By Lemma 1, choose $g_i \in L'_{\varrho_{n_i}}$, finite a.e., such that

$$\varphi_i([f]_{n_i}) = \int f g_i \, d\mu,$$

for all "elements" f of $L_{\varrho_{n_i}}$. For every $f \in F$ we now have

$$\varphi(f) = \sum_{i=1}^{K} \varphi_i(v_{n_i}(f)) = \sum_{i=1}^{K} \varphi_i([f]_{n_i}) = \sum_{i=1}^{K} \int f g_i \, d\mu = \int f\Big(\sum_{i=1}^{K} g_i\Big) d\mu.$$

Hence $\varphi = \varphi_g$ where $g = \sum_{1}^{K} g_i \in F'$.

We conclude with a condition, different from that of Section 2, which is sufficient that F and $F^{\times}$ be in duality under the usual bilinear form $\langle f, g\rangle = \int f g \, d\mu$.

THEOREM 3. *If there exists a sequence which is ϱ_n-exhaustive for all n, then F is saturated and (F, F'), $(F, F^{\times})$ are dual systems under $\langle f, g\rangle$.*

Proof. As in the last proof, F and each L_{ϱ_n} are saturated. Since every ϱ_n has the Fatou null property, L'_{ϱ_n} is a total subspace of the metric dual $L^*_{\varrho_n}$ of L_{ϱ_n} ([4], Note V, Theorem 15.2). By Section 2, Lemma 3, Corollary, we know that for all $g \in F^{\times}$ (and hence for all $g \in F'$) with $g \neq 0$, there is some $f \in F$ with $\langle f, g\rangle \neq 0$. But if $0 \neq f \in F$ then $\varrho_n(f) \neq 0$ for some n, so that there exists $g \in L'_{\varrho_n}$ such that $\langle f, g\rangle \neq 0$. Moreover, ϱ'_n is a function norm ([4], Note IV, Theorem 9.7) so g is finite a.e. and hence belongs to F' and to $F^{\times}$.

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A Cantor–Lebesgue theorem for double trigonometric series

by

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Abstract. Let $\xi = (x, y)$ be points of the plane, $\nu = (m, n)$ – lattice points, and $\langle \nu \cdot \xi\rangle = mx + ny$. It is shown that given any set E of positive measure situated in the square $0 \leqslant x < 1$, $0 \leqslant y < 1$, there is a constant $A = A_E$ such that for any trigonometric polynomial $T(\xi)$ of the form $\sum_{|\nu| = R} c_\nu e^{2\pi i(\nu \cdot \xi)}$ we have

$$\sum |c_\nu|^2 \leqslant A \int_E |T(\xi)|^2 d\xi.$$

In particular, if an infinite series $\sum c_\nu e^{2\pi i(\nu, \xi)}$ converges circularly in a set of positive measure, then $\sum_{|\nu| = R} |c_\nu|^2 \to 0$ as $R \to \infty$.

1. Let $\xi = (x, y) \in R^2$ and let $\nu = (m, n)$ denote lattice points in R^2. Consider a double trigonometric series

$$\sum_{\nu} c_\nu e^{2\pi i \langle \nu \cdot \xi\rangle}, \tag{T}$$

where $\langle \nu \cdot \xi\rangle = mx + ny$, and its circular partial sums

$$T_R(\xi) = \sum_{|\nu| \leqslant R} c_\nu e^{2\pi i \langle \nu \cdot \xi\rangle}.$$

We shall also write

$$A_R(\xi) = \sum_{|\nu| = R} c_\nu e^{2\pi i \langle \nu \cdot \xi\rangle}.$$

Recently, R. L. Cooke proved the following result (see [1]).

THEOREM 1. *If $A_R(\xi) \to 0$ almost everywhere as $R \to \infty$ (and, in particular, if T converges almost everywhere), then $c_\nu \to 0$ as $|\nu| \to \infty$. More generally, we then have*

$$\sum_{|\nu| = R} |c_\nu|^2 \to 0 \qquad (R \to \infty). \tag{1.1}$$

In this note we prove a somewhat more general result.

THEOREM 2. *If $A_R(\xi) \to 0$ at each point ξ of a set of positive measure, we have* (1.1).

2.

LEMMA 1. *Let $E \subset R^2$, $0 < |E| < \infty$, and let*

$$\chi\hat{\ }(\nu) = \int_E e^{-2\pi i\langle\nu\cdot\xi\rangle}\,d\xi.$$

Then there is a strictly positive ε such that

$$|\chi\hat{\ }(\nu)| < |E| - \varepsilon \quad \textit{for all } \nu \neq 0.$$

Proof. Since $\chi\hat{\ }(\nu) \to 0$ as $|\nu| \to \infty$, it is enough to show that each $|\chi\hat{\ }(\nu)|$ is strictly less than $|E|$ if $\nu \neq 0$.

This is a consequence of the elementary fact that if, for any complex-valued f, we have $|\int_E f| = \int_E |f|$, and $f \neq 0$ on E, then $\arg f$ is constant, $\bmod 2\pi$, almost everywhere on E. Hence, in our case, if we had $|\chi\hat{\ }(\nu)| = |E|$ for some $\nu \neq 0$, $\langle\nu\cdot\xi\rangle$ would have to be constant, mod1, almost everywhere on E, almost all points of E would be situated on a finite or denumerable family of straight lines, and we would have $|E| = 0$, contrary to hypothesis.

LEMMA 2. *For any three distinct lattice points λ, μ, ν situated on a circumference of radius R we have*

$$|\lambda-\mu|\,|\mu-\nu|\,|\nu-\lambda| \geqslant 2R. \tag{2.1}$$

This is a corollary of the classical theorem of Elementary Geometry which asserts that if a triangle with sides a, b, c has area S, and if R is the radius of the circle circumscribed on the triangle, then

$$R = abc/4S. \tag{2.2}$$

For in our case a, b, c are the factors on the left in (2.1), and if we write $\lambda = l_1 + il_2$, $\mu = m_1 + im_2$, $\nu = n_1 + in_2$, then S is the absolute value of

$$\frac{1}{2}\begin{vmatrix} 1 & l_1 & l_2 \\ 1 & m_1 & m_2 \\ 1 & n_1 & n_2 \end{vmatrix},$$

and so $S \geqslant 1/2$. This leads to (2.1). Lemma 2, which is essential for the proof of Theorem 2, was first proved (in reply to a question put by the author) by A. Schinzel, in an elementary and purely analytical way. That it is a corollary of (2.2) was later pointed out by A. Pełczyński.

3. Passing to the proof of the theorem we may assume (by Egoroff's theorem) that T_R converges uniformly in E, and that E is contained in the square $0 \leqslant x < 1$, $0 \leqslant y < 1$. Then $\Delta_R(\xi) \to 0$ uniformly in E.

We have

$$\int_E |\Delta_R(\xi)|^2 d\xi = |E| \sum_{|\mu|=R} |c_\mu|^2 + \sum_{\substack{|\mu|=\nu=R \\ \mu\neq\nu}} c_\mu \bar{c}_\nu \chi\hat{\ }(\nu-\mu) = P + Q, \tag{3.1}$$

say.

Let $\Delta = \Delta_R$ be the set of all the differences $\nu - \mu \neq 0$, for $|\nu| = |\mu| = R$.

Let R_0 be so large that

$$\Big(2\sum_{|\lambda|>R_0} |\chi\hat{\ }(\lambda)|^2\Big)^{1/2} < \tfrac{1}{2}\varepsilon, \tag{3.2}$$

where ε is that of Lemma 1 corresponding to our set E. Write $\Delta = \Delta' \cup \Delta''$, where Δ' consists of the elements of Δ of modulus $\leqslant R_0$ and Δ'' is the remainder of Δ.

Correspondingly,

$$Q = Q' + Q''. \tag{3.3}$$

Clearly,

$$\begin{aligned} |Q''| &\leqslant \Big(\sum |c_\mu \bar{c}_\nu|^2\Big)^{1/2} \Big(\sum |\chi\hat{\ }(\nu-\mu)|^2\Big)^{1/2} \quad (|\mu| = |\nu| = R,\ |\mu-\nu| > R_0) \\ &\leqslant \sum_{|\lambda|=R} |c_\lambda|^2 \cdot \Big(\sum_{|\lambda|>R_0} 2|\chi\hat{\ }(\lambda)|^2\Big)^{1/2} \leqslant \tfrac{1}{2}\varepsilon \sum_{|\lambda|=R} |c_\lambda|^2, \end{aligned} \tag{3.4}$$

by (3.2). Here we used the fact that a circle can have at most two chords of prescribed length and direction.

Let $C(0, R)$ be the circumference of center 0 and radius R. The meaning of Lemma 2 is that if two lattice points on $C(0, R)$ are 'close' to each other, then any other lattice point on $C(0, R)$, should it exist, is necessarily 'distant' from those two.

Having fixed R_0 we take R so large that any pair (μ, ν) on $C(0, R)$ with $|\mu-\nu| \leqslant R_0$ is distant by more than R_0 from any other lattice point on $C(0, R)$. Hence the lattice points on $C(0, R)$ can be split into 'distant' pairs (μ, ν) with $|\mu-\nu| \leqslant R_0$. For each such pair (μ, ν), writing $\nu - \mu = \lambda$ we have, by Lemma 1,

$$\begin{aligned} &|c_\mu \bar{c}_\nu \chi\hat{\ }(\lambda) + c_\nu \bar{c}_\mu \chi\hat{\ }(-\lambda)| \\ &\qquad \leqslant \tfrac{1}{2}(|c_\mu|^2 + |c_\nu|^2)(|E|-\varepsilon)\cdot 2 = (|c_\mu|^2 + |c_\nu|^2)(|E|-\varepsilon). \end{aligned}$$

It follows that

$$|Q'| \leqslant \sum_{|\lambda|=R} |c_\lambda|^2 (|E|-\varepsilon). \tag{3.5}$$

Collecting the results (see (3.1), (3.3), (3.4), (3.5)) we obtain

$$\int_E |A_R(\xi)|^2 d\xi = P+Q'+Q''$$
$$\geqslant \sum_{|\lambda|=R} |c_\lambda|^2 \{|E|-\tfrac{1}{2}\varepsilon-(|E|-\varepsilon)\} = \tfrac{1}{2}\varepsilon \sum_{|\lambda|=R} |c_\lambda|^2.$$

Thus

$$\sum_{|\lambda|=R} |c_\lambda|^2 \leqslant 2\varepsilon^{-1} \int_E |A_R(\xi)|^2 d\xi, \tag{3.6}$$

and if $A_R(\xi)$ tends uniformly to 0 on E, $\sum_{|\lambda|=R} |c_\lambda|^2 \to 0$. This completes the proof of Theorem 2.

4. We conclude with a few observations.

a) The proof of Theorem 2 is essentially two-dimensional. Whether an analogue of Theorem 2 (or even only of Theorem 1) holds in higher dimensions remains an open problem.

b) Strictly speaking, the proof of Theorem 2 has little to do with the relation $A_R(\xi) \to 0$ $(\xi \in E, |E| > 0)$. Analyzing the proof (see, in particular, (3.6)) we see that it gives the following result.

THEOREM 3. *Given any set E of positive measure situated in the unit square $0 \leqslant x < 1$, $0 \leqslant y < 1$, we can find a positive number A_E such that*

$$\sum_{|\lambda|=R} |c_\lambda|^2 \leqslant A_E \int_E \Big| \sum_{|\lambda|=R} c_\lambda e^{2\pi i(\lambda\cdot\xi)} \Big|^2 d\xi. \tag{4.1}$$

That (4.1) holds for R sufficiently large, $R \geqslant R_E$, is implicit in the proof of Theorem 2, and for $R < R_E$ follows from the equivalence of norms in spaces of the same finite dimension.

c) One may ask for an estimate analogous to (4.1) for sums $\sum c_\nu e^{2\pi i\langle\nu\cdot\xi\rangle}$ extended over lattice points situated on some plane curve Γ.

The argument of Section 3, where $\Gamma = C(0, R)$, utilizes two properties of the circle: α) it has at most two chords of prescribed length and direction; β) given R_0, for any lattice point $\lambda \in C(0, R)$ there is at most one neighbor $\mu \in C(0, R)$ with $|\lambda-\mu| \leqslant R_0$, provided R is large enough (there may actually be such neighbors; take e.g. the points $(n, n+1)$ and $(n+1, n)$ on $C(0, R)$ with $R^2 = 2n^2+2n+1$).

Property α) was needed to estimate the term Q'' in (3.3) (see (3.4)), property β) — for Q'. As to α), it is certainly satisfied for any strictly convex curve Γ, but it is easily seen to be unnecessarily restrictive: if Γ has at most k chords of prescribed length and direction, the factor 2 in the next to last term of (3.4) can be replaced by k and the argument still works.

Property β) is a little more subtle and we do not propose to study it. The special case of an ellipse is both of independent interest and sufficient simplicity to be considered here and we limit ourselves to it. (Even simpler, though less interesting, is the case when Γ is a convex polygone whose sides make with the x-axis angles incommensurable with π; it is obvious that each side of Γ can contain at most one lattice point.)

Let E_{ab} denote an ellipse with semi-axes a, b, $a > b$, and center 0 (it is obvious that the latter condition is no restriction of generality); the direction of the axes is not specified. Let $C(0, a)$ be the circle circumscribed on E_{ab}. Let λ, μ, ν be three distinct lattice points on E_{ab}, and λ', μ', ν' their projections parallel to the minor axis out to the circumscribed circle (λ', μ', ν' need not be lattice points).

Let S and S' denote respectively the areas of the triangles $\lambda\mu\nu$ and $\lambda'\mu'\nu'$; thus $S \geqslant \frac{1}{2}$. In view of (2.2) we have the relation

$$|\mu'-\lambda'|\,|\mu'-\nu'|\,|\lambda'-\nu'| = 4S'a = 4\cdot\frac{a}{b} S\cdot a \geqslant 2\frac{a^2}{b}. \tag{4.2}$$

Since the passage from S to S' increases the sides by factors $< a/b$, (4.2) leads to the following analogue of (2.1):

$$|\mu-\lambda|\,|\mu-\lambda|\,|\lambda-\nu| \geqslant 2\frac{b^2}{a}. \tag{4.3}$$

If the distance of μ from both λ and ν does not exceed R_0, we deduce from (4.3) that

$$R_0 . R_0 . 2R_0 \geqslant 2\frac{b^2}{a},$$

that is,

$$b \leqslant R_0^{3/2} a^{1/2}.$$

It follows that if $b > R_0^{3/2}a^{1/2}$, then for any lattice point $\mu \in E_{ab}$ there can exist at most one lattice point $\lambda \in E_{ab}$ with $0 < |\lambda-\mu| \leqslant R_0$. Hence, as in the case of a circle, the lattice points on E_{ab} of distance $\leqslant R_0$ can be split into 'distant' pairs, the estimate for Q' holds and we arrive at the following generalization of Theorem 3.

THEOREM 4. *Let E_{ab} be an ellipse with center 0, semi-axes a, b, $a > b$, their direction arbitrary. Then for any set E of positive measure situated in the square $0 \leqslant x < 1$, $0 \leqslant y < 1$ we can find constants $A = A_E$, $K = K_E$ such that if*

$$b > Ka^{1/2}, \tag{4.4}$$

then

$$\sum_{\nu\in E_{ab}} |c_\nu|^2 \leqslant A \int_E \Big| \sum_{\nu\in E_{ab}} c_\nu e^{2\pi i\langle\nu\cdot\xi\rangle} \Big|^2 d\xi. \tag{4.5}$$

Observe that initially we obtain (4.5) under the additional condition that a is large, $a \geqslant L_E$, which later, as in the case of the circle, may be dropped.

Of course, under (4.4) the eccentricity of E_{ab} may tend to 1 as $a \to \infty$.

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