$\tilde{\nu}_p$ is a submultiplicative $p$-norm on $\mathscr{L}(l_{+0})$. With these $p$-norms, the algebra of continuous linear transformations of $l_{+0}$ is in a natural way a locally pseudo-convex, locally multiplicatively convex Fréchet algebra.

The following is an observation of S. Rolewicz. W. Żelazko has proved (unpublished) that $E$ is a normed space if $E$ is locally convex and if there is a topology on $\mathscr{L}(E)$, the algebra of continuous linear operators on $E$ which makes substitution $(u, e) \to u(e)$, $\mathscr{L}(E) \times E \to E$ continuous. Żelazko's result does not extend to the locally pseudo-convex case, nor even to the locally $p$-convex case. The space $l_{+0}$ described above is a locally pseudo-convex counter-example. And the considerations above apply clearly to the space $l_{p+0} = \cap_{p'>p} l_p$ with its obvious Fréchet topology. An algebra topology is defined in this way on $\mathscr{L}(l_{p+0})$. Substitution is again a continuous operation $\mathscr{L}(l_{p+0}) \times l_{p+0} \to l_{p+0}$. But $l_{p+0}$ is locally $p$-convex and not locally bounded.

The last result is trivial. We have the inclusion $l_{+0} \subseteq l_1$, the identity $i\colon l_{+0} \to l_1$ is continuous. A linear mapping $T\colon l_{+0} \to l_{+0}$ is continuous if $i \circ T\colon l_{+0} \to l_1$ is. This is clear, the graph of $T$ is closed in $l_{+0} \times l_1$ and a fortiori in $l_{+0} \times l_{+0}$.

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# Decompositions of non-contractive operator-valued representations of Banach algebras

by

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**Abstract.** The present paper deals with some decompositions of non-contractive operator-valued representations of Banach algebras. These decompositions are closely related to the abstract F. and M. Riesz property. An examination of the Boolean character of this property is basic for our purposes. This, when combined with the Sz.-Nagy–Dixmier theorem concerning similarity of certain Boolean algebras of projections shows that the representation in question is similar to a suitable orthogonally decomposed representation.

Let $T$ be the Hilbert space representation of a function algebra $A$. There are results of Sarason [14] and of Mlak [7], [8] that to every Gleason part of $A$ or intersection of peak sets of $A$ there corresponds a projection which commutes with $T$. This projection is orthogonal for contractive $T$. In this case a full decomposition of $T$ with respect to the totality of all Gleason parts or to the Bishop decomposition of $A$ is available.

In both cases an essential role is played by the F. and M. Riesz property. The point is that this property in an abstract form [13] gives rise to a homomorphism of a certain Boolean algebra of projections in the dual space onto a Boolean algebra of projections commuting with $T$. It seems that this is one of the real reasons why such decompositions as in [7], [8], [14] are available.

Althought our theory concerns representations of general non-commutative algebras, the examples of applications we give in the present paper are commutative. Non-commutative cases will be treated elsewhere.

**1.** Let $B$ be a (not necessarily commutative) Banach algebra with the unit 1. The norm of $u \in B$ is denoted by $\|u\|$. $B^*$ is the dual of $B$. For $u \in B$ and $\mu \in B^*$ we write $\langle \mu, u\rangle$ for $\mu(u)$. $I$ stands for the identity operator in $B^*$.

Let $A$ be a closed subalgebra of $B$ and assume $1 \in A$. If $\langle \mu, u\rangle = 0$ (for all $u \in A$) for $\mu \in B^*$ then we write $\mu \perp A$. For $v \in B$ and $\mu \in B^*$ we define $v\mu$ and $\mu v$ as the elements of $B^*$ given by the formulae: $\langle v\mu, u\rangle = \langle \mu, vu\rangle$, $\langle \mu v, u\rangle = \langle \mu, uv\rangle$, $u \in B$.

Let $Q$ be a bounded projection in $B^*$. We say that $Q$ has the property $R$ (with respect to the subalgebra $A$) if:

(1.1) $$\mu \perp A \quad \text{implies} \quad Q\mu \perp A,$$

(1.2) $$Q(\mu v) = (Q\mu)v, \quad Q(v\mu) = v(Q\mu)$$

for $v \in B$ and $\mu \in B^*$.

Notice that (1.1) and (1.2) yield the following conditions:

(1.3) $$\mu_1 - \mu_2 v \perp A \quad \text{implies } Q\mu_1 - (Q\mu_2)v \perp A,$$

(1.4) $$\mu_1 - v\mu_2 \perp A \quad \text{implies } Q\mu_1 - v(Q\mu_2) \perp A.$$

In what follows only the conditions (1.3) and (1.4) will be needed.

The following is easy to verify by a direct calculation.

PROPOSITION 1.1. *Suppose that $Q, Q_1$ and $Q_2$ have the property $R$ and $Q_1Q_2$ is a projection. Then also $I - Q$ and $Q_1Q_2$ have the property $R$.*

PROPOSITION 1.2. *Let $\mathcal{Q}$ be a set of mutually commuting projections which have the property $R$. Then every projection in the Boolean algebra of projections $\mathcal{B}(\mathcal{Q})$ generated by $\mathcal{Q}$ has the property $R$.* (1)

Proof. According to Proposition 1.1 the projections of the form $Q_{a_1} \wedge \ldots \wedge Q_{a_n} \wedge (I - Q_{\beta_1}) \wedge \ldots \wedge (I - Q_{\beta_m})$ ($Q_{a_i}, Q_{\beta_j}$ belong to $\mathcal{Q}$) has the property $R$. Every $Q \in \mathcal{B}(Q)$ can be represented as a Boolean join of projections of this form (see [12], p. 12).

It follows from the de Morgan formulae and Proposition 1.1 that the Boolean join of projections having the property $R$ also has this property.

One verifies easily the following

PROPOSITION 1.3. *Let $\mathcal{Q}$ be a set of projections having the property $R$. Then every projection in the strong operator closure of $\mathcal{Q}$ has this property.*

**2.** Let $H$ be a complex Hilbert space with the inner product $(x, y)$ $x, y \in H$ and the norm $|x|$. $L(H)$ stands for the algebra of all bounded linear operators in $H$. $|V|$ is the norm, $V^*$ the adjoint of $V \in L(H)$. The identity operator in $H$ is also denoted by $I$. In context, this notation will cause no confusion.

Let $T\colon A \to L(H)$ be a bounded linear map. It is a trivial consequence of the Hahn-Banach extention theorem that there are functionals $\mu_{x,y} \in B^*$ $x, y \in H$, called *elementary functionals* of $T$, such that

(2.1) $$\big(T(u)x, y\big) = \langle \mu_{x,y}, u\rangle, \quad x, y \in H, \; u \in A,$$

(2.2) $$\|\mu_{x,y}\| \leqslant \|T\| \, |x| \, |y|, \quad x, y \in H.$$

(1) For commuting projections $E$, $E_1$ and $E_2$ in a Banach space the Boolean operations: the join $E_1 \vee E_2$, the meet $E_1 \wedge E_2$ and the complement of $E$ are defined respectively as $E_1 + E_2 - E_1E_2$, $E_1E_2$, and $I - E$.



If $T$ is an algebra homomorphism of $A$ in $L(H)$, then the following relations holds true:

(2.3) $$\mu_{x,y}v - \mu_{T(v)x,y} \perp A, \quad x, y \in H, \; v \in A,$$

(2.4) $$v\mu_{x,y} - \mu_{y,T(v)^*y} \perp A, \quad x, y \in H, \; v \in A.$$

It follows from the multiplicativity of $T$ that $T(1)$ is a projection, $T(u) = T(1)T(u) = T(u)T(1)$, and, consequently, the part $T(u)\big(I - T(1)\big)$ is trivial. Without any loss of generality we assume in all what follows that $T(1) = I$.

A bounded algebra homomorphism $T\colon A \to L(H)$ is caled *a representation of $A$ in $H$* if $T(1) = I$.

Suppose that the representation $T$ is a sum of two bounded algebra homomorphisms $T_1$ and $T_2$

(2.5) $$T = T_1 + T_2.$$

Since $I = T_1(1) + T_2(1)$, the projections $T_1(1)$ and $T_2(1)$ split the space $H$ into direct sum of the spaces $T_1(1)H$ and $T_2(1)H$. $T_1$ and $T_2$ are representations in these spaces respectively. Hence, (2.5) may be regarded as a direct sum decomposition.

Since $T(1) = I$, $\|T\| \geqslant 1$. We call $T$ *contractive* if $\|T\| = 1$.

Let $\{\mu_{x,y}\}$ and $\{\mu'_{x,y}\}$ be two systems of elemetary functionals of the bounded linear map $T$. Then $\mu_{x,y} - \mu'_{x,y} \perp A$ and if $Q$ is a projection which satisfies (1.1) (or (1.3) with $v = 1$) then $Q\mu_{x,y} - Q\mu'_{x,y} \perp A$. Now the following definition makes sense: A bounded linear map $T_Q\colon A \to L(H)$ is called the *$Q$ — part of $T$* if

$$\big(T_Q(u)x, y\big) = \langle Q\mu_{x,y}\, u\rangle, \quad u \in A, \; x, y \in H,$$

where $\{\mu_{x,y}\}$ stands for an arbitrary system of elementary functionals of $T$.

Let $S \in L(H)$ be an arbitrary operator with $S^{-1} \in L(H)$ and let $T$ be a bounded linear map of $A$ into $L(H)$. Define $T'$ as

(2.6) $$T'(u) = S^{-1}T(u)S, \; u \in A.$$

Let $\{\mu_{x,y}\}$ and $\{\mu'_{x,y}\}$ be systems of elementary functionals of $T$ and $T'$, respectively. Then, by (2.1), we have

$$\mu'_{x,y} - \mu_{Sx(S-1)^*_y} \perp A, \; \mu_{x,y} - \mu'_{S^{-1}x, S^*y} \perp A.$$

The following proposition is an immediate consequence of the above relations:

PROPOSITION 2.1. *If $T$ and $T'$ are related as in (2.6) and $T_Q, T'_Q$ are their $Q$-parts $\big(Q \cdot$ satisfies (1.1)$\big)$ then $T'_Q(u) = S^{-1}T_Q(u)$, $S$, $u \in A$.*

**3.** Suppose that $T$ is a bounded linear map of $A$ into $L(H)$ and $Q$ is a projection which satisfies (1.1). Let $\{\mu_{x,y}\}$ be a system of elementary functionals of $T$. It is a simple matter to show that (1.1) implies that, for a fixed $u \in A$, $\langle Q\mu_{x,y}\ u\rangle$ is a bilinear form in $x$ and $y$. Indeed, the equality $\big(T(u)x+y, z\big) = \big(T(u)x, z\big)+\big(T(u)y, z\big)$ implies that $\mu_{x+y,z}-\mu_{x,z}-\mu_{y,z} \perp A$. Hence, by (1.1) $Q\mu_{x+y,z}-Q\mu_{x,z}-Q\mu_{y,z} \perp A$ which means that $\langle Q\mu_{x,y}\ u\rangle$ is additive in $x$. Using the same arguments, one easily verifies that these functional are homogeneous in $x$ and antilinear in $y$. On the other hand, by (2.2)

$$\langle Q\mu_{x,y}, u\rangle \leqslant \|Q\|\,\|T\|\,|x|\,|y|. \tag{3.1}$$

Thus $\langle Q\mu_{x,y}\ u\rangle$ is a bounded bilinear form. It follows that there exists the unique $T_Q(u) \in L(H)$ such that

$$\big(T_Q(u)x, y\big) = \langle Q\mu_{x,y}, u\rangle, \qquad u \in A,\ x, y \in H.$$

The mapping $T_Q$: $u \to T_Q(u)$ is thus linear and, by (3.1), bounded. Thus we conclude that for every $Q$ satisfying (1.1) there exists a $Q$-part $T_Q$ of $T$.

The following lemma concerning $Q$-parts of a representation will be of a basic use for our purposes.

LEMMA 3.1. *Let $T$: $A \to L(H)$ be a representation. Let $Q_1$ and $Q_2$ be projections having the property $R$ and let $Q_2Q_1$ be a projection. Then*

$$T_1(u)T_2(v) = T_{12}(uv), \qquad u, v \in A \tag{3.2}$$

*where $T_1, T_2, T_{12}$ are the $Q_1, Q_2, Q_2Q_1$-parts of $T$, respectively.*

Proof. Note first that by Proposition 1.1 $Q_2Q_1$ has the property $R$. Since $\mu_{T(v)x,y}-\mu_{x,y}\ v \perp A$, (1.3) yields $Q_1\mu_{T(v)x,y}-(Q_1\mu_{x,y})v \perp A$. But $\langle Q_1\mu_{T(v)x,y}\ u\rangle = \big(T_1(u)T(v)x, y\big) = \langle \mu_{x,T_1(u)^*y}, v\rangle$. We thus get $\mu_{x,T_1(u)^*y}-u(Q_1\mu_{x,y}) \perp A$. Keeping now $u \in A$ fixed we get, by (1.4), $Q_2\mu_{x,T_1(u)^*y}-u(Q_2Q_1\mu_{x,y}) \perp A$ which simply means that $\big(T_1(u)T_2(v)x,y\big) = \big(T_{12}(uv)x,y\big)$, $x, y \in H$. Q.E.D.

We now can state the following

THEOREM 3.2. *Let $T$: $A \to L(H)$ be a representation and let $Q$ be a projection having the property $R$. Then:*

1° *there exists a $Q$-part $T_Q$ of $T$ and*

$$T_Q(u) = T_Q(1)T(u) = T(u)T_Q(1), \qquad u \in A; \tag{3.3}$$

2° *$T'_Q = I-T_Q$ is a $(I-Q)$-part of $T$ and*

$$T'_Q(u) = T'_Q(1)T(u) = T(u)T'_Q(1), \qquad u \in A;$$

3° *both $T_Q$ and $T'_Q$ are bounded algebra homomorphisms and the representation $T$ is a unique direct sum of its $Q$ and $(I-Q)$-parts.*



Proof. Applying (3.2) with $Q$ and $I$ in place of $Q_1$ and $Q_2$ we obtain (3.3). The projection $I-Q$ has the property $R$ and by the same arguments as above we infer that the assertion 2° holds true. If we put in (3.2) $Q_1 = Q_2 = Q$, we obtain the multiplicitivity of $T_Q$ and $T'_Q$. This completes the proof [2].

Notice that if $T$ is a representation and $T_Q$ its $Q$-part then, by (3.3), $T_Q(u)x = T(u)x$ if $x = T_Q(1)x$. It follows that $|T_Q(u)x| \leqslant |T(u)|\,|x| \leqslant \|T\|\,\|u\|\,|x|$. Consequently $\|T_Q(u)\| \leqslant \|T\|\,\|u\|$. Hence

$$\|T_Q\| \leqslant \|T\| \quad \text{and} \quad |T_Q(1)| \leqslant \|T\|. \tag{3.4}$$

**4.** We say that a bounded linear map $T$: $A \to L(H)$ is *$Q$-supported* if there is a system of elementary functionals $\{\mu_{x,y}\}$ of $T$ such that

$$\mu_{x,y} = Q\mu_{x,y} \qquad x, y \in H. \tag{4.1}$$

It is clear that if $T$ is $Q$-supported then it is identical with its $Q$-part. On the other hand, if $T = T_Q$ then for every elementary functional $\mu_{x,y}$ the functional $Q\mu_{x,y}$ satisfies (2.1) and (4.1). But $Q\mu_{x,y}$ need not be an elementary functional of $T$ because the condition (2.2) can fail unless $\|Q\| = 1$. However, if $T$ is a representation and $T = T_Q$ then $\|Q\mu_{x,y}\| \leqslant \|T_Q\|\,|x|\,|y|$ and, by (3.4), $Q\mu_{x,y}$ is an elementary functional of $T$. Thus $T$ is $Q$-supported.

We remark that it follows from (1.2) that the set

$$I_Q = \{u \in B\,|\,\langle Q\mu, u\rangle = 0,\ \mu \in B^*\} \tag{4.2}$$

is a two-sided closed ideal in $B$.

Indeed, suppose $u \in I_Q$ and $v \in B$. Then, by (1.2), $\langle Q\mu, uv\rangle = \langle (Q\mu)v, u\rangle = 0$, since $\mu v \in B^*$ and $u \in I_Q$. Hence $uv \in I_Q$.

Similarly we infer that $vu$ belongs to $I_Q$.

Moreover, if $T$ is $Q$-supported then $T(u) = 0$ for $u \in I_Q$. This establishes the following

PROPOSITION 4.1. *Suppose that $Q$ has the property $R$. If $T$: $A \to L(H)$ is a $Q$-supported linear map, then there exists a linear map $\tilde{T}$ (with the same norm as that of $T$) of the quotient algebra $A/I_Q$ into $L(H)$ such that $\tilde{T}(\tilde{u}) = T(u)$, where $\tilde{u}$ is the canonical image of $u$ in $A/I_Q$.*

We will now consider a stronger condition in place of (1.1), namely the following one

$$\textit{If } \mu \perp A \textit{ then } Q\mu = 0 \tag{4.3}$$

or, in the equivalent but more convenient form,

$$\mu_1-\mu_2 \perp A \textit{ impies } Q\mu_1 = Q\mu_2. \tag{4.4}$$

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[2] The arguments of this section work well if $H$ is replaced by a reflexive Banach space like in [13]. We take here the opportunity to point out that the last part of the proof of Theorem 1 of [13] needs some rather trivial correction.

THEOREM 4.2. *Suppose that the projection $Q$ satisfies* (4.3) *and* (1.2). *If $T\colon A \to L(H)$ is a $Q$-supported representation then there exists a unique $Q$-supported representation $T^{\text{ext}}\colon B \to L(H)$ such that $T^{\text{ext}}(u) = T(u)$, $u \in A$. Moreover, $T^{\text{ext}}$ has the same norm as that of $T$.*

Proof. Let $\{\mu_{x,y}\}$ be a system of elementary functionals of $T$ satisfying (4.1). The condition (4.4) proves that such $\mu_{x,y}$ are uniquely determined. Moreover, the condition (4.4) shows that $\langle \mu_{x,y}, u\rangle$ for every fixed $u \in B$ is a bilinear form in $x$ and $y$. This bilinear form is bounded by $\|T\|\,\|u\|$. Consequently there exists a unique $T^{\text{ext}}(u) \in L(H)$ such that

$$\langle \mu_{x,y}, u\rangle = \big(T^{\text{ext}}(u)x, y\big) \tag{4.5}$$

for $x, y \in H$ and $u \in B$. $T^{\text{ext}}$ is linear in $u$. The mapping $T^{\text{ext}}\colon u \to T^{\text{ext}}(u)$ is a unique $Q$-supported extention of $T$ and $\|T^{\text{ext}}\| = \|T\|$.

The proof that $T^{\text{ext}}$ is multiplicative goes as follows. Let us fix $u \in A$. Since $T$ is multiplicative, $\mu_{x,T(u)^*y} - u\mu_{x,y} \perp A$. By (4.4), (4.1) and (1.4), $\mu_{x,T(u)^*y} = u\mu_{x,y}$. This means, by (4.5), that the equality $T(u)T^{\text{ext}}(v) = T^{\text{ext}}(uv)$ holds true for $u \in A$ and $v \in B$. Keeping $v \in B$ fixed we infer by the last equality that $\mu_{T^{\text{ext}}(v)x,y} - \mu_{x,y}v \perp A$. Applying (4.4), (1.3) and (4.1) we infer that $\mu_{T^{\text{ext}}(v)x,y} = \mu_{x,y}v$ which simply means, by (4.5), that $T^{\text{ext}}(u)T^{\text{ext}}(v) = T^{\text{ext}}(uv)$ for $u, v \in B$. Q.E.D.

Note that by Proposition 4.1 $T^{\text{ext}}$ is in fact a representation of the quotient algebra $B/I_Q$ where $I_Q$ is defined by (4.2). Theorem 4.2 generalizes the results of [6] and [12].

**5.** Let $\mathcal{Q}$ be a set of mutually commuting projections in $B$ which have the property $R$. Recall that $\mathcal{B}(\mathcal{Q})$ denotes the Boolean algebra of projections generated by $\mathcal{Q}$. Since $Q \in \mathcal{B}(\mathcal{Q})$ share the property $R$ (Proposition 1.2), the $Q$-part of the represention $T$ is well defined for every $Q$ in $\mathcal{B}(\mathcal{Q})$. Let us define

$$P_Q = T_Q(1), \qquad Q \in \mathcal{B}(\mathcal{Q}) \tag{5.1}$$

The reformulation of Lemma 3.1 reads now as follows:

(*) *the mapping $Q \to P_Q\big(Q \in \mathcal{B}(\mathcal{Q})\big)$ is a Boolean algebra homomorphism of $\mathcal{B}(\mathcal{Q})$ onto the Boolean algebra $\mathcal{P}(\mathcal{Q}) = \{P_Q \mid Q \in \mathcal{B}(\mathcal{Q})\}$ of projections.*

Notice that $\mathcal{P}(\mathcal{Q})$ is generated by the set $\{P_Q \mid Q \in \mathcal{Q}\}$.

This is a result of Sz.-Nagy [10] as later developed by Dixmier in [2] that for every bounded Boolean algebra $\mathcal{B}$ of projections in the Hilbert space $H$ there exists an $S \in L(H)$ such that:

(a) $S^{-1}$ exists and belongs to $L(H)$

(b) $S^{-1}PS$ is an orthogonal projection for every $P \in \mathcal{B}$.

It follows from (3.4) that $|P_Q| \leqslant \|T\|$ for $Q \in \mathcal{B}(\mathcal{Q})$. Applying the above mentioned Sz.-Nagy-Dixmier theorem we get the following



THEOREM 5.1. *Let $T$ be a representation of $A$ in $H$ and let $\mathcal{Q}$ be a set of mutually commuting projection with the property $R$. Then there is an invertible operator $S$ in $H$ such that $S^{-1}P_QS$ is an orthogonal projection for every $Q \in \mathcal{B}(\mathcal{Q})$ $\big(P_Q$ is defined by (5.1)$\big)$.*

COROLLARY 5.2. *Assume that the assumptions of Theorem 5.1 hold true. Define now a representation $T'$ by $T'(u) = S^{-1}T(u)S$, $u \in A$, where $S$ is given by Theorem 5.1. Then, by Proposition 2.1, $T'_Q(u) = S^{-1}T_Q(u)S$, $u \in A$ and, in particular, $P'_Q = T'_Q(1) = S^{-1}P_QS$. Moreover, $P'_Q$ is an orthogonal projection.*

Suppose $\{Q_\alpha\}$ is an indexed set of projections in $\mathcal{B}(\mathcal{Q})$ such that

$$Q_\alpha Q_\beta = 0 \quad \text{if } \alpha \neq \beta. \tag{5.2}$$

We define

$$T_\alpha = T_{Q_\alpha}, \quad T'_\alpha = T'_{Q_\alpha}, \quad P_\alpha = P_{Q_\alpha}, \quad P'_\alpha = P'_{Q_\alpha}. \tag{5.3}$$

The condition (5.2) implies, by (*),

$$P'_\alpha P'_\beta = 0 \quad \text{if } \alpha \neq \beta. \tag{5.4}$$

Thus

$$P'_0 = I - \oplus P_\alpha \tag{5.5}$$

is an orthogonal projection and

$$P'_0 P_\alpha = 0 \quad \text{for every } \alpha. \tag{5.6}$$

A trivial verification shows that $T'(u)P'_0 = P'_0T'(u)$, $u \in A$. Thus $T'_0 = T'P'_0$ is a bounded algebra homomorphism. By (5.4) and (5.5) we have

$$T'(u) = \big(\oplus T'_\alpha(u)\big) \oplus T'_0(u), \qquad u \in A.$$

Coming back to the representation $T$ we get

THEOREM 5.3. *Let $\mathcal{Q}$ and $S$ be as in Theorem 5.1. Let $\{Q_\alpha\}$ be an indexed set of projections belonging to $\mathcal{B}(\mathcal{Q})$ which satisfy* (5.2). *Then*

$$S^{-1}TS = (\oplus S^{-1}T_\alpha S) \oplus T'_0,$$

*where $T_\alpha$ is the $Q_\alpha$ — part of $T$ and $T'_0$ is some representation.*

Notice that if $T$ is contractive then $S = I$.

**6.** The purpose of this section is the description of the representation $T'_0$ of Theorem 5.3 in terms of its elementary functionals. Suppose $\{Q_\alpha\}$ is contained in some complete Boolean algebra of projections having the property $R$. (3) Denote by $Q_0$ the projection $\bigwedge(I - Q_\alpha) = I - \bigvee Q_\alpha$.

(3) The Boolean algebra $\mathcal{B}$ of projections in the Banach space $X$ is called complete (cf. [1]) if for every subset $\{Q_\alpha\} \subset \mathcal{B}$ there exists a projection (denoted by $E$) with range $\bigwedge(Q_\alpha X)$ and null space $\cap\,(I - Q_\alpha)X$, and if this projection belongs to $B$. Then $I - \bigwedge(I - Q_\alpha)\ (= \bigwedge Q_\alpha)$ is in $\mathcal{B}$, $(\bigwedge Q_\alpha)X = \cap\, Q_\alpha X$ and $(I - \bigwedge Q_\alpha)X = \bigvee\big((I - Q_\alpha)X\big)$.

We shall show that $T'_0$ is a $Q_0$ — part of $T'$. Denote this part by $T_Q$ and set $P_{Q_0} = T_{Q_0}(1)$. Since $Q_0 Q_\alpha = 0$ for all $\alpha$, we have, by (*) and (5.6),

$$(6.1) \qquad P'_{Q_0} P'_0 = P'_{Q_0}(I - \oplus P_\alpha) = P'_{Q_0}.$$

Our assertion will follow if we show that

$$(6.2) \qquad P'_{Q_0} P'_0 = P'_0.$$

To this end we prove the following

(6.3) *If $Q_\alpha \mu \perp A$ for all $\alpha$ then $\vee Q_\alpha \mu \perp A$.*

Note that if $Q_{\alpha_i}\mu \perp A$ for a finite set of indices then $(\vee Q_{\alpha_i})\mu \perp A$. Let $\mu \in B^*$ be such that $Q_\alpha \mu \perp A$ for all $\alpha$. Since $Q'_0 = \vee Q_\alpha$ is a projection on $\vee (Q_\alpha B^*)$, for every $\varepsilon > 0$ there is $\bar{\mu} = \Sigma Q_{\alpha_i}\mu_i$ such that $\|Q'_0\mu - \bar{\mu}\| < \varepsilon$. Since (5.2) holds true, then $(\vee Q_{\alpha_i})\bar{\mu} = \bar{\mu}$ and $(\vee Q_{\alpha_i}) Q_0 = \vee Q_{\alpha_i}$. Keeping all the above in mind we can write for $u \in A$

$$\begin{aligned} |\langle Q'_0\mu, u\rangle| &= |\langle Q'_0\mu, u\rangle - \langle \vee Q_{\alpha_i}\mu, u\rangle| \\ &\leqslant |\langle Q'_0\mu, u\rangle - \langle\bar{\mu}, u\rangle| + |\langle\bar{\mu}, u\rangle - \langle(\vee Q_{\alpha_i})\mu, u\rangle| \\ &= |\langle Q'_0\mu - \bar{\mu}, u\rangle| + |\langle \vee Q_{\alpha_i}(Q'_0\mu - \bar{\mu}), u\rangle| \leqslant \varepsilon\|u\|(k+1), \end{aligned}$$

where $K$ is a bound for the complete Boolean algebra in question (this bound is finite by Theorem 2.1 of [1]). So $Q_0\mu \perp A$ which proves (6.3).

Since $P'_\alpha P'_0 = 0$ for all $\alpha$, we have $0 = \langle Q\alpha\mu_{P_0x,y}, u\rangle = \langle (I - Q_0)\mu_{P_0x,y}, u\rangle = \big((T(u) - T_{Q_0}(u))P'_0\, x, y\big)$. This implies (6.2) and together with (6.1) proves our assumption that $T'_0$ is the $Q_0$ — part of $T'$.

If $B^*$ is weakly sequentially complete and the Boolean algebra $\mathscr{B}(\mathscr{Q})$ is bounded, i.e. $\sup\{\|Q\| \,|\, Q \in \mathscr{B}(\mathscr{Q})\} < +\infty$, then according to Corrolary 2.10 of [1], the strong operator closure $\overline{\mathscr{B}(\mathscr{Q})}_s$ of $\mathscr{B}(\mathscr{Q})$ is a complete Boolean algebra of projections which have the property $R$ (cf. Proposition 1.3).

Using Proposition 2.1 we obtain the following:

THEOREM 6.1. *Let $\mathscr{Q}$, $S$, $\{Q_\alpha\}$ and $T_0$ be as in Theorem 5.3. Suppose $\mathscr{B}(\mathscr{Q})$ is a bounded Boolean algebra of projections and $B^*$ is weakly complete. Then $T'_0 = S^{-1}T_0 S$, where $T_0$ is the $\wedge(I - Q_\alpha)$ — part of the representation $T$.*

Remark 6.2. In the case when $B^*$ is weakly complete, one can consider in Section 5 just the Boolean algebra $\overline{\mathscr{B}(\mathscr{Q})}_s$ in place of $\mathscr{B}(\mathscr{Q})$. In particular, one may assume that $Q_\alpha \in \overline{\mathscr{B}(\mathscr{Q})}_s$.

**7.** In this section we will consider the case when $B = C(X) =$ the sup norm algebra of all continuous complex functions on the compact Hausdorff space $X$. Then (via the Riesz representation theorem) $B^* = M(X) =$ the space of all regular Borel measures on $X$ with total variation as the norm. The space $M(X)$ is weakly sequentialy complete



(see [3], Chapter 4). Let $A$ be a closed subalgebra of $C(X)$ containing constants and separating the points of $X$.

Let $G$ be a Gleason part of $A$. Denote by $\mu_G$(for $\mu \in B^*$) the $G$-continuous part of $\mu$ (for definition and related matters see [4]). Let $\{G_\alpha\}$ be an indexed set of all Gleason parts of $A$ ($G_\alpha \neq G_\beta$ if $\alpha \neq \beta$). Then [4]:

1° for every $\mu \in M(X)$ there exists a unique $\mu_0 \in M(X)$ called the completely singular part of $\mu$, and a sequence $\{\alpha_n\}$ such that $\mu = \Sigma\mu_{G_{\alpha_i}} + \mu_0$ in the norm of $M(X)$;

2° the projections $Q_\alpha$ and $Q_0$ defined by $Q_\alpha\mu = \mu_{Q_\alpha}$, $Q_0\mu = \mu_0$ satisfy (1.1);

3° $Q_\alpha Q_\beta = 0$ if $\alpha \neq \beta$, and, $Q_0 Q_\alpha = 0$ for all $\alpha$.

There is no trouble in checking that (1.2) holds true. Thus $Q_\alpha$ and $Q_0$ have the property $R$.

By a direct calculation we show that, by 3°, every $Q$ in $\mathscr{B}(\mathscr{Q})$ is of the form $Q\beta_1 + \ldots + Q\beta_k$ or $I - (Q\beta_1 + \ldots + Q\beta_k)$. One may deduce from 1° and 3° that $\|u\| = \|Q\beta_1 + \ldots + Q\beta_k\| + \|\sum_{\alpha_n \neq \beta_i} Q_{\alpha_n}\mu + \mu_0\|$. This implies that $\|Q_{\beta_1} + \ldots + Q_{\beta_k}\| \leqslant 1$ and $\|I - (Q_{\beta_1} + \ldots + Q_{\beta_k})\| = \|\sum_{\alpha \neq \beta_i} Q_\alpha + Q_0\| \leqslant 1$. Consequently $\mathscr{B}(\mathscr{Q})$ is bounded.

We are now in a position to apply Theorems 5.3 and 6.1 and just get the following

THEOREM 7.1. *Let $T\colon A \to L(H)$ be a representation. Then there exists an invertible $S \in L(H)$ such that*

$$S^{-1}TS = \big(\oplus(S^{-1}T_\alpha S)\big) \oplus (S^{-1}T_0 S),$$

*where $T_\alpha$ is the $G_\alpha$ — continuous part of $T$* (i.e. *the $Q_\alpha$ part of $T$*) *and $T_0$ is the completely singular part of $T$* (i.e. *the $Q_0$-part of $T$*).

Notice that, by 1°, Theorem 7.1 may be deduced immediately from Theorem 5.3 only. In particular, the fact that $M(X)$ is weakly complete is needless. In case when $T$ is contractive Theorem 7.1 reduces to Theorem 2.4 of [7].

We define the projection $Q_\alpha$ ($\alpha$ is a Borel subset of $X$) by the formulae: $Q_\alpha\mu = \mu_\alpha$, $\mu_\sigma(\sigma) = \mu(\alpha \cap \sigma)$ ($\sigma$ a Borel set), $\mu \in M(X)$. Let $\mathscr{A}$ be the totality of all intersections of peak sets of $A$. Set $\mathscr{Q} = \{Q_\alpha | \alpha \in \mathscr{A}\}$. Since $\|Q_\alpha\| = 1$, $\overline{\mathscr{B}(\mathscr{Q})}_s$ is a complete Boolean algebra of projections. Moreover, it is known (see, for instance, [5]) that $\mu \perp A$ implies $\mu_\alpha \perp A$ if $\alpha \in \mathscr{A}$. Thus $Q_\alpha \in \overline{\mathscr{B}(\mathscr{Q})}$ (Proposition 1.3) has the property $R$.

Let the family $\mathscr{A}_0$ of subsets of $X$ be the Bishop decomposition of $X$ relative to $A$. Then $\mathscr{A}_0 \subset \mathscr{A}$ [5]. Since $\alpha \cap \beta = \emptyset$ for $\alpha \neq \beta$ $\alpha, \beta \in \mathscr{A}_0$, we have $Q_\alpha Q_\beta = 0$.

Using Theorem 5.3 and 6.1 we get the following

THEOREM 7.2. *Let $T\colon A\to L(H)$ be a representation and let $\mathscr{A}_0$ be the Bishop decomposition of $X$ relative to $A$. Then there exists an invertible $S\in L(H)$ such that*

$$S^{-1}TS = \Big(\bigoplus_{a\in\mathscr{A}_0}(S^{-1}T_aS)\Big)\oplus(S^{-1}T_0S),$$

*where $T_a$ is the $a$-part of $T$* (i.e. *$Q_a$-part of $T$*) *and $T_0$ is the $\bigwedge_{a\in\mathscr{A}_0}(I-Q_a)$-part of $T$.*

Notice that $T_0$ has a system of elementary measures vanishing on every $a\in\mathscr{A}_0$.

Theorem 7.2 for T contractive was proved by Mlak in [8].

It is evident that the mapping $a\colon a\to Q_a$ is a Boolean homomorphism. The mapping $a$ is also injective. Indeed, $Q_a\mu_x=\mu_x\neq 0$ where $\mu_x$ is the point mass at $x\in a$. Hence, if $Q_a=0$, then $a=\emptyset$. Let $\mathscr{B}_\sigma(\mathscr{A})$ denote the $\sigma$-algebra of sets generated by $\mathscr{A}$. Since $a$ is a Boolean isomorphism between $\mathscr{B}_\sigma(\mathscr{A})$ and $a\big(\mathscr{B}_\sigma(\mathscr{A})\big)=\mathscr{B}$, $\mathscr{B}$ is a $\sigma$-complete Boolean algebra, and $\mathscr{B}(\mathscr{Q})\subset\mathscr{B}\subset\overline{\mathscr{B}(\mathscr{Q})_s}$ By Propositions 1.2 and 1.3, every $Q_a$ in $\mathscr{B}$ satisfies (1.1) and, consequently, every $a$ in $\mathscr{B}_\sigma(\mathscr{A})$ satisfies

$$\mu\perp A \quad\text{implies } \mu_a\perp A. \tag{7.1}$$

It is a result of Gliksberg ([5], Theorem 4.8) which says that every closed $a$ satisfying (7.1) belongs to $\mathscr{A}$. Denoting by $\mathscr{F}$ the familly of all closed subsets of $X$ we get finally

THEOREM 7.3. *$\mathscr{F}\cap\mathscr{B}_\sigma(\mathscr{A})=\mathscr{A}$.*

This statement includes that of [5], p. 435.

Let $X\subset C$ (the complex plane). $R(X)$ denotes the closure in $C(X)$ of restrictions to $X$ of the algebra of all rational functions with poles off $X$. The set $X$ is called a $K$-spectral set of $V\in L(H)$ if

$$|u(V)|\leqslant K\sup_x|u|$$

for every rational function $u$ with poles off $X$. Let $u_1$ be the function $u_1(z)=z$. If $X$ is a $K$-spectral set of $V\in L(H)$, then there exists a unique representation $T\colon R(X)\to L(H)$ such that $T(u_1)=V$ and $\|T\|\leqslant K$. Conversely, if $T\colon R(X)\to L(H)$ is a representation then $X$ is a $\|T\|$ — spectral set of $T(u_1)$.

Using the arguments of [9], in a much similar way as in Theorems 7.1 and 7.2 we may obtain the following results.

THEOREM 7.4. *Suppose that $X\subset C$ is a $K$-spectral set of $V\in L(H)$. Then there exists an invertible $S\in L(H)$ such that*

$$S^{-1}VS=\oplus(S^{-1}V_iS),$$



*where $\{G_i\}_{i>0}$ is the sequence of all non-peak point Gleason parts of $R(X)$:*

(i) *$V_i$ has $G_i$ as a $K$-spectral set for $i>0$,*

(ii) *the representation of $R(\overline{G_i})$ generated by $V_i (i>0)$ is $G_i$-continuous.*

(iii) *$V_0$ is normal with spectrum carried by $\partial X$ and $V_0=V_0'+V_0''$, where $V_0'$ is normal with completely singular spectral measure and $V_0''=\oplus z_jI_j$ ($I_j$ — the identity in suitable $H_j$) with $z_j$ ranging over the totality of all peak points of $R(X)$.*

Let $X=\bigcup_{k=0}^{\infty}a_k$ be the Bishop decomposition of $X$ relative to $R(X)$ such that $a_k (k>0)$ are peak sets with positive planar measure and $a_0$ is the union of all peak point maximal sets of antisymetry (for details we refer [5])

THEOREM 7.5. *Suppose that $X\subset C$ is a $K$-spectral set of $V\in L(H)$. Then there exists an invertible $S\in L(H)$ such that*

$$S^{-1}VS=\oplus(S^{-1}V_kS),$$

*where $V_k$ has $a_k$ as its $K$-spectral set for $k>0$ and $V_0$ is normal with spectral measure carried by $\partial X$.*

These results for 1-spectral sets are due to Mlak [9].

Our last result is based on the fact the if $A=B$ then every projection in $B^*$ satisfies (1.1). In this way we obtain almost gratuidously a theorem on decomposition of a spectral measure.

Recall that the mapping $F\colon\mathscr{B}(X)\to L(H)$ ($\mathscr{B}(X)$ — the algebra of all Borel subsets of $X$) is a spectral measure if (i) the mapping $F_{x,y}\colon$ $:\sigma\to\big(F(\sigma)x,y\big)$ $(x,y)\in H$ is a complex regular measure on $X$; (ii) $F(\varrho\cap\sigma)=F(\varrho)F(\sigma)$ $\varrho,\sigma\in\mathscr{B}(X)$; (iii) $|F(\sigma)|\leqslant K$ for every $\sigma$ and some $K$. It is well known that:

(S) to every spectral measure $F$ there corresponds a unique representation of $C(X)$ on $H$ which is given by

$$T(u)=\int_x u\,dF,\qquad u\in C(X). \tag{7.2}$$

and, conversely, to every representation $T\colon C(X)\to L(H)$ there corresponds a unique spectral measure such that (7.2) holds true.

If $T$ and $F$ are related as in (7.2) then $\|T\|\leqslant K$. Moreover, the projection $P$ commutes with $T$ if and only if it commutes with every $F(\sigma)$.

We say that the spectral measure $F$ is absolutely continuous (singular) with respect to the positive measure $m$, in symbols $F\ll m$ ($F\perp m$), if $F_{x,y}\ll m$ ($F_{x,y}\perp m$) for all $x,y\in H$.

Applying Theorems 5.3 and 6.1 and using (S) we get the following

Theorem 7.6. *Let $F$ be a spectral measure and $\{m_\iota\}$ a familly of mutually singular positive Borel measures on $X$. Then there exists an invertible $S \in L(H)$ such that*

$$S^{-1}FS = \oplus(S^{-1}F_\iota S)\oplus(S^{-1}F_0S),$$

*where $F_\iota \ll m_\iota$, $F_0 \perp m_\iota$ for all $\iota$ and the spectral measures $S^{-1}F_\iota S$ and $S^{-1}F_0S$ are self-adjoint.*

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