set* M *is or an one-point set or a real analytic submanifold of* M.

This follows immediately by an application to $\Delta(\xi) = b^2(\xi) - c(\xi)$ of the following

Lemma. *Let* $M \subset R^n$ *be open and let* Δ *be non-negative real-analytic function on* M (*or, respectively, a non-negative function on* M *of the class* C^k, *where* $k = n, n+1, \ldots$ *or* $k = \infty$). *Let*

$$N = \{\xi\colon \ \xi \in M, \ \Delta(\xi) = 0\} \neq \emptyset$$

and suppose that there is a positive constant k, *such that for every* $\xi \ni M \setminus N$ *we have*

$$(*') \qquad \Delta(\xi) \geqslant kd^2(\xi, N),$$

where $d(\xi, N)$ *denotes the distance from* ξ *to* N. *Then every maximal connected part of* N *is or an one-point set or a proper real-analytic submanifold of* M (*or, respectively, a proper submanifold of* M *of the class* C^{k-1}).

Let us remark, that if Δ is an arbitrary real function analytic on open set $M \subset R^n$, then according to the famous theorem of Łojasiewicz (see [4] or [5], Chapter 4, § 4) for every compact subset K of M there are positive constants k and α, such that

$$|\Delta(\xi)| \geqslant kd^\alpha(\xi, N) \quad \text{for every} \quad \xi \in K \setminus N.$$

Of course, if Δ is non negative and $N \neq \emptyset$, then α cannot be less then 2.

Proof of the lemma. We shall proceed by an induction with respect to n. For $n = 1$ the lemma is obvious. Suppose that it is true for any dimension less then n and consider the n-dimensional case. Of course

$$\operatorname{grad} \Delta(\xi) = 0 \quad \text{for every} \quad \xi \in N.$$

For any $\xi \in M$ let $S(\xi)$ be the symmetric $n \times n$ matrix, which element in ith row and jth column is $\dfrac{\partial^2 \Delta(\xi)}{\partial \xi_i \partial \xi_j}$. Let now $\xi_0 \in N$. Then, by $(*')$, $S(\xi_0)$ has positive rank r and if $r = n$, then clearly ξ_0 is an isolated point of N. Therefore suppose that $1 \leqslant r < n$. Because $S(\xi_0)$ is symmetric, it has a non vanishing principal minor of degree r. We may assume, that this non vanishing principal minor is the determinant of $\left(\dfrac{\partial^2 \Delta}{\partial \xi_i \partial \xi_j}(\xi_0)\right)$ $i, j = 1, \ldots, r$. Put

$$\text{(i)} \qquad N^* = \left\{\xi\colon \ \xi \in M, \frac{\partial \Delta(\xi)}{\partial \xi_k} = 0 \text{ for every } k = 1, 2, \ldots, r\right\}.$$

It follows from the implicit function theorem that there is an open neighbourhood U of ξ_0, such that

$$\text{(ii)} \qquad U \cap N^* = \{(g(\xi'), \xi')\colon \ \xi' \in V\},$$

where $V = \Pi(U)$ is the image of U under the projection $\pi\colon (\xi_1, \ldots, \xi_n) \to (\xi_{r+1}, \ldots, \xi_n)$ and g is a real-analytic mapping of V into R^r. (or, respectively, a mapping of V into R^r of the class C^{k-1}). Because obviously $N \subset N^*$, our proof will be complete if we shall show that $U' \cap N^* \subset N$, where U' is any open bounded subset of U, such that $\overline{U'} \subset U$, or, which is the same, that $\Delta'(\xi') \equiv \Delta(g(\xi'), \xi')$ vanishes in $M' = \Pi(U')$.

Put
$$N' = \{\xi' : \xi' \in M',\ \Delta'(\xi') = 0\}.$$

We have to prove that $N' = M'$. Suppose that this is not true. Then $M' \setminus N' \neq \emptyset$. From the facts, that $\overline{U}'$ is compact, N is closed and $N' = \Pi(N \cap U')$, by (*′) and (ii), it follows that

(iii) $$k' d^2(\xi', N') \leqslant k' d^2\big((g(\xi'), \xi'),\ N \cap \overline{U}'\big) \leqslant \Delta\big(g(\xi'), \xi'\big) = \Delta'(\xi')$$

for every $\xi' \in M' \setminus N'$, where k' is a positive constant. Therefore, according to our inductive assumption, any maximal connected part of N' is or a single point or a proper submanifold of M'. Now we shall obtain a contradiction with (iii), showing that all the second order partial derivatives of Δ' vanish at $\xi_0' = \pi(\xi_0)$. By (i) and (ii), for every $i, j = r+1, \ldots, n$ and $\xi' \in M'$ we have
$$\frac{\partial^2 \Delta'(\xi')}{\partial \xi_i \partial \xi_j} = \Big\langle S\big(g(\xi'), \xi'\big) \circ \frac{\partial}{\partial \xi_i}\big(g(\xi'), \xi'\big), \frac{\partial}{\partial \xi_j}\big(g(\xi'), \xi'\big)\Big\rangle.$$

The first r components $s_\nu(\xi')$, $\nu = 1, \ldots, r$, of the vector $S\big(g(\xi'), \xi'\big) \circ \circ \dfrac{\partial}{\partial \xi_i}\big(g(\xi'), \xi'\big)$ are
$$s_\nu(\xi') = \sum_{\mu=1}^{r} \frac{\partial^2 \Delta}{\partial \xi_\mu \partial \xi_\nu}\big(g(\xi'), \xi'\big) \frac{\partial g_\mu(\xi')}{\partial \xi_i} + \frac{\partial^2 \Delta}{\partial \xi_i \partial \xi_\nu}\big(g(\xi'), \xi'\big) = \frac{\partial}{\partial \xi_i}\left[\frac{\partial \Delta}{\partial \xi_\nu}\big(g(\xi'), \xi'\big)\right]$$
and, by (i) and (ii), we see that these components vanish in M'. Because the rank of $S\big(g(\xi_0'), \xi_0'\big) = S(\xi_0)$ is r and $\det\left(\dfrac{\partial^2 \Delta}{\partial \xi_\mu \partial \xi_\nu}(\xi_0)\right)_{\mu,\nu=1,\ldots,r} > 0$, it follows from the equalities $s_\nu(\xi_0') = 0$, $\nu = 1, \ldots, r$, that the vector $S\big(g(\xi'), \xi'\big) \circ \dfrac{\partial}{\partial \xi_i}\big(g(\xi'), \xi'\big)$ vanishes at $\xi' = \xi_0'$. Therefore
$$\frac{\partial^2 \Delta'}{\partial \xi_i \partial \xi_j}(\xi_0') = 0 \quad \text{for every} \quad i, j = r+1, \ldots, n.$$

But this is in contradiction to the facts that the maximal connected part of N' containing ξ_0' is or $\{\xi_0'\}$ or a proper submanifold of M' and that (iii) holds. Therefore N' cannot be less then M' and the proof is complete.

2.3. Consider the homogenous differential operator
$$P(D) = E\frac{\partial}{\partial t} - \sum_{k=1}^{n} A_k \frac{\partial}{\partial x_k},$$

where coefficients A_k are square matrices of degree m with constant complex elements and E is the unit matrix of degree m. Consider also non homogenous operators of the form
$$P(D) - B,$$
where B is a square matrix of degree m with constant complex elements. The operator $P(D) - B$ is called *hyperbolic with respect to the vector* $(t, x_1, \ldots, x_n) = (1, 0, \ldots, 0)$ if it has a fundamental solution with support contained in a cone $\{(t, x_1, \ldots, x_n) : t, x_1, \ldots, x_n \in R^1, t \geqslant C(x_1^2 + \ldots + x_n^2)^{1/2}\}$, where $C = \text{const} > 0$.

It is known from papers of Kreiss [2], [3] and Svensson [7] that following three conditions are equivalent:

(a) for every square matrix B of degree m with constant complex elements the operator $P(D) - B$ is hyperbolic with respect to the vector $(1, 0, \ldots, 0)$,

(b) the matrix-valued function $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \to \sum_{k=1}^{n} \xi_k A_k$ is uniformly symmetrizable on R^n,

(c) the differential operator $\sum_{k=1}^{n} A_k \dfrac{\partial}{\partial x_k}$ considered on the domain $D = \left\{u : u \in L^2(R^n; C^m),\ \sum_{k=1}^{n} A_k \dfrac{\partial}{\partial x_k} \in L^2(R^n; C^m)\right\}$, where the derivatives $\dfrac{\partial}{\partial x_k}$ are taken in distributional sense, is infinitesimal generator of a strongly continuous one parameter group of bounded linear operators in the space $L^2(R^n; C^m)$.

2.4. We shall show by an example, that the inequality (*′) plays an essential role in theorem from Section 1.

For any $\xi = (\xi_1, \xi_2) \in R^2$ let
$$A(\xi) = \begin{bmatrix} \xi_1 & \xi_2 & 0 \\ \xi_2 & 0 & \xi_2 \\ 0 & 0 & 0 \end{bmatrix}$$
and let
$$M = \{(\xi_1, \xi_2) : \tfrac{1}{2} < \sqrt{\xi_1^2 + \xi_2^2} < \tfrac{3}{2},\ \xi_1 > -\tfrac{1}{4}\}.$$

The characteristic roots of $A(\xi)$ are then $\lambda_1(\xi) \equiv 0$, $\lambda_2(\xi) = \frac{1}{2}\xi_1 - \frac{1}{2}\sqrt{\xi_1^2 + 4\xi_2^2}$ and $\lambda_3(\xi) = \frac{1}{2}\xi_1 + \frac{1}{2}\sqrt{\xi_1^2 + 4\xi_2^2}$, so that $\lambda_3(\xi)$ has multiplicity 1 in whole M, while, for $\xi \in M$, $\lambda_1(\xi)$ and $\lambda_2(\xi)$ are equal if $\xi_2 = 0$ and distinct if

$\xi_2 \neq 0$. From this we see, that $A(\xi)$ always has purely diagonal and real canonical Jordan form. Furthermore, the factorisation of the type described in theorem from Section 1 holds in present situation with $b(\xi) = \frac{1}{2}\lambda_2(\xi)$ and $c(\xi) = 0$, and we have now

$$N = \{(\xi_1, \xi_2)\colon \tfrac{1}{2} < \xi_1 < \tfrac{3}{2},\ \xi_2 = 0\}.$$

The inequality (*) is not satisfied, because if $\xi = (1, \xi_2) \in M$, then $d(\xi, N) = |\xi_2|$ and $b^2(\xi) - c(\xi) = \frac{1}{4}\lambda_2^2(\xi) = \frac{1}{4}\left(\frac{\xi_2^2}{\sqrt{1+4\theta\xi_2^2}}\right)^2$, where $\theta \in (0,1)$, so that $b^2(\xi) - c(\xi) = O\big(d^4(\xi, N)\big)$ as $\xi = (1, \xi_2)$ and $\xi_2 \to 0$.

The matrix function A in this example is not uniformly symmetrizable on the compact set $K = \{(\xi_1, \xi_2)\colon \xi_1^2 + \xi_2^2 = 1,\ \xi_1 \geqslant 0\} \subset M$. Indeed, in the contrary case, by homogeneity, this function would be uniformly symmetrizable on the whole R^2. But then, by equivalence (a) $\Leftrightarrow$ (b) from Section 2.3, the differential operator

$$D = \begin{bmatrix} \frac{\partial}{\partial t} - \frac{\partial}{\partial x_1}, & -\frac{\partial}{\partial x_2}, & 0 \\ -\frac{\partial}{\partial x_2}, & \frac{\partial}{\partial t}, & -\frac{\partial}{\partial x_2} \\ 0, & 1, & \frac{\partial}{\partial t} \end{bmatrix}$$

should be hyperbolic with respect to the vector $(t, x_1, x_2) = (1, 0, 0)$. However this is not true. The operator D is an example, given by Petrovsky [6], of an operator, which is not hyperbolic with respect to the vector $(1, 0, 0)$, although its main part is hyperbolic with respect to this vector. This example of a non hyperbolic operator with hyperbolic main part is presented with details also in [7].

All the remainder of this paper is devoted to the proof, unfortunately long, of the theorem from Section 1. In Sections 3, 4 and 5 we construct and investigate some matrices, which are used in this proof, given in Section 6.

3. Symmetric positive matrices defined by separating pairs of polynomials. Let

$$p(\lambda) = \lambda^m + p_{m-1}\lambda^{m-1} + \ldots + p_1\lambda + p_0 = \prod_{k=1}^{m}(\lambda - \lambda_k),$$

and

$$q(\lambda) = \lambda^{m-1} + q_{m-2}\lambda^{m-2} + \ldots + q_1\lambda + q_0 = \prod_{k=1}^{m-1}(\lambda - \lambda_k')$$

be polynomials with real coefficients. If (p, q) is a pair of polynomials of such a form, having all the roots λ_k and λ_k' real and satisfying the inequalities

$$\lambda_1 \leqslant \lambda_1' \leqslant \lambda_2 \leqslant \lambda_2' \leqslant \ldots \leqslant \lambda_{m-1} \leqslant \lambda_{m-1}' \leqslant \lambda_m, \tag{3.1}$$

then we shall say that (p, q) is a separating pair of polynomials. If all the inequalities in (3.1) are strong, then we shall say that the pair (p, q) is strongly separating. For instance, if all the roots λ_k of p are real and simple, then the pair $\left(p, \frac{1}{m}\frac{\partial p}{\partial\lambda}\right)$ is strongly separating.

Separating pairs of polynomials were applied for obtaining "*a priori* estimates" of solutions of hyperbolic partial differental equations by Leray and Gårding. Here we devellop a method of symmetrization of some special matrices (see Lemma 3.2), which in fact is contained in *a priori* estimations by separating polynomials.

If (p, q) is a separating pair of polynomials and p has degree m, then there is exactly one square matrix K of degree $m+1$, such that

$$\left\langle K\begin{bmatrix}1\\ \lambda\\ \vdots\\ \lambda^m\end{bmatrix}, \begin{bmatrix}1\\ \lambda\\ \vdots\\ \lambda^m\end{bmatrix}\right\rangle = p(\lambda)q(\bar\lambda) - p(\bar\lambda)q(\lambda)$$

for every $\lambda \in C$, where $\langle\, ,\, \rangle$ stands for the scalar product in C^m, and there is exactly one square matrix S of degree m such that

$$K = \begin{bmatrix}0 & \\ \vdots & S \\ 0 & \cdots\ 0\end{bmatrix} - \begin{bmatrix}0\cdots 0 & \\ & \vdots \\ S & 0\end{bmatrix}.$$

As easy to see, K is real and skew-adjoint, so that S is real and symmetric. Moreover, S is the unique square matrix of degree m, such that

$$(\lambda - \bar\lambda)\left\langle S\begin{bmatrix}1\\ \lambda\\ \vdots\\ \lambda^{m-1}\end{bmatrix}, \begin{bmatrix}1\\ \lambda\\ \vdots\\ \lambda^{m-1}\end{bmatrix}\right\rangle = p(\lambda)q(\bar\lambda) - p(\bar\lambda)q(\lambda) \tag{3.2}$$

for every $\lambda \in C$. We shall denote this matrix by $S(p, q)$. Let

$$S(p, q) = \begin{bmatrix} S_{00} & S_{01} & \ldots & S_{0,m-1} \\ S_{10} & S_{11} & \ldots & S_{1,m-1} \\ \ldots & \ldots & \ldots & \ldots \\ S_{m-1,0} & S_{m-1,1} & \ldots & S_{m-1,m-1} \end{bmatrix}.$$

Then

$$S_{m-1,\nu} = S_{\nu,m-1} = q_\nu \quad \text{for} \quad \nu = 0, 1, \ldots, m-2, \qquad S_{m-1,m-1} = 1. \tag{3.3}$$

For any real λ let

$$T_m(\lambda) = \begin{bmatrix} -\lambda & 1 & & & 0 \\ & -\lambda & 1 & & \\ & & \ddots & \ddots & \\ 0 & & & -\lambda & 1 \end{bmatrix} \tag{3.4}$$

be a matrix with m rows and $m+1$ columns and let

$$U_m(\lambda) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \lambda & 1 & 0 & \cdots & 0 \\ \lambda^2 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \lambda^{m-1} & \lambda^{m-2} & \lambda^{m-3} & \cdots & 1 \end{bmatrix}$$

be a matrix with $m+1$ rows and m columns. We then have

$$T_m(\lambda)\, U_m(\lambda) = E, \tag{3.5}$$

the unit matrix of degree m.

LEMMA 3.1. *Let (p, q) be a separating pair of polynomials and $m =$ degree of p. Let λ_0 be real and*

$$\tilde{p}(\lambda) = (\lambda-\lambda_0)p(\lambda), \qquad \tilde{q}(\lambda) = (\lambda-\lambda_0)q(\lambda), \qquad \lambda \in C.$$

Then

$$S(\tilde{p}, \tilde{q}) = T_m^*(\lambda_0)\, S(p, q)\, T_m(\lambda_0)$$

and consequently, by (3.5),

$$S(p, q) = U_m^*(\lambda_0)\, S(\tilde{p}, \tilde{q})\, U_m(\lambda_0).$$

Proof. By (3.2), for any $\lambda \in C$ we have

$$\left\langle S(\tilde{p}, \tilde{q}) \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^m \end{bmatrix}, \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^m \end{bmatrix} \right\rangle = |\lambda-\lambda_0|^2 \left\langle S(p, q) \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{m-1} \end{bmatrix}, \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{m-1} \end{bmatrix} \right\rangle$$

$$= \left\langle S(p, q)\, T_m(\lambda_0) \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^m \end{bmatrix}, T_m(\lambda_0) \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^m \end{bmatrix} \right\rangle.$$

LEMMA 3.2. *Let (p, q) be a separating pair of polynomials and let $p(\lambda) = \lambda^m + p_{m-1}\lambda^{m-1} + \ldots + p_1\lambda + p_0$. Put*

$$G(p) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -p_0 & -p_1 & -p_2 & \cdots & -p_{m-2} & -p_{m-1} \end{bmatrix}.$$

Then the real matrix $S(p, q)G(p)$ is symmetric.

Proof. By (3.3), $\sum_{\nu=0}^{m-1} S_{m-1,\nu}\lambda^\nu = q(\lambda)$ and therefore

$$\left\langle S(p, q) \begin{bmatrix} 0 \\ \vdots \\ 0 \\ p(\lambda) \end{bmatrix}, \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{m-1} \end{bmatrix} \right\rangle$$

$$= \left\langle \begin{bmatrix} 0 \\ \vdots \\ 0 \\ p(\lambda) \end{bmatrix}, S(p, q) \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{m-1} \end{bmatrix} \right\rangle = p(\lambda)q(\bar{\lambda}).$$

Consequently, by (3.2)

$$\operatorname{Im} \left\langle S(p, q)G(p) \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{m-1} \end{bmatrix}, \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{m-1} \end{bmatrix} \right\rangle$$

$$= \operatorname{Im} \left\langle S(p, q) \begin{bmatrix} \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{m-1} \\ \lambda^m - p(\lambda) \end{bmatrix}, \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{m-1} \end{bmatrix} \right\rangle$$

$$= \operatorname{Im} \left(\lambda \left\langle S(p, q) \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{m-1} \end{bmatrix}, \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{m-1} \end{bmatrix} \right\rangle - p(\lambda)q(\bar{\lambda}) \right) = 0$$

for every $\lambda \in C$, which proves that $S(p, q)G(p)$ is symmetric.

LEMMA 3.3. *If (p, q) is a strongly separating pair of polynomials, then the real symmetric matrix $S(p, q)$ is positive. In particular, if $p(\lambda) = \lambda^m + p_{m-1}\lambda^{m-1} + \ldots + p_1\lambda + p_0 = \prod_{k=1}^{m}(\lambda-\lambda_k)$ has only real and simple roots $\lambda_1, \lambda_2, \ldots, \lambda_m$, then*

$$\det S\left(p, \frac{1}{m}\frac{\partial p}{\partial \lambda}\right) = m^{-m} \prod_{1 \leqslant i < j \leqslant m} (\lambda_i - \lambda_j)^2. \tag{3.6}$$

Proof. The outline of the following reasoning is taken from [1], Chapter 2, proof of the Lemma 5.2. Let $q(\lambda) = \prod_{k=1}^{m-1}(\lambda-\lambda'_k)$. Then

$$\lambda_1 < \lambda'_1 < \lambda_2 < \lambda'_2 < \ldots < \lambda_{m-1} < \lambda'_{m-1} < \lambda_m. \tag{3.7}$$

Let

$$r_k(\lambda) = \frac{p(\lambda)}{\lambda-\lambda_k} = \sum_{l=0}^{m-1} r_{kl}\lambda^l, \qquad \lambda \in C, \qquad k = 1, 2, \ldots, m.$$

Then, by (3.7), we have

$$\gamma_k = \frac{\prod_{i=1}^{m-1}(\lambda_k-\lambda'_i)}{r_k(\lambda_k)} > 0 \tag{3.8}$$

for very $k = 1, \ldots, m$ and

$$q(\lambda) = \sum_{k=1}^{m} \gamma_k r_k(\lambda),$$

so that

$$p(\lambda)q(\bar{\lambda}) - p(\bar{\lambda})q(\lambda) = (\lambda-\bar{\lambda})\sum_{k=1}^{m}\gamma_k|r_k(\lambda)|^2$$

and consequently

$$\left\langle S(p,q)\begin{bmatrix}1\\ \lambda\\ \vdots\\ \lambda^{m-1}\end{bmatrix}, \begin{bmatrix}1\\ \lambda\\ \vdots\\ \lambda^{m-1}\end{bmatrix}\right\rangle = \sum_{k=1}^{m}\gamma_k|r_k(\lambda)|^2$$

for every $\lambda \in C$. It follows that

$$S(p,q) = R^*\Gamma R,$$

where

$$R = \begin{bmatrix} r_{10} & r_{11} & \cdots & r_{1,m-1}\\ r_{20} & r_{21} & \cdots & r_{2,m-1}\\ \cdots & \cdots & \cdots & \cdots\\ r_{m0} & r_{m1} & \cdots & r_{n,m-1}\end{bmatrix}, \qquad \Gamma = \begin{bmatrix}\gamma_1 & & & 0\\ & \gamma_2 & & \\ & & \ddots & \\ 0 & & & \gamma_m\end{bmatrix}$$

from which, by (3.8), we see that the positivity of $S(p,q)$ will be proved, if we shall show that

$$\det R = (-1)^{\frac{m(m-1)}{2}} \prod_{1 \leqslant i < j \leqslant m}(\lambda_j-\lambda_i). \tag{3.9}$$

The formula (3.6) also follows from (3.9), because if $q(\lambda) = \frac{1}{m}\frac{\partial p(\lambda)}{\partial\lambda}$, then $\gamma_k = \frac{1}{m}$ for every $k = 1, 2, \ldots, m$.

In order to prove (3.9) put

$$\tau_0 = 1, \quad \tau_k = \sum_{1 \leqslant \nu_1 < \ldots \nu_k \leqslant m} \lambda_{\nu_1}\lambda_{\nu_2}\ldots\lambda_{\nu_k}, \quad k = 1, 2, \ldots, m,$$

$$\tau_0^i = 1, \qquad \tau_k^i = \sum_{\substack{1 \leqslant \nu_1 < \ldots < \nu_k \leqslant m\\ \nu_1, \ldots, \nu_k \neq i}} \lambda_{\nu_1}\lambda_{\nu_2}\ldots\lambda_{\nu_k}, \quad i = 1, \ldots, m, \ k = 1, \ldots, m-1.$$

Then

$$\tau_k = \lambda_i\tau_{k-1}^i + \tau_k^i$$

and

$$r_{i,m-1-k} = (-1)^k\tau_k^i$$

for $i = 1, \ldots, m$, $k = 0, 1, \ldots, m-1$. Consequently

$$\sum_{j=0}^{k}(-\lambda_i)^j\tau_{k-j} = (-\lambda_i)^k + \sum_{j=0}^{k-1}(-\lambda_i)^j\tau_{k-j}^i - \sum_{j=0}^{k-1}(-\lambda_i)^{j+1}\tau_{k-j-1}^i$$
$$= \sum_{j=0}^{k}(-\lambda_i)^j\tau_{k-j}^i - \sum_{j=1}^{k}(-\lambda_i)^j\tau_{k-j}^i = \tau_k^i$$

and so

$$r_{i,k} = (-1)^{m-1-k}\tau_{m-1-k}^i = \sum_{j=0}^{m-1-k}(-\lambda_i)^{m-1-k-j}\tau_{m-1-k-j}.$$

Therefore

$$R = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^{m-1}\\ 1 & \lambda_2 & \lambda_2^2 & \ldots & \lambda_2^{m-1}\\ \cdot & \cdot & \cdot & \cdots & \cdot\\ 1 & \lambda_m & \lambda_m^2 & \ldots & \lambda_m^{m-1}\end{bmatrix}\begin{bmatrix} (-1)^{m-1}\tau_{m-1} & (-1)^{m-2}\tau_{m-2} & \cdots & -\tau_1 & 1\\ \vdots & \vdots & & & \\ \tau_2 & -\tau_1 & & & \\ -\tau_1 & 1 & & & \\ 1 & & & 0 & \end{bmatrix}$$

and (3.9) follows from well known formula for Vandermonde's determinant.

Lemma 3.4. *For every separating pair (p, q) of polynomials the real symmetric matrix $S(p,q)$ is non negative.*

Proof. Let m = degree of p. Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be all the roots of $p(\lambda)$ and let $\lambda'_1, \lambda'_2, \ldots, \lambda'_{m-1}$ be all the roots of $q(\lambda)$, labelled in such manner that

$$\lambda_1 < \lambda'_1 < \lambda_2 < \lambda'_2 < \ldots < \lambda_{m'-1} < \lambda'_{m'-1} < \lambda_{m'}$$

and

$$\lambda'_{m'} = \lambda_{m'+1}, \quad \lambda'_{m'+1} = \lambda_{m'+2}, \ldots, \lambda'_{m-1} = \lambda_m.$$

For any $k = 1, \ldots, m'$ let

$$r_k(\lambda) = \frac{p(\lambda)}{\lambda - \lambda_k} = \sum_{l=0}^{m-1} r_{k,l}\lambda^l$$

and

$$\gamma_k = \frac{\prod\limits_{i=1}^{m'-1} (\lambda_k - \lambda'_i)}{\prod\limits_{\substack{i=1 \\ i\neq k}}^{m'} (\lambda_k - \lambda_i)}.$$

Then

(3.10) $$\gamma_k > 0 \quad \text{for} \quad k = 1, \ldots, m'$$

and

$$q(\lambda) = \sum_{k=1}^{m'} \gamma_k r_k(\lambda)$$

so that, by (3.2),

$$\left\langle S(p, q)\begin{bmatrix}1\\ \lambda\\ \vdots\\ \lambda^{m-1}\end{bmatrix}, \begin{bmatrix}1\\ \lambda\\ \vdots\\ \lambda^{m-1}\end{bmatrix}\right\rangle = \sum_{k=1}^{m'} \gamma_k |r_k(\lambda)|^2$$

for every $\lambda \in C$ and therefore

(3.11) $$S(p, q) = R^* \Gamma R,$$

where

$$R = \begin{bmatrix} r_{10} & r_{11} & \ldots\ldots & r_{1,m-1} \\ r_{20} & r_{21} & \ldots\ldots & r_{2,m-1} \\ \ldots & \ldots & \ldots\ldots & \ldots \\ r_{m'0} & r_{m'1} & \ldots\ldots & r_{m',m-1} \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \gamma_1 & & & 0 \\ & \gamma_2 & & \\ & & \ddots & \\ 0 & & & \gamma_{m'} \end{bmatrix}$$

It follows from (3.10) and (3.11) that $S(p, q)$ is non negative.

LEMMA 3.5. *Let (p, q) be a strongly separating pair of polynomials. If $\tilde{p}(\lambda) = (\lambda - \lambda_0)p(\lambda)$, $\tilde{q}(\lambda) = (\lambda - \lambda_0)q(\lambda)$, where λ_0 is real, then the $(m+1)\times(m+1)$ matrix $S(\tilde{p}, \tilde{q})$ has rank m.*

Proof. By Lemma 3.1, the rank of $S(\tilde{p}, \tilde{q})$ is equal to the rank of $S(p, q)$, which, by Lemma 3.3, is real symmetric positive matrix of degree m.

4. The matrices $B_k(A, p)$. Let A be a complex $m \times m$ matrix and let

$$p(\lambda) = \lambda^{m'} + a_{m'-1}\lambda^{m'-1} + \ldots a_1\lambda + a_0$$

be a polynomial divisible by the minimal polynomial of A^*, i.e. let p be such that

$$p(A^*) = 0.$$

Then there is unique square matrix $\boldsymbol{B}(A, p, \lambda)$ of degree m, which elements are polynomials in λ of degree $m'-1$, such that

$$(\lambda E - A^*)\boldsymbol{B}(A, p, \lambda) = \boldsymbol{B}(A, p, \lambda)(\lambda E - A^*) = p(\lambda)E,$$

where E is the unit matrix of degree m. We define the matrices $B_k(A, p)$, $k = 1, \ldots, m$, with constant complex elements by the condition that

$$\boldsymbol{B}(A, p, \lambda) = \left[B_1(A, p)\begin{bmatrix}1\\ \lambda\\ \vdots\\ \lambda^{m'-1}\end{bmatrix}, B_2(A, p)\begin{bmatrix}1\\ \lambda\\ \vdots\\ \lambda^{m'-1}\end{bmatrix}, \ldots, B_m(A, p)\begin{bmatrix}1\\ \lambda\\ \vdots\\ \lambda^{m'-1}\end{bmatrix}\right]$$

for every $\lambda \in C$. In other words, $B_k(A, p)\begin{bmatrix}1\\ \lambda\\ \vdots\\ \lambda^{m'-1}\end{bmatrix}$ is the k-th column of $\boldsymbol{B}(A, p, \lambda)$. Each of the matrices $B_k(A, p)$ has m rows and m' columns.

LEMMA 4.1. *If A is a square matrix of degree m, $p(\lambda) = \lambda^{m'} + a_{m'-1}\lambda^{m'-1} + \ldots + a_1\lambda + a_0$ is a polynomial divisible by the minimal polynomial of A^* and $\tilde{p}(\lambda) = (\lambda - \lambda_0)p(\lambda)$, then*

$$B_k(A, \tilde{p}) = B_k(A, p)T_{m'}(\lambda_0)$$

for every $k = 1, \ldots, m$, where $T_{m'}(\lambda_0)$ is defined by (3.4).

Proof. We have $\boldsymbol{B}(A, \tilde{p}, \lambda) = (\lambda - \lambda_0)\boldsymbol{B}(A, p, \lambda)$, so that

$$B_k(A, \tilde{p})\begin{bmatrix}1\\ \lambda\\ \vdots\\ \lambda^{m'}\end{bmatrix} = (\lambda - \lambda_0)B_k(A, p)\begin{bmatrix}1\\ \lambda\\ \vdots\\ \lambda^{m'-1}\end{bmatrix} = B_k(A, p)T_{m'}(\lambda_0)\begin{bmatrix}1\\ \lambda\\ \vdots\\ \lambda^{m'}\end{bmatrix}$$

for every $\lambda \in C$.

LEMMA 4.2. *If A is a square matrix of degree m, $p(\lambda) = \lambda^{m'} + a_{m'-1}\lambda^{m'-1} + \ldots + a_1\lambda + a_0$ is a polynomial divisible by the minimal polynomial of A^* and*

$$G(p) = \begin{bmatrix} 0 & 1 & 0 & & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & & 0 & 0 \\ -a_{m-1} & -a_{m-2} & -a_{m-3} & \ldots & -a_1 & -a_0 \end{bmatrix}$$

then, for every $k = 1, \ldots, m$,

$$A^* B_k(A, p) = B_k(A, p) G(p).$$

Proof. For every $\lambda \in C$ we have

$$[A^* B_k(A,p) - B_k(A,p)G(p)] \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{m'-1} \end{bmatrix}$$

$$= A^* B_k(A,p) \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{m'-1} \end{bmatrix} - B_k(A,p) \begin{bmatrix} \lambda \\ \vdots \\ \lambda^{m'-1} \\ \lambda^{m'} - p(\lambda) \end{bmatrix}$$

$$= B_k(A,p) \begin{bmatrix} 0 \\ \vdots \\ 0 \\ p(\lambda) \end{bmatrix} - (\lambda E - A^*) B_k(A,p) \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{m'-1} \end{bmatrix}$$

$$= p(\lambda) B_k(A,p) \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} - (\text{the } k\text{-th column of } p(\lambda)E).$$

Because $B_k(A, p) \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$ is the column of the coefficients close to $\lambda^{m'-1}$ in the k-th column of the matrix $\boldsymbol{B}(A, p, \lambda) = \lambda^{m'-1}E + \ldots$, we have

$$B_k(A,p) \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \text{the } k\text{-th column of } E.$$

It follows that for every $\lambda \in C$ we have

$$[A^* B_k(A,p) - B_k(A,p)G(p)] \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{m'-1} \end{bmatrix} = 0.$$

5. The matrices $H_k(A)$ and $H(A)$. Let A be a square matrix of degree m with complex elements, having purely diagonal and real canonical Jordan form. Let $p(\lambda) = \lambda^{m'} + a_{m'-1}\lambda^{m'-1} + \ldots a_1\lambda + a_0$ be the minimal polynomial of A and let $q(\lambda) = \frac{1}{m'} \frac{\partial p(\lambda)}{\partial \lambda}$. Because the canonical Jordan form of A is purely diagonal, $p(\lambda)$ has only simple roots and therefore, according to Lemma 3.3, the real symetric matrix $S(p, q)$ is positive and hence inversible. For every $k = 1, \ldots, m$ we put

$$H_k(A) = B_k(A, p) S^{-1}(p, q) B_k^*(A, p).$$

Then $H_k(A)$, $k = 1, \ldots, m$, are hermitean non negative matrices of degree m.

We put

$$H(A) = H_1(A) + H_2(A) + \ldots + H_m(A).$$

LEMMA 5.1. $\inf H(A) \geqslant 1$.

Proof. Let $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, e_m = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$ be columns with m elements and let $e'_{m'} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$ be column with m' elements. Then, by (3.3) we have

$$\langle S(p, q) e'_{m'}, e'_{m'} \rangle = 1$$

and, moreover, for any $k = 1, \ldots, m$ we have

$$B_k(A, p) e'_{m'} = e_k,$$

because each of two sides of this equality is the column of the coefficients close to $\lambda^{m'-1}$ in the kth column of the matrix $\boldsymbol{B}(A, p, \lambda) = \lambda^{m'-1}E + \ldots$

For any $k = 1, \ldots, m$ put

$$C_k = S^{-1}(p, q) B_k^*(A, p).$$

Then, for any $x = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \in C^m$, by Schwartz's inequality we have

$$\langle H(A)x, x \rangle |x|^2 = \Big(\sum_{k=1}^{m} \langle S(p, q) C_k x, C_k x \rangle \Big) \cdot \Big(\sum_{k=1}^{m} \langle S(p, q) x_k e'_{m'}, x_k e'_{m'} \rangle \Big)$$

$$\geqslant \Big| \sum_{k=1}^{m} \langle S(p, q) C_k x, x_k e'_{m'} \rangle \Big|^2 = \Big| \sum_{k=1}^{m} \langle x, x_k B_k(A, p) e'_{m'} \rangle \Big|^2$$

$$= \Big| \sum_{k=1}^{m} \langle x, x_k e_k \rangle \Big|^2 = |x|^4,$$

which proves the lemma.

LEMMA 5.2. *The matrix $H(A)A$ is hermitean.*

Proof. By Lemma 4.2 we have

$$H_k(A)A = B_k(A,p)S^{-1}(p,q)B_k^*(A,p)A = B_k(A,p)S^{-1}(p,q)G^*(p)B_k^*(A,p)$$

and, because, by Lemma 3.2, the matrix $S^{-1}(p,q)G^*(p)$ is real and symmetric, it folows that $H_k(A)A$ is hermitean for every $k=1,\dots,m$.

6. Proof of the theorem from Section 1. Under assumptions of theorem from Section 1, for any $\xi\in M$ put

$$H(\xi) = H(A(\xi)),$$

where the hermitean positive matrix $H(A(\xi))$ is defined as in Section 5. Then it follows from Lemmas 5.1 and 5.2, that $\|H^{-1}(\xi)\| = (\inf H(\xi))^{-1}\leqslant 1$ and that $H(\xi)A(\xi)$ is hermitean for every $\xi\in M$. Therefore our theorem will be proved, if we shall show that the function

$$m(\xi) = \|H(\xi)\| = \sup H(\xi), \quad \xi\in M,$$

is locally bounded on M. For any $\lambda\in C$ and $\xi\in M$ put

$$p(\lambda,\xi) = (\lambda^2+2b(\xi)\lambda+c(\xi))\prod_{k=1}^{m'}(\lambda-\lambda_k(\xi)),$$

$$p'(\lambda,\xi) = (\lambda+b(\xi))\prod_{k=1}^{m'}(\lambda-\lambda_k(\xi)).$$

Then $p(\lambda,\xi)$ and $p'(\lambda,\xi)$ are polynomials in λ, the coefficients of which, by Remark 1 from Section 2, are real functions of ξ, analytic in M. The minimal polynomial of the matrix $A(\xi)$ is

$$p(\lambda,\xi) \quad \text{if} \quad \xi\in M\setminus N,$$
$$p'(\lambda,\xi) \quad \text{if} \quad \xi\in N.$$

From the construction presented in Sections 3–5 it is clear, that $H(\xi)$ is an analytic function of ξ on the set N, which, as we know from Remark 1 from Section 2, is a sum of disjoint analytic submanifolds of M. Similary, $H(\xi)$ is an analytic function of on the set $M\setminus N$, which is an open subset of M. Therefore $m(\xi)$ is locally bounded on N, and it is also locally bounded on $M\setminus N$. It remains to prove that $m(\xi)$ is locally bounded in a neighbourhood of N.

For any $\xi\in M$, $\lambda\in C$ and $k=1,\dots,m$ put

$$q(\lambda,\xi) = \frac{1}{m'+2}\frac{\partial}{\partial\lambda}p(\lambda,\xi),$$

$$S(\xi) = S(p(\cdot,\xi),q(\cdot,\xi)), \quad B_k(\xi) = B_k(A(\xi),p(\cdot,\xi)).$$

Then, by Lemmas 3.3 and 3.4, $S(\xi)$ is a real symmetric square matrix of degree $m'+2$, which is non negative if $\xi\in N$ and positive if $\xi\in M\setminus N$. For any $\xi\in M$ let $S^{\frac12}(\xi)$ be the symmetric non negative square root of $S(\xi)$. If $\xi\in M\setminus N$, then the matrices $S(\xi)$ and $S^{\frac12}(\xi)$ are positive, and then we denote its inverse matrices by $S^{-1}(\xi)$ and $S^{-\frac12}(\xi)$. For any $\xi\in M\setminus N$ and $k=1,\dots,m$ put

$$D_k(\xi) = S^{-\frac12}(\xi)B_k^*(\xi),$$

which is a matrix which $m'+2$ rows and m columns. We then have

$$H(\xi) = \sum_{k=1}^{m} D_k^*(\xi)D_k(\xi)$$

and therefore the theorem from Section 1 will be proved, if we shall show that, for any $k=1,\dots,m$, $D_k(\xi)$ is locally bounded in a neighbourhood of N. In that order consider some auxiliary matrix-valued functions defined on N. Namely, for any $\eta\in N$ and $\lambda\in C$ put

$$q'(\lambda,\eta) = \frac{q(\lambda,\eta)}{\lambda+b(\eta)}, \quad S_0(\eta) = S(p'(\cdot,\eta),q'(\cdot,\eta)),$$
$$B_k^0(\eta) = B_k(A(\eta),p'(\cdot,\eta)).$$

Then, by Lemma 3.3, for any $\eta\in N$, $S_0(\eta)$ is a real symmetric positive matrix of degree $m'+1$, so that, for any $\eta\in N$ and $k=1,\dots,m$ we may define

$$D_k(\eta) = S^{\frac12}(\eta)U_{m'+1}(-b(\eta))S_0^{-1}(\eta)B_k^{0*}(\eta).$$

According to Lemmas 3.1 and 4.1, for $\eta\in N$ and $k=1,\dots,m$, we have

$$S(\eta) = T_{m'+1}^*(-b(\eta))S_0(\eta)T_{m'+1}(-b(\eta))$$

and

$$B_k(\eta) = B_k^0(\eta)T_{m'+1}(-b(\eta)),$$

so that, by (3.5),

$$\begin{aligned}S^{\frac12}(\eta)D_k(\eta) &= S(\eta)U_{m'+1}(-b(\eta))S_0^{-1}(\eta)B_k^{0*}(\eta)\\ &= T_{m'+1}^*(-b(\eta))S_0(\eta)T_{m'+1}(-b(\eta))U_{m'+1}(-b(\eta))S_0^{-1}(\eta)B_k^{0*}(\eta)\\ &= T_{m'+1}^*(-b(\eta))B_k^{0*}(\eta) = B_k^*(\eta).\end{aligned}$$

It follows that

$$D_k(\xi) = D_k(\eta)+S^{-\frac12}(\xi)[B_k^*(\xi)-B_k^*(\eta)+(S^{\frac12}(\eta)-S^{\frac12}(\xi))D_k(\eta)] \tag{6.1}$$

for every $k=1,\dots,m$, $\xi\in M\setminus N$ and $\eta\in N$. This equality is basic for our proof of local boundedness of $D_k(\xi)$ in a neighbourhood of N. Namely, from analicity of $A(\xi)$ and from analicity of the coefficients of polynomials $p(\lambda, \xi)$ and $q(\lambda, \xi)$ it follows that the matrix-valued functions $\xi \to S(\xi)$ and $\xi \to B_k(\xi)$ are analytic in M. Similary, the matrix-valued functions $\eta \to D_k(\eta)$ are analytic on N. Let $\xi_0 \in N$ and let V_{ξ_0} be an open bounded neighbourhood of ξ_0 in M, such that $\overline{V}_{\xi_0} \subset M$. Then there is a constant $\mathscr{M}$ such that

$$\|D_k(\eta)\| \leqslant \mathscr{M} \quad \text{and} \quad \|B_k^*(\xi) - B_k^*(\eta)\| \leqslant \mathscr{M} d(\xi, \eta)$$

for every $\eta \in N \cap V_{\xi_0}$, $\xi \in V_{\xi_0}$ and $k = 1, \ldots, m$, $\|\ \|$ being the norm of corresponding matrix as operator of C^m into $C^{m'+2}$ and $d(\xi, \eta)$ is the distance from ξ to η. Therefore, by (6. 1), for $\xi \in V_{\xi_0} \setminus N$ and $\eta \in V_{\xi_0} \cap N$ we have

$$\|D_k(\xi)\| \leqslant \mathscr{M} + \mu_0^{-\frac{1}{2}}(\xi)\, \mathscr{M}\big(d(\xi, \eta) + \|S^{\frac{1}{2}}(\xi) - S^{\frac{1}{2}}(\eta)\|\big), \tag{6.2}$$

where $\mu_0(\xi) > 0$ is the smaller characteric root of the real symmetric positive matrix $S(\xi)$, $\|D_k(\xi)\|$ is the norm of $D_k(\xi)$ as operator of C^m into $C^{m'+2}$ and $\|S^{\frac{1}{2}}(\xi) - S^{\frac{1}{2}}(\eta)\|$ is the norm of $S^{\frac{1}{2}}(\xi) - S^{\frac{1}{2}}(\eta)$ as operator of $C^{m'+2}$ into itself.

Let $\mu_k(\xi)$, $k = 1, \ldots, m'+1$, be the other characteristic roots of $S(\xi)$. Then, for $\xi \in V_{\xi} \setminus N$, by Lemma 3.3,

$$\begin{aligned}\mu_0(\xi) &= \frac{\det S(\xi)}{\mu_1(\xi)\mu_2(\xi)\ldots\mu_{m'+1}(\xi)} \geqslant \frac{\det S(\xi)}{\big(\sup S(\xi)\big)^{m'+1}} \\ &= \big(\sup S(\xi)\big)^{-m'-1}(m'+2)^{-m'-2} \prod_{1\leqslant i<j\leqslant m'+2} \big(\lambda_i(\xi) - \lambda_j(\xi)\big)^2,\end{aligned}$$

where $\lambda_k(\xi)$, $k = 1, \ldots, m'$, are roots of $\det\big(\lambda E - A(\xi)\big)$, distinquished in factorisation (1. 1) and $\lambda_{m'+1}(\xi)$ and $\lambda_{m'+2}(\xi)$ are roots of $\lambda^2 + 2b(\xi)\lambda + c(\xi)$. Thus, from assumptions 1° and 2° it follows that there is a constant $C > 0$, such that

$$C^2 \mu_0(\xi) \geqslant d^2(\xi, N)$$

for every $\xi \in V_{\xi_0} \setminus N$, so that, by (6.2),

$$\|D_k(\xi)\| \leqslant \mathscr{M} + \mathscr{M}C\big(1 + d^{-1}(\xi, N) \inf_{\eta \in V_{\xi_0} \cap N} \|S^{\frac{1}{2}}(\xi) - S^{\frac{1}{2}}(\eta)\|\big)$$

for every $\xi \in V_{\xi_0} \setminus N$. Therefore in order to prove the boundedness of $D_k(\xi)$ in $V_{\xi_0} \setminus N$ it remains to show that the matrix-valued function $\xi \to S^{\frac{1}{2}}(\xi)$ is lipschitzian in V_{ξ_0}, i.e. that there is a constant $\mathscr{L}$, such that

$$\|S^{\frac{1}{2}}(\xi) - S^{\frac{1}{2}}(\eta)\| \leqslant \mathscr{L} d(\xi, \eta)$$

for every ξ and η in V_{ξ_0}. Since, by Lemma 3.5, the rank of real symmetric non negative matrix $S(\xi)$ is equal $m'+2$, when $\xi \in M \setminus N$ and is equal $m'+1$, when $\xi \in N$, it follows that if the characteristic roots of $S(\xi)$ are $\mu_0(\xi) \leqslant \mu_1(\xi) \leqslant \ldots \leqslant \mu_{m'+1}(\xi)$, then there is a positive constant r, such that

$$\mu_k(\xi) \geqslant 6r \quad \text{for } k = 1, \ldots, m+1 \quad \text{and} \quad \xi \in V_{\xi_0}$$

while

$$\mu_0(\xi) \begin{cases} = 0 & \text{if } \xi \in V_{\xi_0} \cap N, \\ > 0 & \text{if } \xi \in V_{\xi_0} \setminus N. \end{cases}$$

Let

$$\begin{aligned} V'_{\xi_0} &= \{\xi\colon \xi \in V_{\xi_0},\ \mu_0(\xi) > 2r\}, \\ V''_{\xi_0} &= \{\xi\colon \xi \in V_{\xi_0},\ \mu_0(\xi) < 3r\}, \\ R &= \sup\{\mu_{m'+1}(\xi)\colon \xi \in V_{\xi_0}\}. \end{aligned}$$

Let, for any $\varrho \in (0, R]$, $\mathscr{C}_\varrho$ be the rectangular contour in complex plane with edges $\varrho - i$, $R+1-i$, $R+1+i$, $\varrho + i$. Then we have

$$S^{\frac{1}{2}}(\xi) = \frac{1}{2\pi i}\int_{\mathscr{C}_r} z^{\frac{1}{2}}\big(zE - S(\xi)\big)^{-1} dz,$$

when $\xi \in V'_{\xi_0}$ and

$$S^{\frac{1}{2}}(\xi) = \mu_0^{\frac{1}{2}}(\xi)\frac{1}{2\pi i}\int_{|z|=4r}\big(zE - S(\xi)\big)^{-1} dz + \frac{1}{2\pi i}\int_{\mathscr{C}_{5r}} z^{\frac{1}{2}}\big(zE - S(\xi)\big)^{-1} dz$$

when $\xi \in V''_{\xi_0}$, where $\mu_0^{\frac{1}{2}}(\xi)$ is non negative square root of $\mu_0(\xi)$ and $|\arg z^{\frac{1}{2}}| < \frac{\pi}{2}$ under the integrals. From the first of these formulas we see that $S^{\frac{1}{2}}(\xi)$ depends analytically on ξ in $\overline{V'_{\xi_0}}$. In the second formula the integrals represent matrix-valued functions of ξ, analytic in $\overline{V''_{\xi_0}}$. Moreover, for $\xi \in \overline{V''_{\xi_0}}$, $\mu_0(\xi)$ is a simple characteristic root of the matrix $S(\xi)$, which depends analytically on ξ. Therefore $\mu_0(\xi)$ is an analytic function of ξ in $\overline{V''_{\xi_0}}$, so that $\mu_0^{\frac{1}{2}}(\xi)$ is a lipschitzian function of ξ in V''_{ξ_0} and consequently the matrix-valued function $\xi \to S^{\frac{1}{2}}(\xi)$ is lipschitzian in V''_{ξ_0}. Because $V_{\xi_0} = V'_{\xi_0} \cup V''_{\xi_0}$, it follows that $\xi \to S^{\frac{1}{2}}(\xi)$ is a lipschitzian function in V_{ξ_0}, which completes the proof.

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Generalized invariant subspaces for linear operators*

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Summary. The Banach space $\{Y, \| \|\}$ is said to be an invariant Banach subspace for the bounded linear operator A on $\{X, |.|\}$ if Y is continuously imbedded in X and $AY \subset Y$. It is shown that every bounded linear operator A on X has a nontrivial invariant Hilbert subspace $\mathscr{H}$ which is nuclearly imbedded in X and on which $A|_{\mathscr{H}}$ is a positive multiple of a simple unilateral shift. If A is quasinilpotent then it has invariant Banach subspaces Y on which the restricted operator $A|_Y$ is compact. These invariant spaces may in addition be chosen to be Hilbert spaces with nuclear imbedding into X. As a consequence, by the theory of interpolation between Hilbert spaces, every quasinilpotent operator A with a cyclic vector on a Hilbert space $\mathscr{H}_0$ has nontrivial invariant Hilbert subspaces $\mathscr{H}_\alpha$ $(0 < \alpha \leqslant 1)$ "arbitrarily close" to $\mathscr{H}_0$ which are compactly imbedded in $\mathscr{H}_0$ and on which $A|_{\mathscr{H}_\alpha}$ is quasinilpotent and compact.

1. The purpose of this note is to introduce the notion of invariant Banach or Hilbert subspace for a bounded linear operator on a Banach space. This notion is intermediate to the usual notion of invariant subspace (closed in the original norm) and that of an invariant linear manifold. Its study seems justified in view of the fact that the problem of existence of (ordinary) invariant subspaces for arbitrary operators is still unsolved. The present paper contains a general existence theorem for invariant Hilbert subspaces and some further results for operators of the class (Q).

DEFINITIONS. Let $\{X, |.|\}$ be a separable, complex Banach space and A a bounded linear operator on X. The Banach space $\{Y, \| \|\}$ is said to be a *Banach subspace of* X if $Y \subset X$ and the injection of Y into X is continuous. If in addition $AY \subset Y$ then Y is called an *invariant Banach subspace* for A. (In that case $A|_Y$ is continuous in the norm $|.|$, by the closed graph theorem.) If Y is not dense in X and is invariant under A, then the closure of Y in X is an ordinary invariant subspace for A. If the $\{Y, \| \|\}$ above is a Hilbert space $\mathscr{H} = Y$, it is called an *invariant Hilbert subspace*.

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