$\mathscr{H}_0$ and $\mathscr{H}_1$. By general interpolation theory (cf. [4] for instance) it follows from the compactness of the imbedding $J\colon \mathscr{H}_1 \subset \mathscr{H}_0$ and from Theorem 3 that, for every $\alpha \in]0,1]$, 1° $\mathscr{H}_\alpha$ is compactly imbedded in $\mathscr{H}_0$, 2° $T\mathscr{H}_\alpha \subset \mathscr{H}_\alpha$, and 3° $T|_{\mathscr{H}_\alpha}$ is compact in the norm $|.|_\alpha$. We can formulate the following.

THEOREM 4. *Let $\mathscr{H}$ be a Hilbert space and $A \in B(\mathscr{H})$ of class (Q). Then there exist Hilbert subspaces $\mathscr{K}$ for A with compact imbedding into $\mathscr{H}$, which can be chosen "arbitrarily close" to $\mathscr{H}$, and so that $A|_{\mathscr{K}}$ is compact.*

With regard to ordinary invariant subspaces of A in $\mathscr{H}$, we propose the following.

PROBLEM. *Let $\mathscr{H}$ and $A \in B(\mathscr{H})$ be as in Theorem 4. Find some $\mathscr{K}$ as indicated there and a maximal chain of invariant subspaces (or at least one invariant subspace) for the compact $A|_{\mathscr{K}}$, none of which is dense in $\mathscr{H}$.*

Added in proof. Some of the results of this paper have been announced in "Sous-espaces hilbertiens invariants pour un opérateur linéaire", C. R. Acad. Sci. Paris, Sér. A-B 272 (1971), pp. 251–253.

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On reflexivity and summability

by

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Abstract. We construct a reflexive Banach space containing a weak null sequence such that no subsequence has strongly convergent (C, 1) means.

Let E be a Banach space. We shall say that E has property (BS) if each bounded sequence in E possesses a subsequence whose $(C,1)$ means converge strongly.

Banach and Saks [1] proved that $L_p(0,1)$ and l_p have (BS) for $1<p<\infty$, while Schreier [7] showed that $C[0,1]$ does not. Kakutani [3] proved that every uniformly convex space has (BS). Nishiura and Waterman proved that every (BS) space is reflexive, and showed, in the other direction, that for each bounded sequence in a reflexive space there is some positive regular summability method T and a subsequence whose T-means converge strongly. This led Sakai [6] to ask if there exist reflexive spaces which are not (BS). Klee [4] exhibited certain non-(BS) spaces, but Waterman, Ito, Barber, and Ratti [8] showed later that these are also non-reflexive.

The following construction provides an affirmative answer to Sakai's question. Denote by γ a finite non-empty set of positive integers such that the cardinality of γ is $\leqslant$ the smallest element of γ. Let Γ be the set of all such γ. Write $\gamma<\gamma'$ if the largest element of γ is $<$ the smallest of γ'. For $\gamma \in \Gamma$ and $x=\{x_i\}_{i=1}^\infty$ a sequence of real numbers, define

$$\sigma(x,\gamma)=\sum_{i\in\gamma}|x_i|.$$

For $\{\gamma_k\}$ a sequence in Γ with $\gamma_k<\gamma_{k+1}$ $(k\geqslant 1)$ define

$$\sigma(x,\{\gamma_k\})=\Big(\sum_{k=1}^{\infty}\sigma(x,\gamma_k)^2\Big)^{1/2} \tag{1}$$

and define

$$\|x\|=\sup\sigma(x,\{\gamma_k\}),$$

where the sup is taken over all such sequences $\{\gamma_k\}$.

Let E be the set of all x with norm $\|x\| < \infty$. It can be shown in the usual way that E is a Banach space. Let $e_n (n = 1, 2, \ldots)$ be the sequence with $(e_n)_i = 1$ for $i = n$, $(e_n)_i = 0$ for $i \neq n$. Then $\|e_n\| = 1$. We claim that

(2) *$\{e_n\}$ is a boundedly complete shrinking basis for E.*

If we assume (2) then, by a theorem of James [2, p. 71], E is reflexive. Moreover, $\{e_n\}$ is a weak null sequence, since it is a shrinking basis. Let r be a strictly increasing function from the positive integers into themselves. If the strong limit of

$$s_n = n^{-1} \sum_{i=1}^{n} e_{r(i)}$$

exists, it must be zero. This is impossible, since the choice $\gamma = \{r(n+1), r(n+2), \ldots, r(2n)\}$ shows that $\|s_{2n}\| > 1/2$ for all n. We conclude that E does not have property (BS).

Proof of (2). We first show that $\{e_n\}$ is a (Schauder) basis. For $x \in E$ write $T^n x = x - \sum_{i=1}^{n} x_i e_i$. If $\{e_n\}$ is not a basis there exists $x \in E$ with $\|T^n x\| > 1$ for all n. In particular, $\|x\| > 1$, so we can find $\gamma_1 < \gamma_2 < \ldots < \gamma_{p(1)}$ in Γ with

$$\sum_{k=1}^{p(1)} \sigma(x, \gamma_k)^2 > 1.$$

Let m be larger than the largest element of $\gamma_{p(1)}$. Since $\|T^m x\| > 1$, we can find $\gamma_{p(1)+1} < \gamma_{p(1)+2} < \ldots < \gamma_{p(2)}$ with $\gamma_{p(1)} < \gamma_{p(1)+1}$ and

$$\sum_{k=p(1)+1}^{p(2)} \sigma(x, \gamma_k)^2 = \sum_{k=p(1)+1}^{p(2)} \sigma(T^m x, \gamma_k)^2 > 1.$$

Hence

$$\|x\|^2 \geqslant \sum_{k=1}^{p(2)} \sigma(x, \gamma_k)^2 > 2.$$

Continuing this procedure, we deduce $\|x\| = \infty$.

Assume next that $x = \{x_i\}_{i=1}^{\infty}$ is a real sequence with

$$\sup_n \Big\|\sum_{i=1}^{n} x_i e_i\Big\| < \infty.$$

An easy argument shows that $x \in E$. Hence

$$\sum_{i=1}^{\infty} x_i e_i = x,$$

since $\{e_n\}$ is a basis. In particular, the series converges. Thus $\{e_n\}$ is boundedly complete.

It remains to be shown that $\{e_n\}$ is a shrinking basis. This means that

(3) $$\sup_{\|x\| \leqslant 1} \Big| u\Big(\sum_{i=n}^{\infty} x_i e_i\Big)\Big| \to 0 \qquad (n \to \infty)$$

for each $u \in E^*$. Assume that (3) is false. Then there exists $\delta > 0$, a strictly increasing sequence $\{p(m)\}_{m=1}^{\infty}$ of positive integers with $p(1) = 1$, and a sequence $\{x^{(m)}\}$ in E such that $\|x^{(m)}\| \leqslant 1$, $u(x^{(m)}) \geqslant \delta$ and $x_i^{(m)} = 0$ unless $p(m) \leqslant i < p(m+1)$. Put $Q(0) = 1$, and then define

$$Q(n) = Q(n-1) + p\big(Q(n-1)\big) \qquad (n = 1, 2, \ldots).$$

Define

$$w^{(n)} = \frac{1}{n} \frac{1}{Q(n) - Q(n-1)} \sum_{m=Q(n-1)}^{Q(n)-1} x^{(m)}$$

and let x be the sequence defined by

$$x_i = w_i^{(n)} \qquad \Big(p\big(Q(n-1)\big) \leqslant i < p\big(Q(n)\big)\Big).$$

Our proof is complete if we can show that $x \in E$, for then $x = \sum_{n=1}^{\infty} w^{(n)}$, with convergence in E, and this is incompatible with $u(w^{(n)}) \geqslant n^{-1}\delta$.

Let $\{\gamma_k\}$ be a sequence as in (1). For $n = 1, 2, \ldots$ let

$$A(n) = \{k\colon \text{smallest element of } \gamma_k \text{ is in } [p\big(Q(n-1)\big),\ p\big(Q(n)\big)]\}$$

and let $\mu(n)$ be the largest element of $A(n)$. Since $\sigma(x, \gamma_k) = \sigma(w^{(n)}, \gamma_k)$ for $k \in A(n)$, $k < \mu(n)$, we have

(4) $$\sum_{k \in A(n)} \sigma(x, \gamma_k)^2 \leqslant \|w^{(n)}\|^2 + \sigma(x, \gamma_{\mu(n)})^2.$$

Now $\|w^{(n)}\| \leqslant n^{-1}$. To estimate the other term on the right, write $\gamma_{\mu(n)} = \gamma' \cup \gamma''$, where $\gamma' = \gamma \cap \big(0, p\big(Q(n)\big)\big)$, $\gamma = \gamma'' - \gamma'$. Then $\sigma(x, \gamma') = \sigma(w^{(n)}, \gamma') \leqslant n^{-1}$. Moreover, $\gamma_{\mu(n)} \in \Gamma$, so $\sigma(x, \gamma'')$ is the sum of less than $p\big(Q(n)\big)$ terms, each of which has the form $aN^{-1}[Q(N) - Q(N-1)]^{-1}$, where $0 \leqslant a \leqslant 1$ and $N \geqslant n+1$. Since $Q(N) - Q(N-1) = p\big(Q(N-1)\big)$, and p is increasing, each term must be less than $n^{-1}[p\big(Q(n)\big)]^{-1}$, hence $\sigma(x, \gamma'') < n^{-1}$, hence

$$\sigma(x, \gamma_{\mu(n)})^2 = [\sigma(x, \gamma') + \sigma(x, \gamma'')]^2 < 4n^{-2}$$

hence the sum in (4) is $< 5n^{-2}$. Thus

$$\sigma(x, \{\gamma_k\})^2 = \sum_{n=1}^{\infty} \sum_{k \in A(n)} \sigma(x, \gamma_k)^2$$

has a bound independent of $\{\gamma_k\}$, and thus $x \in E$.

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About the space $\cap\, l_p$, $p > 0$.

by

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Abstract. We give a few properties of the space of sequences (a_n) such that $\sum |a_n|^p$ converges for all $p > 0$. The fact that the algebra of continuous linear transformations of this space has a natural, locally psudo-convex, locally multiplicatively convex, Fréchet topology is rather unexpected. This space also provides a negative answer to a question of W. Żelazko.

l_{+0} is the space of sequences $(a_n)_{n\in N}$ such that $\sum |a_n|^p = \nu_p(a)$ is finite for all positive p, with the Fréchet locally pseudo-convex topology determined by the pseudo norms ν_p. The reader may find the following observations about this space amusing.

The elements of l_{+0} are the sequences whose decreasing rearrangements belong to the space s of rapidly decreasing sequence. If we equip s with the usual topology determined by the norms $\sup n^k |a_n|$, the identity mapping $s \to l_{+0}$ is continuous. Permutations of N induce on l_{+0} an equicontinuous family of linear transformations. A translation invariant topology $\mathscr{T}$, on l_{+0} is weaker than the given one if it induces on s a weaker topology than its usual one, and if permutations of N induce on l_{+0} a $\mathscr{T}_1$-equicontinuous system of transformations at the origin.

These facts are either trivial, well known, or follow from the observation that $|a'_n| < \varepsilon n^{-1/p}$ if (a'_n) is the decreasing rearrangement of (a_n) and $\sum |a_n|^p < \varepsilon^p$.

Let $T\colon l_{+0} \to l_{+0}$ be a continuous linear transformation. Let B_p be the set of sequences $(a_n) \in l_{+0}$ such that $\nu_p\big((a_n)\big) \leqslant 1$. Then B_p is closed, absolutely p-convex, and a neighbourhood of the origin. $\overset{-1}{T}(B_p)$ is then also a closed, absolutely p-convex neighbourhood of the origin in l_{+0}. Being a neighbourhood of the origin, it contains $\varepsilon B_{p'}$ for some $\varepsilon > 0$, $p' > 0$. Further, the closed, absolutely p-convex hull of $B_{p'}$ is B_p when $p' < p$, so that $\overset{-1}{T} B_p \supseteq \varepsilon B_p$.

In other words, T extends to a continuous linear transformation of l_p, for all p, $0 < p \leqslant 1$. We can define $\tilde{\nu}_p(T)$ by

$$\tilde{\nu}_p(T) = \sup\{\nu_p(Tx) \mid \nu_p(x) \leqslant 1\}.$$