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Generalized invariant subspaces for linear operators*

by

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Summary. The Banach space $\{Y, \|\ \|\}$ is said to be an invariant Banach subspace for the bounded linear operator A on $\{X, |.|\}$ if Y is continuously imbedded in X and $AY \subset Y$. It is shown that every bounded linear operator A on X has a nontrivial invariant Hilbert subspace $\mathscr{H}$ which is nuclearly imbedded in X and on which $A|_{\mathscr{H}}$ is a positive multiple of a simple unilateral shift. If A is quasinilpotent then it has invariant Banach subspaces Y on which the restricted operator $A|_Y$ is compact. These invariant spaces may in addition be chosen to be Hilbert spaces with nuclear imbedding into X. As a consequence, by the theory of interpolation between Hilbert spaces, every quasinilpotent operator A with a cyclic vector on a Hilbert space $\mathscr{H}_0$ has nontrivial invariant Hilbert subspaces $\mathscr{H}_\alpha$ $(0 < \alpha \leqslant 1)$ "arbitrarily close" to $\mathscr{H}_0$ which are compactly imbedded in $\mathscr{H}_0$ and on which $A|_{\mathscr{H}_\alpha}$ is quasinilpotent and compact.

1. The purpose of this note is to introduce the notion of invariant Banach or Hilbert subspace for a bounded linear operator on a Banach space. This notion is intermediate to the usual notion of invariant subspace (closed in the original norm) and that of an invariant linear manifold. Its study seems justified in view of the fact that the problem of existence of (ordinary) invariant subspaces for arbitrary operators is still unsolved. The present paper contains a general existence theorem for invariant Hilbert subspaces and some further results for operators of the class (Q).

Definitions. Let $\{X, |.|\}$ be a separable, complex Banach space and A a bounded linear operator on X. The Banach space $\{Y, \|\ \|\}$ is said to be a *Banach subspace of* X if $Y \subset X$ and the injection of Y into X is continuous. If in addition $AY \subset Y$ then Y is called an *invariant Banach subspace* for A. (In that case $A|_Y$ is continuous in the norm $|.|$, by the closed graph theorem.) If Y is not dense in X and is invariant under A, then the closure of Y in X is an ordinary invariant subspace for A. If the $\{Y, \|\ \|\}$ above is a Hilbert space $\mathscr{H} = Y$, it is called an *invariant Hilbert subspace.*

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2. DEFINITIONS. The vector $x \in X$ is called *cyclic for* $A \in B(X)$ if $\{A^n x\colon n = 0, 1, 2, \ldots\}$ spans a dense subspace of X. The operator $A \in B(X)$ is said to belong to the *class* (Q) if it has a cyclic vector x for which

$$\lim_{n\to\infty} |A^n x|^{1/n} = 0. \tag{1}$$

By a theorem of C. Apostol [1] and T. A. Gillespie [3], an operator A of class (Q) has non-trivial invariant subspaces if there exists a non-zero compact operator which is uniform limit of polynomials in A. Without any additional condition, we have the following result for the class (Q).

THEOREM 1. *Let $A \in B(X)$ be of class (Q). Then A has an invariant Banach subspace $\{Y, \|\ \|\}$ which is dense in X. The injection of Y into X is compact, and $A|_Y$ is compact. If A has no non-zero eigenvalues in X, then $A|_Y$ is quasi-nilpotent on Y.*

Proof. We adapt a similar construction found in [2]. Let x be a cyclic vector for A satisfying (1). We may suppose $|A| \leqslant 1$ and $|x| = 1$. Let $\beta_k = (\max\{|A^n x|^{1/n}\colon n \geqslant k\})^k$. Then $|A^k x| \leqslant \beta_k$ and

$$\lim_{k\to\infty} (\beta_{k+1}/\beta_k) = 0. \tag{2}$$

Set $a_k = \beta^{-\frac12} A^k x$ $(k = 0, 1, 2, \ldots)$ and let K be the closed absolutely convex hull of the a's in X. K is contained in the unit ball of X, and thus $Y = \{y \in X\colon \|y\| < \infty\}$ with the norm $\|y\| = \inf\{\lambda\colon \lambda^{-1} y \in K\}$ is an invariant Banach subspace for A. (Completeness of $\{Y, \|\ \|\}$ follows from the closedness of K in X.) Since $|a_k| \leqslant \beta_k^{\frac12} \to 0$, the set K is compact in X, and therefore Y is compactly imbedded in X. To see that $A|_Y$ is compact, observe $A a_k = \beta_k^{-\frac12} A^{k+1} x = (\beta_{k+1}/\beta_k)^{\frac12} a_{k+1}$, hence $\|A a_k\| \leqslant (\beta_{k+1}/\beta_k)^{\frac12}$. By (2), $\{A a_k\colon k = 0, 1, 2, \ldots\}$ is totally bounded, and its closed absolutely convex hull in $\{Y, \|\ \|\}$ contains the image AK of the unit ball K of Y; hence $A|_Y$ is compact. Finally, any eigenvalue of $A|_Y$ is also an eigenvalue of A in X, and any non-zero point in the spectrum of $A|_Y$ must be an eigenvalue. Thus the absence of non-zero eigenvalues of A implies that $A|_Y$ is quasi-nilpotent.

Remark. Since $A|_Y$ is compact, it has "sufficiently many" non-trivial (ordinary) invariant subspaces in Y; unfortunately every one of these might happen to be dense in X.

3. When one wants to make use of an invariant Banach subspace Y for an operator $A \in B(X)$, it will be convenient if Y is actually a Hilbert space. We can prove a general existence theorem for invariant Hilbert subspaces, and in addition it turns out that the space Y of Theorem 1 may be replaced by a Hilbert space which has the same properties with respect to A as the Y above.

If $\{a_n\colon n = 0, 1, 2, \ldots\}$ is a sequence of vectors in X satisfying $\sum |a_n| < \infty$, then $S\{\xi_n\} = \sum \xi_n a_n$ defines a nuclear map $S\colon l^2 \to X$, and its image $Sl^2 = \mathscr{K}$, endowed with the norm $\|h\| = \inf\{|\xi|_{l^2}\colon S\xi = h, \xi \in l^2\}$ is a Hilbert subspace with nuclear imbedding in X.

THEOREM 2. *Every $A \in B(X)$ has an invariant Hilbert subspace $\mathscr{H}$ with nuclear imbedding into X. If A has a cyclic vector, $\mathscr{H}$ can be chosen to be dense in X and so that $A|_{\mathscr{H}}$ is a positive multiple of a unilateral shift.*

Proof. Without loss of generality assume $|A| = q < 1$, and take any x, $0 \neq x \in X$. Then form the Hilbert subspace $\mathscr{K}$ as described above, with $a_n = A^n x$; it is clearly invariant for A. If x is cyclic for A, it is also cyclic for $A|_{\mathscr{K}}$; we may suppose $\|A|_{\mathscr{K}}\| < 1$. By a result of B. Sz.-Nagy and C. Foiaş [5], $A|_{\mathscr{K}}$ has among its quasi-affine transformations a unilateral shift V, but V may be realized as the restriction of A to an appropriate Hilbert subspace $\mathscr{H}$ of $\mathscr{K}$ which is invariant for A and dense in $\mathscr{K}$ (and in X).

THEOREM 3. *Let $A \in B(X)$ be of class (Q). Then there exists an invariant Hilbert subspace $\mathscr{H}$ of A with nuclear imbedding into X and such that $A|_{\mathscr{H}}$ is compact. If A has no non-zero eigenvalues on X, then $A|_{\mathscr{H}}$ is quasi-nilpotent on $\mathscr{H}$.*

Proof. We imitate the proof of Theorem 1. Let x be cyclic for A, satisfying (1). Suppose $|A| \leqslant 1$, $|x| = 1$ and define $\beta_k = (\max\{|A^n x|^{1/n}\colon n \geqslant k\})^k$ as before. Choose any q in $0 < q < 1$ and set $\beta_k^{-q} A^k = a_k$. Then $|a_k| \leqslant \beta_k^{1-q}$, and $\sum |a_k| < \infty$ by (2). Define $\mathscr{K}$ as above and take $\mathscr{H} = \mathscr{K}$. Clearly $A\mathscr{H} \subset \mathscr{H}$. We show that $A|_{\mathscr{H}}$ is compact. First note that S is an isometric isomorphism of $l^2 \ominus N(S) = \mathscr{G}$ ($N(S)$ the nullspace of S) onto $\mathscr{H}$. Hence $\|S\xi\|^2 = \sum |\xi_k|^2$ for $\xi = \{\xi_k\} \in \mathscr{G}$. Let V be the weighted shift operator in l^2 given by $\eta_{k+1} = (\beta_{k+1}/\beta_k)^q \xi_k$ if $V\{\xi_j\} = \{\eta_k\}$. By (2), V is compact. Now let u_j converge to u weakly in $\mathscr{H}$; we show that Au_j converges to Au strongly in $\mathscr{H}$. Since $Aa_k = (\beta_{k+1}/\beta_k)^q a_{k+1}$, we have $A|_{\mathscr{H}} = SVJS^{(-1)}$ where $S^{(-1)}$ denotes the inverse of $S|_{\mathscr{G}}\colon \mathscr{G} \to \mathscr{H}$ and J the injection of $\mathscr{G}$ into l^2. Let $x_j, x_0 \in \mathscr{G}$ so that $Sx_j = u_j$, $Sx_0 = u$; then $x_j \to x_0$ weakly. As $S^{(-1)}$ and J are isometries, we see that $\|Au_j - Au\| \leqslant |Vx_j - Vx_0|_{l^2} \to 0$ because V is compact. Thus $A|_{\mathscr{H}}$ is compact. Quasi-nilpotency of $A|_{\mathscr{H}}$ in the absence of non-zero eigenvalues of A holds as before.

4. If $X = \mathscr{H}_0$ is a Hilbert space and $\mathscr{H} = \mathscr{H}_1$ is constructed as above for $A \in B(\mathscr{H}_0)$ of class (Q), we can strengthen Theorem 3 as follows. Let J denote the injection of $\mathscr{H}_1$ into $\mathscr{H}_0$ and let H be the unique partial isometry of $\mathscr{H}_0$ onto $\mathscr{H}_1$ which is given by

$$(u, v)_0 = (u, HJHv)_1 \quad \text{for all } u \in \mathscr{H}_1,\ v \in \mathscr{H}_0.$$

Since $\mathscr{H}_1$ is dense in $\mathscr{H}_0$, H is injective. Let $\{\mathscr{H}_\alpha = H^\alpha \mathscr{H}_0\colon 0 \leqslant \alpha \leqslant 1\}$ with norms $|x|_\alpha = |(JH)^{-\alpha} x|_0$ be the Hilbert scale interpolating between

$\mathscr{H}_0$ and $\mathscr{H}_1$. By general interpolation theory (cf. [4] for instance) it follows from the compactness of the imbedding $J: \mathscr{H}_1 \subset \mathscr{H}_0$ and from Theorem 3 that, for every $a \in]0, 1]$, 1° $\mathscr{H}_a$ is compactly imbedded in $\mathscr{H}_0$, 2° $T\mathscr{H}_a \subset \mathscr{H}_a$, and 3° $T|_{\mathscr{H}_a}$ is compact in the norm $|.|_a$. We can formulate the following.

THEOREM 4. *Let $\mathscr{H}$ be a Hilbert space and $A \in B(\mathscr{H})$ of class (Q). Then there exist Hilbert subspaces $\mathscr{K}$ for A with compact imbedding into $\mathscr{H}$, which can be chosen "arbitrarily close" to $\mathscr{H}$, and so that $A|_{\mathscr{K}}$ is compact.*

With regard to ordinary invariant subspaces of A in $\mathscr{H}$, we propose the following.

PROBLEM. *Let $\mathscr{H}$ and $A \in B(\mathscr{H})$ be as in Theorem 4. Find some $\mathscr{K}$ as indicated there and a maximal chain of invariant subspaces (or at least one invariant subspace) for the compact $A|_{\mathscr{K}}$, none of which is dense in $\mathscr{H}$.*

Added in proof. Some of the results of this paper have been announced in "Sous-espaces hilbertiens invariants pour un opérateur linéaire", C. R. Acad. Sci. Paris, Sér. A-B 272 (1971), pp. 251–253.

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On reflexivity and summability

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Abstract. We construct a reflexive Banach space containing a weak null sequence such that no subsequence has strongly convergent (C, 1) means.

Let E be a Banach space. We shall say that E has property (BS) if each bounded sequence in E possesses a subsequence whose $(C, 1)$ means converge strongly.

Banach and Saks [1] proved that $L_p(0, 1)$ and l_p have (BS) for $1 < p < \infty$, while Schreier [7] showed that $C[0, 1]$ does not. Kakutani [3] proved that every uniformly convex space has (BS). Nishiura and Waterman proved that every (BS) space is reflexive, and showed, in the other direction, that for each bounded sequence in a reflexive space there is some positive regular summability method T and a subsequence whose T-means converge strongly. This led Sakai [6] to ask if there exist reflexive spaces which are not (BS). Klee [4] exhibited certain non-(BS) spaces, but Waterman, Ito, Barber, and Ratti [8] showed later that these are also non-reflexive.

The following construction provides an affirmative answer to Sakai's question. Denote by γ a finite non-empty set of positive integers such that the cardinality of γ is $\leqslant$ the smallest element of γ. Let Γ be the set of all such γ. Write $\gamma < \gamma'$ if the largest element of γ is $<$ the smallest of γ'. For $\gamma \in \Gamma$ and $x = \{x_i\}_{i=1}^{\infty}$ a sequence of real numbers, define

$$\sigma(x, \gamma) = \sum_{i \in \gamma} |x_i|.$$

For $\{\gamma_k\}$ a sequence in Γ with $\gamma_k < \gamma_{k+1} (k \geqslant 1)$ define

$$\sigma(x, \{\gamma_k\}) = \Big(\sum_{k=1}^{\infty} \sigma(x, \gamma_k)^2\Big)^{1/2} \tag{1}$$

and define

$$\|x\| = \sup \sigma(x, \{\gamma_k\}),$$

where the sup is taken over all such sequences $\{\gamma_k\}$.