STUDIA MATHEMATICA, T. XLIII. (1972)

Two-norm spaces and decompositions of Banach spaces, I

by

P. K. SUBRAMANIAN (Missouri)

Abstract. The only known applications of *two-norm spaces*, a recent invention of the Polish school, is to summability theory. The aim of this paper is to apply two-norm space theory to Banach spaces with bases and decompositions. To every such Banach space we associate a two-norm space the study of the properties of which illustrates the structure of the Banach space. Corollaries df our results include those of Ruckle, Sanders, Singer and others.

Introduction. The theory of *two-norm spaces* is a relatively new invention of the Polish school. A two-norm space $X_s = (X, |\ |_1, |\ |_2)$ is defined to be a linear set X (over the real or complex field) with two norms $|\ |_1$ and $|\ |_2$. Closely related to the notion of a two-norm space X_s is a metric space (S, d), where S is the closed unit ball of the normed linear space $(X, |\ |_1)$ and d is the metric induced by the norm $|\ |_2$. Such metric spaces, when they are complete, have been called *Saks spaces* by Orlicz, after S. Saks who first used this idea in [11] and [12].

In [7] Orlicz showed the use of Saks spaces in the theory of summability. A. K. Snyder has used two-norm spaces to characterise conull FK spaces (see, for instance, [18], p. 941). Conway [4] gave a simple proof of Schur's theorem (sequential weak convergence in l implies norm convergence). Although he does not explicitly mention Saks spaces, his proof makes use of the closed unit ball of the space m with the metric d being induced by $|\{a_n\}|_2 = \sum\limits_n |a_n|/2^n$. Using Saks spaces, Rothman [9] has constructed a whole class of Banach spaces with this property.

In [1], [2] and [3] Alexiewicz and Semadeni developed an elegant theory of two-norm spaces but did not indicate any applications. Our aim in this paper is to show an application of their theory to Banach spaces with Schauder bases and decompositions. Sections 1 and 2 contain the necessary basic definitions and results pertaining to two-norm spaces and Schauder decompositions respectively. The *canonical two-norm space* is introduced in section 3 and the concept of *k-duals* in section 4. These two sections contain the main theorems of this paper. In section 5 we show that theorems of Ruckle [10], Sanders [13] and others can be obtained as corollaries of our results. We also prove duality theorems between shrinking and boundedly complete Schauder decompositions. In section 6 we use two-norm spaces to extend Singer's results [15] on basic constants.

1. Two-norm spaces. A *two-norm space* $X_s = (X, |\ |_1, |\ |_2)$ is a linear set X with two homogeneous norms $|\ |_1, |\ |_2$ although the homogeneity is not necessary in general. We shall also assume that X is over the reals $\mathbf{R}$, the extension to the case when the field of scalars is the complex numbers C being routine. We shall say X_s *satisfies* (n_0) if for every sequence $\{x_n\}$ of points in X_s with $|x_n|_1 \to 0$ we also have $|x_n|_2 \to 0$.

A sequence $\{x_n\}$ of points in X_s is said to be *γ-convergent* to x in X if both

$$\sup_n |x_n|_1 < \infty \quad \text{and} \quad \lim_n |x_n - x|_2 = 0.$$

A *γ-Cauchy* sequence is defined in an analogous manner and X_s is called *γ-complete* if it is sequentially complete for the convergence (γ). A *γ-linear functional* f on X_s is a real valued functional such that (i) $f(ax+by) = af(x)+bf(y)$ for all a, b in $\mathbf{R}$, x, y in X and (ii) $x_n \xrightarrow{\gamma} x \Rightarrow f(x_n) \to f(x)$. The set of all γ-linear functionals on X_s is denoted by $A(X_s)$.

The space X_s is called *quasi-normal* if there is a constant $C(X_s) \geqslant 1$ such that for any sequence $\{x_n\}$ of points in X_s,

$$x_n \xrightarrow{\gamma} x \Rightarrow |x|_1 \leqslant C(X_s) \liminf_n |x_n|_1.$$

The smallest constant $C(X_s)$ is called the *constant of quasi-normality* of X_s. When $C(X_s)$ is 1, X_s is called *normal.*

Let $(X_j^*, |\ |_j^*) = (X, |\ |_j)^*$, $j = 1, 2$. It is known [8], p. 57, that whenever X_s satisfies (n_0), $X_2^* \subseteq A(X_s)$ and that $A(X_s)$ is a closed subspace of X_1^*. If X_s is quasi-normal as well, it is shown in [3], p. 118, that $(X_2^*, |\ |_1^*)$ is dense in $(A(X_s), |\ |_1^*)$. The equality $A(X_s) = X_1^*$ may occur in non-trivial cases. In this case X_s is called *saturated.* An example is the space $c_{0s} = (c_0, \sup_n |\{a_n\}|, \sum_n |a_n|/2^n)$, c_0 being the space of null sequences. Here $A(c_{0s}) = c_0^* = l$, where l is the space of absolutely summable sequences.

Suppose X_s is quasi-normal that it satisfies (n_0). Then the canonical embedding $J\colon X \to A(X_s)^*$, defined for x in X by $J(x)f = f(x)$ for all $f \in A(X_s)$, is a linear isomorphism (that is, a topological isomorphism) [3], p. 119. If X_s is normal, J is also an isometry [2], p. 279. When ever X_s is normal and J is onto $A(X_s)^*$, X_s is called *γ-reflexive.*

2. Schauder decompositions. A sequence $\{M_i\}$ of non-trivial subspaces of a Banach space X is (*weak*) *decomposition* of X if for each x in X there exists a unique sequence $\{x_i\}$ such that $x_i \in M_i$ for each i and $x = \lim_n \sum_{i \leqslant n} x_i$ in X (in the weak topology of X respectively). If all subspaces M_i are closed, then $\{M_i\}$ is called a (*weak*) *Schauder decomposition* of X. When each M_i is of dimension 1, the decomposition reduces to the familiar notion of a basis for X.

Associated with a decomposition $\{M_i\}$ of a Fréchet space X is a sequence $\{P_i\}$ of projection operators, $P_i\colon X \to M_i$ such that $P_i(x) = x_i$. It is known [10], II. 2, that if $\{M_i\}$ is a Schauder decomposition of X, each P_i is continuous. We shall denote the Schauder decomposition $\{M_i\}$ of a Fréchet space X by $\{M_i, P_i\}$, $\{X, P_i\}$ or $\{X, M_i\}$. We shall sometimes write $X = \oplus \sum_i M_i$ and refer to M_i as the *i-th component* of X.

Let $(X, |\ |_1)$ be a Banach space (referred to as B-space hereafter) with a Schauder decomposition $\{M_i\}$. Then there exists a constant $K_1 \geqslant 1$ such that

$$\Big|\sum_{i \leqslant n} x_i\Big|_1 \leqslant K_1 \Big|\sum_{i \leqslant m} x_i\Big|_1$$

for all n, m with $n \leqslant m$ and for all sequences $\{x_i\}$ with $x_i \in M_i$ [6], p. 93. The smallest constant K_1 will be called the *decomposition constant* of $\{M_i\}$. When $K_1 = 1$ we shall say $\{M_i\}$ is a *monotone* decomposition. For $x = \sum_k x_k \in X$ let us write $s(n, x) = \sum_{i \leqslant n} x_i$. For a monotone decomposition then, $|x|_1 = \sup_n |s(n, x)|_1$.

Although a given decomposition $\{M_k\}$ for a B-space $(X, |\ |_1)$ may not be monotone, the following theorem is of interest. The proof is similar to the case of a B-space with a basis and is, therefore, omitted (see [17], p. 207).

THEOREM 2.1. *Let $(X, |\ |_1)$ be a B-space with a Schauder decomposition $\{M_k\}$. Then $|x|_0 = \sup_n |s(n, x)|_1$ is a norm on X equivalent to $|\ |_1$, and with respect to which $\{M_k\}$ is monotone.*

3. The canonical two-norm space. Let $(X, |\ |_1)$ be a B-space with a Schauder decomposition $\{M_k\}$. For $x = \sum_k x_k \in X$, $|x|_2 = \sum_k |x_k|_1/2^k$ is easily seen to be a norm on X. We shall call $|\ |_2$ the *canonical second norm* of X. The two-norm space $X_s = (X, |\ |_1, |\ |_2)$ will be called the *canonical two-norm space* of X. The following theorem shows that the canonical two-norm space has some desirable properties.

THEOREM 3.1. *Let $(X, |\ |_1)$ be a B-space with a Schauder decomposition $\{M_k\}$. Then the canonical two-norm space X_s has property (n_0) and is quasi-normal.*

Proof. Let $|x|_1' = \sup_n |s(n, x)|_1$. By Theorem 2.1, there exist positive numbers k and K such that for any $x \in X$,

$$k|x|_1 \leqslant |x|_1' \leqslant K|x|_1.$$

For $x = \sum_k x_k \in X$, $|x_k|_1 \leqslant 2|x|_1'$, so that $|x|_2 = \sum_k |x_k|_1/2^k \leqslant 2|x|_1' \leqslant 2K|x|_1$ whence it follows that (n_0) is satisfied. To show that X_s is quasi-normal, observe that the canonical second norm $|\ |_2'$ induced by $|\ |_1'$ is equivalent to $|\ |_2$ so that it suffices to show that the two-norm space $X_s' = (X, |\ |_1', |\ |_2')$ is normal. Let $x_n \xrightarrow{\gamma} x$ in X_s'. We may assume that $|x_n|_1' \leqslant 1$. Let j be arbitrary but fixed. Since

$$|x_{n,k} - x_k|_1' \to 0 \quad \text{for } k = 1, 2, \ldots$$

(where $x_n = \sum_k x_{n,k}$), we have

$$|s(j, x_n) - s(j, x)|_1' \leqslant \sum_{k \leqslant j} |x_{n,k} - x_k|_1' \to 0$$

as $n \to \infty$. It follows by the monotonicity of $\{M_k\}$ with respect to $|\ |_1'$ (Theorem 2.1) that

$$|s(j, x)|_1' = \lim_n |s(j, x_n)|_1' \leqslant \liminf_n |x_n|_1' \leqslant 1$$

and hence $|x|_1' \leqslant 1$. This proves the normality of X_s' and completes the proof.

Remark 3.2. As an immediate consequence of Theorem 3.1, we note that $A(X_s) \subseteq X_1^*$. Let K_1 be the decomposition constant of $\{M_k\}$ and $C(X_s)$ the constant of quasi-normality of X_s. The proof of Theorem 3.1 may be used to show that $C(X_s) \leqslant K_1$. Recall that the Dixmier characteristic $r(A(X_s))$ of $A(X_s)$ is the largest positive number r such that $\{f \in A(X_s) : |f|_1^* \leqslant 1\}$ is $\sigma(X_1^*, X)$ dense in $\{f \in X_1^* : |f|_1^* \leqslant r\}$. It follows from [3], p. 117, that

$$r(A(X_s)) = 1/C(X_s) > 0.$$

Since $0 < r(A(X_s)) \leqslant 1$, it follows that $1 \leqslant C(X_s) \leqslant K_1$. From these considerations it follows that the canonical map J from X into $A(X_s)^*$, defined for $x \in X$ by

$$J(x)f = f(x), \quad f \in A(X_s)$$

is an isomorphism. When $K_1 = 1$, $C(X_s) = 1$ so that the monotonicity of $\{M_k\}$ implies the normality of X_s. As we shall see later (Remark 6.4), the converse is false: X_s may be normal without the decomposition being monotone. We also observe that the normality of X_s implies that $r(A(X_s)) = 1$ and in this case J is an isometry.

Definition 3.3. A Schauder decomposition $\{M_k\}$ for a B-space $(X, |\ |_1)$ is called *boundedly complete* if for any sequence $\{x_k\} \subset X$, $x_k \in M_k$,

$$\sup_n \Big|\sum_{k \leqslant n} x_k\Big|_1 < \infty \Rightarrow \sum_k x_k = x$$

for some x in X. The decomposition $\{M_k\}$ is called *shrinking for* $f \in X_1^*$, if the norm $|f|_{1,n}^*$ of f evaluated on $X_n = \oplus \sum_{k>n} M_k$ approaches 0 as $n \to \infty$, that is,

$$\lim_n \Big\{\sup \Big\{|f(x)| : |x|_1 \leqslant 1,\ x \in \oplus \sum_{k>n} M_k\Big\}\Big\} = 0.$$

It is called *shrinking for* $\Sigma \subseteq X_1^*$ if it is shrinking for each f in Σ. When $\{M_k\}$ is shrinking for X_1^*, we shall merely say that *$\{M_k\}$ is shrinking.*

We note that if $|\ |_0$ is a norm equivalent to $|\ |_1$ on X, then $\{M_k\}$ is boundedly complete for $(X, |\ |_0)$ if and only if it is boundedly complete for $(X, |\ |_1)$. A similar remark applies to the shrinkingness of $\{M_k\}$.

Theorem 3.4. *Let $\{M_k\}$ be a Schauder decomposition for a B-space $(X, |\ |_1)$. Then $\{M_k\}$ is shrinking for $f \in X_1^*$ if and only if $f \in A(X_s)$.*

Proof. By Theorem 2.1 and our remarks above, we may assume without loss of generality that $\{M_k\}$ is monotone. Let S and X_s be the closed unit ball and the canonical two-norm space respectively of $(X, |\ |_1)$.

To prove the 'if' part, suppose that $f \in A(X_s)$ and choose $\varepsilon > 0$. There exists $\delta > 0$ such that

$$x \in S, \quad |x|_2 < \delta \Rightarrow |f(x)| < \varepsilon.$$

Let N be a positive integer such that $\sum_{k>N} 2^{-k+1} < \delta$. Then for $x = \sum_{k>N} x_k \in X_N \cap S$ we obtain,

$$|x|_2 = \sum_{k>N} |x_k|_1/2^k \leqslant \sum_{k>N} 2^{-k+1} < \delta$$

since $|x_k|_1 \leqslant 2$ by the monotonicity of $\{M_k\}$. Hence for $n > N$, we now get

$$|f|_{1,n}^* = \sup\{|f(x)| : x \in X_n \cap S\} < \varepsilon,$$

so that $\{M_k\}$ is indeed shrinking for f.

Conversely let $f \in X_1^*$ such that $|f|_{1,n}^* \to 0$. Let $\{x_n\}$ be a sequence of points in X which γ converges to 0. We may assume that $|x_n|_1 \leqslant 1/2$. Let $\varepsilon > 0$ and choose a positive integer N such that $|f|_{1,N}^* < \varepsilon/2$. Now $|x_n|_2 \to 0$ implies that $|x_{n,k}|_1 \to 0$ for $k = 1, 2, \ldots$, where $x_n = \sum_k x_{n,k}$. Since $f \in X_1^*$, there exists a positive integer M such that

$$n > M \Rightarrow |f(x_{n,k})| < \varepsilon/2N, \quad k = 1, 2, \ldots, N.$$

By the monotonicity of $\{M_k\}$,

$$\Big|\sum_{k>N} x_{n,k}\Big|_1 \leqslant 2|x_n|_1 \leqslant 1.$$

Since $\sum_{k>N} x_{n,k} \in X_N$, we obtain for $n > M$,

$$|f(x_n)| \leqslant \Big|f\Big(\sum_{k\leqslant N} x_{n,k}\Big)\Big| + \Big|f\Big(\sum_{k>N} x_{n,k}\Big)\Big|$$
$$< (\varepsilon/2N)N + \varepsilon/2 = \varepsilon$$

showing that $f \in A(X_s)$. This completes the proof.

COROLLARY 3.5. *Let $\{M_k\}$ be a Schauder decomposition for a B-space $(X, |\ |_1)$. Then $\{M_k\}$ is shrinking if and only if the canonical two-norm space X_s of X is saturated.*

Proof. Recall that X_s is saturated if $A(X_s) = X_1^*$. Now apply Theorem 3.4.

The following theorem shows the close relation between a boundedly complete decomposition and the canonical two-norm space.

THEOREM 3.6. *A Schauder decomposition $\{M_k\}$ for a B-space $(X, |\ |_1)$ is boundedly complete if and only if the canonical two-norm space is γ-complete.*

Proof. Without loss of generality we assume that $\{M_k\}$ is monotone. Assume that $\{M_k\}$ is boundedly complete and let $\{x_n\}$ be a γ-Cauchy sequence in X_s. We write $x_n = \sum_k x_{n,k}$. Tnen there exists a positive number K such that $|x_n|_1 \leqslant K$ and $|x_n - x_m|_2 \to 0$ as $n, m \to \infty$. Hence $\{x_{n,k}\}$ is a Cauchy sequence in M_k for $k = 1, 2, \ldots$ By the completeness of M_k, there is a y_k in M_k such that $|x_{n,k} - y_k|_1 \to 0$ as $n \to \infty$, $k = 1, 2, \ldots$ For any fixed $m \geqslant 1$ we have,

$$\Big|\sum_{k=1}^{m} y_k\Big|_1 = \lim_n \Big|\sum_{k=1}^{m} x_{n,k}\Big|_1 \leqslant \varliminf_n |x_n|_1 \leqslant K.$$

By the boundedly completeness of $\{M_k\}$, $\sum_k y_k = \xi$ for some $\xi \in X$. It is clear that $x_n \xrightarrow{\gamma} \xi$ so that X_s is γ-complete.

To prove the otherway, let $\{x_k\}$ be a sequence in X such that $x_k \in M_k$ and $\sup_n \{|\sum_{k\leqslant n} x_k|_1\} = K < \infty$. Let $y_n = \sum_{k\leqslant n} x_k$. Now for $n \leqslant m$,

$$|y_n - y_m|_2 = \sum_k 2^{-k}|y_{n,k} - y_{m,k}|_1 = \sum_{k=n+1}^{m} |x_k|_1 \cdot 2^{-k}$$
$$< \sum_{k\geqslant n} 2K/2^k \to 0 \quad \text{as } n \to \infty.$$

Hence $\{y_n\}$ is γ-Cauchy in X_s. By the γ-completeness of X_s, we have a z in X_s such that $y_n \xrightarrow{\gamma} z$, so that in particular, $|y_{n,k} - z_k|_1 \to 0$ for $k = 1, 2, \ldots$, where $y_n = \sum_k y_{n,k}$ and $z = \sum_k z_k$. Since $y_{n,k} = x_k$ for $k \leqslant n$ and equal to 0 for $k > n$, it follows that $x_k = z_k$ for each k, that is, $\sum_k x_k$ converges to z. This completes the proof.

4. Schauder decomposition for $A(X_s)$ and the k-duals. We shall now show that a Schauder decomposition of a B-space X induces a Schauder decomposition on $A(X_s)$.

THEOREM 4.1. *Let $(X, |\ |_1)$ be a B-space with Schauder decomposition $\{M_k, P_k\}$ and canonical two-norm space X_s. Then $\{P_k^*(X_1^*)\}$ is a Schauder decomposition for $A(X_s)$ and a weak-* Schauder decomposition for X_1^*.*

Proof. Clearly P_k^*, the adjoint of P_k, is both norm and weak-* continuous. For any $f \in X_1^*$, $P_k^*(f) = f$ on M_k and vanishes on M_j for $j \neq k$. Thus $P_k^*(f)$ is unique. It is easy to see that $\{P_k^*(X_1^*)\}$ is a weak-* decomposition of X_1^*. It remains, therefore, to verify that it is a decomposition of $A(X_s)$.

To show that $P_k^*(X_1^*) \subseteq A(X_s)$ for each k, let $\{x_n\}$ be a sequence of points in X_s, $x_n = \sum_k x_{n,k}$, which is γ-convergent to 0. Since $|x_n|_2 \to 0$, $|x_{n,k}|_1 \to 0$ so that $P_k^*(f)x_n = f(x_{n,k}) \to 0$ for any f in X_1^*, showing that $P_k^*(f) \in A(X_s)$.

Let $f \in A(X_s)$. We shall show that

$$\Big|f - \sum_{k=1}^{n} P_k^*(f)\Big|_1^* \to 0 \quad \text{as } n \to \infty.$$

By Theorem 2.1, the norm $|\ |_0$ on X defined by $|x|_0 = \sup_n |s(n, x)|_1$ is equivalent to $|\ |_1$. Thus there exist positive numbers k and K such that

$$k|x|_1 \leqslant |x|_0 \leqslant K|x|_1.$$

By Theorem 3.4, $\{M_k\}$ is shrinking for f. Thus given $\varepsilon > 0$ we can find a positive integer N such that for $n > N$, we have $|f|_{1,n}^* < \varepsilon/C$, where $C = (1+K)$. For any $x = \sum_k x_k$ in X with $|x|_1 \leqslant 1$, we have

$$\Big|\sum_{k>n} x_k\Big|_1 = |x - s(n, x)|_1 \leqslant |x|_1 + |x|_0$$
$$\leqslant (1+K)|x|_1 \leqslant C.$$

Since $x - s(n, x) \in X_n$, we have for $n > N$,

$$\Big|f - \sum_{k=1}^{n} P_k^*(f)\Big|_1^* = \sup\Big\{\Big|f(x) - \sum_{k=1}^{n} P_k^*(f)x\Big|:\ |x|_1 \leqslant 1\Big\}$$
$$= \sup\Big\{\Big|\sum_{k>n} f(x_k)\Big|:\ |x|_1 \leqslant 1\Big\}$$
$$= \sup\{|f(x - s(n, x))|:\ |x|_1 \leqslant 1\}$$
$$\leqslant C \sup\{|f(y)|:\ |y|_1 \leqslant 1,\ y \in X_n\}$$
$$= C \cdot |f|_{1,n}^* < \varepsilon.$$

It follows that $f = \sum_k P_k^*(f)$ in $A(X_s)$, that is, $A(X_s) = \oplus \sum_k P_k^*(X_1^*)$ concluding the proof.

As a special case of the above theorem we note that the biorthogonal functionals associated with a basis in B-space X form a basis for $A(X_s)$, in particular they form a basic sequence.

Theorem 4.1 suggests the following definition.

Definition 4.2. Let $(X, |\ |_1)$ be a B-space with Schauder decomposition $\{M_k\}$ and canonical two-norm space X_s. Then the *k-dual* of X_s is defined to be the canonical two-norm space of $A(X_s)$ and is denoted by $k-X_s$. Higher k-duals are defined recursively and are denoted by k^2-X_s, k^3-X_s and so on.

Notation 4.3. We shall denote the canonical second norm of $A(X_s)$ by $|\ |_2^*$. As a subspace of X_1^*, $A(X_s)$ inherits the norm $|\ |_1^*$ so that we may write $k-X_s = \big(A(X_s), |\ |_1^*, |\ |_2^*\big)$. Let $\big(A(X_s), |\ |_1^*\big)^* = \big(A(X_s)^*, |\ |_1^{**}\big)$. Since $A(k-X_s)$ is a subspace of $A(X_s)^*$, we have $k^2-X_s = \big(A(k-X_s), |\ |_1^{**}, |\ |_2^{**}\big)$ the second norm being the canonical second norm. Finally we write $k^3-X_s = \big(A(k^2-X_s), |\ |_1^{***}, |\ |_2^{***}\big)$, where $|\ |_1^{***}$ is the norm in $A(k-X_s)^*$ and $|\ |_2^{***}$ its canonical second norm. Thus

$$X_s = (X, |\ |_1, |\ |_2), \quad k-X_s = \big(A(X_s), |\ |_1^*, |\ |_2^*\big),$$

and

$$k^2-X_s = \big(A(k-X_s), |\ |_1^{**}, |\ |_2^{**}\big), \quad k^3-X_s = \big(A(k^2-X_s), |\ |_1^{***}, |\ |_2^{***}\big).$$

For $n \geqslant 1$ we shall sometimes write $A(k^n-X_s)_j$ for the j-th component of the Schauder decomposition of $A(k^n-X_s)$. Although the sequence of continuous projections associated with $A(X_s)$ is $\{P_k^*\}$, to avoid cumbersome notation, we shall sometimes write p_k for P_k^*. Thus, $\{p_k^*\}$ is the sequence associated with $A(k-X_s)$.

We are now ready to prove that the k-duals have the characteristic property of dual spaces, namely that J maps X_s into k^2-X_s.

Theorem 4.4. *Let $\{M_k\}$ be a Schauder decomposition for a B-space $(X, |\ |_1)$. If $\{M_k\}$ is monotone, J maps X isometrically into $A(k-X_s)_j$. If $\{M_k\}$ is non-monotone, J reduces to an isomorphism.*

Proof. First consider the case when $\{M_k\}$ is monotone. By Theorem 3.1, X_s is normal whence, by Remark 3.2, J is an isometry from X into $A(X_s)^*$. To show that $J(X) \subseteq A(k-X_s)$, it suffices to show that $J(M_j) \subseteq A(k-X_s)_j$ for each j and then consider the closed linear span of $\bigcup_j M_j$ and $\bigcup_j J(M_j)$ in X and $A(k-X_s)$ respectively.

Let $x_j \in M_j$. For any $f = \sum_k P_k^*(f) \in A(X_s)$,

$$p_j^*\big(J(x_j)f\big) = J(x_j)P_j^*(f) = f\big(P_j(x_j)\big) = J(x_j)f$$

so that $p_j^*\big(J(x_j)\big) \equiv J(x_j)$, that is, p_j^* leaves $J(X_j)$ fixed. This proves that $J(M_j) \subseteq A(k-X_s)_j$, $j = 1, 2, \ldots$, and completes the proof in this case.

If $\{M_k\}$ is not monotone, we may equip X with the norm $|\ |_0$ equivalent to $|\ |_1$, and with respect to which $\{M_k\}$ is monotone. From what was just proved it is immediate that J is an isomorphism mapping X into $A(k-X_s)_j$. This completes the proof.

Remark 4.5. We may point out here that in Theorem 4.4 the monotonicity of $\{M_k\}$, while sufficient, is not necessary for J to be an isometry. It is enough that X_s be normal (see Section 2 and Remark 6.4).

This suggests the following definition.

Definition 4.6. Let $\{M_k\}$ be a Schauder decomposition for a B-space $(X, |\ |_1)$. The canonical two-norm space X_s of X is called *k-reflexive* if the canonical map J from X into $A(k-X_s)$ is onto.

A characterisation of k-reflexivity of X_s is given by

Theorem 4.7. *Let X_s be the canonical two-norm space of a B-space $(X, |\ |_1)$ with Schauder decomposition $\{M_k\}$. Then X_s is k-reflexive if and only if each M_k is reflexive.*

We shall find the following Lemma useful in the proof of Theorem 4.7.

Lemma 4.8. *Let $(X, |\ |_1)$ be a B-space with Schauder decomposition $\{M_k\}$ and suppose that $\{A(X_s)_j\}$ and $\{A(k-X_s)_j\}$ denote the Schauder decompositions of $A(X_s)$ and $A(k-X_s)$ respectively. Then $A(X_s)_j$ is linearly isomorphic to M_j^* and $A(k-X_s)_j$ to $A(X_s)_j^*$. Let φ_j denote the linear isomorphism between $A(k-X_s)_j$ and M_j^{**}. Then the restriction of φ_j to $J(M_j)$ is the identity map.*

Proof of Lemma. Clearly $P_j^*\colon X_1^* \to A(X_s)_j$ with kernel $M_j^{\perp}$, the annihilator of M_j in X_1^*. Since P_j^* is continuous and $X_1^*/M_j^{\perp}$ is isometrically isomorphic to M_j^*, the isomorphism between $A(X_s)_j$ and M_j^* follows. (This fact has also been observed in [10], p. 552). The isomorphism between $A(k-X_s)_j$ and $\big(A(X_s)_j\big)^*$ follows likewise.

Let $\varphi_j\colon M_j^{**} \to A(k-X_s)_j = p_j^*[A(X_s)^*]$. Suppose π_j is the isometry between M_j^* and $X_1^*/M_j^{\perp}$, where for $f_0 \in M_j^*$, $\pi_j(f_0) = f + M_j^{\perp}$ and $f(x) = f_0(x)$ for $x \in M_j$. We then have the following diagram:

$$M_j^* \xrightarrow{\pi_j} X_1^*/M_j^{\perp} \xrightarrow{P_j^*} P_j^*(X_1^*). \tag{1}$$

It follows that $(P_j^* \circ \pi_j)^*$ is a linear isomorphism and

$$[P_j^*(X_1^*)]^* \xrightarrow{(P_j^* \circ \pi_j)^*} M_j^{**}. \tag{2}$$

Let $x_j \in M_j$. Then $J(x_j) \in M_j^{**}$ and the equations

$$J(x_j)P_j^*(f) = P_j^*(f)x_j = f(x_j),$$

where $f \in X_1^*$, show that $J(x_j)$ belongs to the left-hand side of (2) as well. We next observe that $(P_j^* \circ \pi_j)^*$ leaves $J(x_j)$ fixed. For if $f_j \in M_j^*$, $\pi_j(f_j) = f + M_j^\perp$, we have

$$\begin{aligned}(P_j^* \circ \pi_j)^* J(x_j) f_j &= J(x_j)(P_j^* \circ \pi_j) f_j \\ &= J(x_j)\big(P_j^*(f+M_j^\perp)\big) = J(x_j)\big(P_j^*(f)\big) \\ &= P_j^*(f) x_j = f(x_j) = f_j(x_j) = J(x_j) f_j\end{aligned}$$

and this proves our assertion.

Similarly considering the natural isometry $\bar{\pi}_j$ between $[A(X_s)_j]^* = [P_j^*(X_1^*)]^*$ and $A(X_s)^*/[A(X_s)_j]^\perp$, we are led to the sequence of maps

$$[P_j^*(X_1^*)]^* \xrightarrow{\bar{\pi}_j^*} A(X_s)^*/[A(X_s)_j]^\perp \xrightarrow{p_j^*} A(k-X_s)_j. \tag{3}$$

From Theorem 4.4 and what was just seen above, $J(x_j)$ belongs to the extreme sides of (3) for every $x_j \in M_j$. A proof similar to the one in the last paragraph may be used to show that $(p_j^* \circ \bar{\pi}_j)$ also leaves $J(x_j)$ fixed for every $x_j \in M_j$.

It follows from the above considerations that each of the maps in the sequence

$$M_j^{**} \xrightarrow{\alpha} [A(X_s)_j]^* \xrightarrow{\beta} A(k-X_s)_j,$$

where α is the inverse of $(P_j^* \circ \pi_j)^*$ and $\beta = (p_j^* \circ \bar{\pi}_j)$ leaves $J(x_j)$ fixed for each $x_j \in M_j$. Clearly $\varphi_j = \beta \circ \alpha$ and this completes the proof of the Lemma.

Proof of Theorem 4.7. Suppose X_s is k-reflexive. Then J maps X onto $A(k-X_s)$ and M_j onto $A(k-X_s)_j$. Hence the map φ_j of the Lemma maps M_j^{**} identically onto $J(M_j)$, that is M_j is reflexive for every j. Conversely when every M_j is reflexive, we may retrace our steps to see that J maps M_j onto $A(k-X_s)_j$ for each j. By Remark 4.5 this means that X_s is k-reflexive and we are done.

COROLLARY 4.8. *With the hypotheses of Theorem 4.7, X_s is k-reflexive if and only if $k^n - X_s$ is k-reflexive for $n = 1, 2, \ldots$*

COROLLARY 4.9. *The canonical two-norm space of a B-space with a basis is k-reflexive.*

If we write $k^0 - X_s$ for X_s, we note that in the case of a B-space with a basis, all the spaces $k^{2n} - X_s$ are linearly isomorphic (with respect to both norms) for $n = 0, 1, 2, \ldots$ A similar result holds for the spaces $k^{2n+1} - X_s$, $n = 0, 1, 2, \ldots$ As we shall see later (Remark 6.4) all k-duals are always normal so that, in the second case, this isomorphism (J) is actually an isometry. This is true in the first case also for $n \geq 1$, and if X^s

is normal, for all n. As an illustration, consider the space $(c_0, \sup_n |a_n|)$ for which the unit vectors $e_n = (0, 0, \ldots, 1, 0, \ldots)$, with the 1 in the n-th position, form a monotone basis. In this case $k^{2n} - X_s$ is isometric to $(c_0, \sup_n |a_n|, \sum_n |a_n|/2^n)$ for $n = 0, 1, \ldots$ and $k^{2n+1} - X_s$ is isometric to $(l, \sum_n |a_n|, \sum_n |a_n|/2^n)$ for $n = 0, 1, \ldots$ Here c_0 is the space of null sequences and l is the space of absolutely summable sequences.

COROLLARY 4.10. *Let S denote the set of all B-spaces with a Schauder decomposition, K the subset of those spaces X for which the canonical two-norm space X_s is k-reflexive and G the subset of those spaces X for which X_s is γ-reflexive. Then $G \subset K \subset S$.*

Proof. The γ-reflexivity of X_s means $J(X) = (X_s)^*$. Since $J(X) \subseteq A(k-X_s) \subseteq A(X_s)^*$ in general, it follows that $JX = A(k-X_s)$, that is, X_s is k-reflexive. The classical example B of James [5] shows that G is properly contained in K. The other proper containment is obvious.

Remark 4.11. Let $\{M_k\}$ be the Schauder decomposition of X and suppose that X_s is normal. There exists a closed subspace $C(X)$ of $A(X_s)^*$ for which $\{J(M_k)\}$ is a decomposition and with the properties that $C(X)_s = \big(C(X), |\ |_1^{**}, |\ |_2^{**}\big)$ is γ-complete and normal (see [16] for details). There is a natural imbedding of X_s into $C(X)_s$ such that X_s is γ-dense in it. In this sense $C(X)_s$ is called a 'γ-completion' of X_s. It is shown in [16] that $C(X) = A(X_s)^*$ and $J(X) = C(X)$ if and only if X_s is k-reflexive and γ-complete respectively.

In the aforementioned example B of James, although B_s is k-reflexive it is not γ-reflexive. This is so since a γ-reflexive space is clearly γ-complete while B_s is not γ-complete (the basis for B is not boundedly complete). The following result, observed in [16], shows that in the presence of normality and γ-completeness, k-reflexivity implies γ-reflexivity.

THEOREM 4.12. *Let $(X, |\ |_1)$ be a B-space with Schauder decomposition $\{M_k\}$. Then X_s is γ-reflexive if and only if it is normal, γ-complete and k-reflexive.*

Proof. X_s is γ-reflexive $\Leftrightarrow$ J is an isometry of X onto $A(X_s)^* \Leftrightarrow X_s$ is normal, $J(X) = C(X)$ and $C(X) = A(X_s)^*$ from our Remark 4.11 $\Leftrightarrow X_s$ is normal, γ-complete and k-reflexive.

Since each M_k is reflexive in the case of a B-space with a basis, we have the following

COROLLARY 4.13. *The canonical two-norm space of B-space with a basis is γ-reflexive if and only if it is both normal and γ-complete.*

5. Applications. In this section we shall derive the theorems of Sanders, Ruckle and others as corollaries of our results proved in the previous sections.

THEOREM 5.1 ([10], p. 551). *Let X be a reflexive B-space. Then $\{M_k, P_k\}$ is a Schauder decomposition of $X \Leftrightarrow \{P_k^*(X_1^*), P_k^*\}$ is a Schauder decomposition of X_1^*.*

Proof. X is reflexive if and only if X_s is normal, γ-reflexive and saturated [2], p. 281, so that $A(X_s) = X_1^* = \oplus\sum_k P_k^*(X_1^*)$ and the 'only if' part is follows. For the converse replace X by X_1^*, X_1^* by X_1^{**}, P_k^* by P_k^{**} in the part just proved and note that X reflexive $\Rightarrow X_1^*$ is reflexive and $P_k^{**} = P_k$.

THEOREM 5.2 ([10], p. 551). *Let $\{M_k\}$ be a Schauder decomposition for a B-space X. Then $\{P_k^*(X_1^*)\}$ is a Schauder decomposition of X_1^* if and only if $\{M_k\}$ is shrinking.*

Proof. By Theorem 4.1, $A(X_s) = \sum_k P_k^*(X_1^*) = X_1^*$ if and only if X_s is saturated and this, by Corollary 3.5, is possible if and only if $\{M_k\}$ is shrinking.

THEOREM 5.3 ([10, p. 552). *Let $\{M_k\}$ be a Schauder decomposition for a B-space $(X, |\ |_1)$. If each M_k is reflexive and $\{M_k\}$ is boundedly complete, thenX is topologically isomorphic to the dual space of $\sum_k P_k^*(X_1^*)$.*

Proof. First af all it is clear by Theorem 4.1 and ([8], p. 57 that $\sum_k P_k^*(X_1)$ is closed in X_1^*. We replace $|\ |_1$ by an equivalent norm $|\ |_0$ with respect to which $\{M_k\}$ is monotone and this does not affect the hypotheses on $\{M_k\}$. By Theorems 3.6, 4.7 and 4.12, X_s is γ-reflexive, that is, J is an isometry from $(X, |\ |_0)$ onto $A(X_s)^* = \big(\sum_k P_k^*(X_1^*)\big)^*$ and the theorem follows.

In connection with Theorem 5.3 let us mention a question of Sanders [13], p. 205: "Is every B-space with a boundedly complete Schauder decomposition isometric to a dual space?" The answer is in the negative and many counter-examples are known. To this list we wish to add the following particularly simple example.

Let $(X_k, \|\ \|_k)$, $k = 1, 2, \ldots$, be a sequence of B-spaces and let X denote the l-sum of $\sum_k X_k$, that is, the space

$$l\Big(\sum_k X_k\Big) = \Big\{x\colon x = \{x_k\},\ x_k \in X_k,\ \sum_k \|x_k\|_k < \infty\Big\}$$

with the norm $\|x\| = \sum_k \|x_k\|_k$. If we choose $X_k = c_0$, the space of null sequences, $k = 1, 2, \ldots$, it is easily seen that $l\big(\sum_k c_0\big)$ is a B-space with a boundedly complete Schauder decomposition. It is not isometric to a dual space since the unit ball has no extreme points.

If we choose $X_k = l$ for each k, we get the space $l\big(\sum_k l\big)$ which has a boundedly complete Schauder decomposition and is isometric to a dual space (since it is isometric with l under a natural isometry). This example shows that the reflexivity of the spaces M_k is not necessary, in general, for X to be isomorphic to a dual space. Nevertheless, when X_s is normal, the converse of the second part of Theorem 5.3 is true and is easily derived from Theorem 4.12.

THEOREM 5.4 ([13], p. 205). *A B-space X with Schauder decomposition $\{M_k\}$ is reflexive if and only if* (1) *each M_k is reflexive,* (2) *$\{M_k\}$ is boundedly complete and* (3) *$\{M_k\}$ is shrinking.*

Proof. Without loss of generality we may assume that $\{M_k\}$ is monotone. Then the canonical two-norm space X_s is normal and by [2], p. 281, X is reflexive if and only if X_s is saturated and γ-reflexive. By Corollary 3.5, Theorem 4.12, Theorem 3.6 and Theorem 4.7 this is possible if and only if (1), (2) and (3) are satisfied.

Next we prove a duality theorem consisting of two parts which form the analogues in the theory of Schauder decompositions to theorems known for B-spaces with bases. The basis analogue of the first part was proved by Singer [14], Proposition 5, that of the second part by Wilansky [18], Theorem 2.

THEOREM 5.5. *Let $(X, |\ |_1)$ be a B-space with Schauder decomposition $\{M_k\}$ and suppose that each M_k is reflexive.*

(1) *$\{M_k\}$ is boundedly complete if and only if $\{P_k^*(X_1^*)\}$ is shrinking and*

(2) *$\{M_k\}$ is shrinking if and only if $\{P_k^*(X_1^*)\}$ is boundedly complete, where $\{P_k\}$ is the associated sequence of projections.*

Proof. There is no loss of generality in assuming that $\{M_k\}$ is monotone. We prove (1) and obtain (2) as a Corollary. By Theorem 4.7, X_s is k-reflexive.

Now $\{M_k\}$ is boundedly complete $\Leftrightarrow X_s$ is γ-complete (Theorem 3.6) $\Leftrightarrow X_s$ is γ-reflexive (Theorem 4.12) $\Leftrightarrow J(X) = A(k - X_s) = A(X_s)^*$. The last equality shows that $k - X_s$ is saturated which, by Corollary 3.5, is equivalent to the shrinkingness of $\{P_k^*(X_1^*)\}$. This proves (1).

To prove (2), observe that X_s is k-reflexive if and only if $k - X_s$ is k-reflexive. Thus each $P_k^*(X_1^*)$ is reflexive. By the k-reflexivity of X_s, $J(X) = A(k - X_s)$ so that $\{J(M_k)\}$ is the Schauder decomposition for $A(k - X_s)$. We use (1) replacing X by $A(X_s)$ and M_k by $P_k^*(X_1^*)$. This gives that $\{P_k^*(X_1^*)\}$ is boundedly complete $\Leftrightarrow \{J(M_k)\}$ is shrinking, that is, $\{M_k\}$ is shrinking.

THEOREM 5.6. *Let X be a B-space with Schauder decomposition $\{M_k, P_k\}$. Then $\{M_k\}$ is shrinking if and only if a bounded sequence $\{y_k\}$ in X converges weakly to 0 whenever $\lim_k P_j(y_k) = 0$, $j = 1, 2, \ldots$*

Proof. A bounded sequence $\{y_k\}$ satisfying the hypothesis of the Theorem is clearly γ-convergent to 0. Hence $y_k \to 0$ weakly $\Leftrightarrow$ every $f \in X_1^*$ is a γ-linear functional on $X_s \Leftrightarrow A(X_s) = X_1^* \Leftrightarrow X_s$ is saturated $\Leftrightarrow \{M_k\}$ is shrinking by Corollary 3.5.

The special case of the above Theorem when $\{M_k\}$ reduces to a basis may be found in [6], p. 36.

6. Decomposition constants. We conclude this paper with some remarks on decomposition constants. Let $(X, |\ |_1)$ be a B-space with Schauder decomposition $\{M_k\}$. Suppose that the decomposition constant of $\{M_k\}$ is K_1, that of $\{P_k^*(X_1^*)\}$ is K_2 and that $C(X_s)$ is the constant of quasi-normality of X_s. We observed in our Remark 3.2 that $1 \leqslant C(X_s) \leqslant K_1$. We shall now show that $K_1 \geqslant K_2$.

THEOREM 6.1. *For a B-space $(X, |\ |_1)$ with Schauder decomposition $\{M_k\}$, $K_1 \geqslant K_2$.*

Proof. Let $\xi > 0$ be any number such that

$$\Big|\sum_{k\leqslant n} x_k\Big|_1 \leqslant \xi \Big|\sum_{k\leqslant m} x_k\Big|_1$$

for all sequences $x_1, x_2, \ldots, x_m$, where $x_j \in M_j$ and $n \leqslant m$. Let $f_1, f_2, \ldots, f_m$ be any sequence in $A(X_s)$ such that $f_j \in A(X_s)_j$, where $A(X_s)_j$ is the j-th component of the Schauder decomposition of $A(X)_s$. Denoting the closed unit sphere of $(X, |\ |_1)$ by S, we have for $n \leqslant m$,

$$\begin{aligned}\Big|\sum_{k\leqslant n} f_k\Big|_1^* &= \sup\Big\{\Big|\sum_{k\leqslant n} f_k(x)\Big| : x = \sum_k x_k \in S\Big\}\\ &= \sup\Big\{\Big|\sum_{k\leqslant n} f_k(x_k)\Big| : x = \sum_k x_k \in S\Big\}\\ &\leqslant \sup\Big\{\Big|\sum_{k\leqslant m} f_k\Big(\sum_{k=1}^{n} x_k\Big)\Big| : \Big|\sum_{k=1}^{n} x_k\Big|_1 \leqslant \xi\Big\}\\ &= \xi \sup\Big\{\Big|\sum_{k=1}^{m} f_k\Big(\sum_{k=1}^{n} x_k\Big)\Big| : \sum_{k=1}^{n} x_k \in S\Big\} \leqslant \xi \Big|\sum_{k\leqslant m} f_k\Big|_1^*\end{aligned}$$

whence it follows that $K_2 \leqslant K_1$.

COROLLARY 6.2. *If the Schauder decomposition for the B-space X is monotone, so is the Schauder decomposition for $A(X_s)$.*

Let K_{n+2} denote the decomposition constant of $\{A(k^n - X_s)_j\}$ (the Schauder decomposition for $A(k^n - X_s)$) for $n \geqslant 1$. The following Theorem gives a relation between K_1, K_2 and $C(X_s)$.

THEOREM 6.3. *Let $\{M_k, P_k\}$ be a Schauder decomposition for a B-space $(X, |\ |_1)$. Then*

(i) $K_2 = K_{n+2}$ *for* $n \geqslant 1$ *and*

(ii) $1 \leqslant r(A(X_s)) \cdot K_1 \leqslant K_2 \leqslant K_1$.

Proof. Since J maps X_s into $k^3 - X_s$, $J(X) \subseteq A(k - X_s) \subseteq A(X_s)^*$ and it is easy to verify that $r(J(X)) = 1 = r(A(k - X_s))$. Hence $k - X_s$ is normal and the map J from $k - X_s$ into $k^2 - X_s$ is an isometry with respect to both norms. To prove (i), we have by definition,

$$K_4 = \sup\Big\{\Big|\sum_{k=1}^{n} \Phi_k\Big|_1^{***} \Big/ \Big|\sum_{k=1}^{m} \Phi_k\Big|_1^{***} : \Phi_j \in A(k^2 - X_s)_j,\ n \leqslant m\Big\}$$

$$\geqslant \sup\Big\{\Big|\sum_{k=1}^{n} J(f_k)\Big|_1^{***} \Big/ \Big|\sum_{k=1}^{m} J(f_k)\Big|_1^{***} : f_j \text{ is in } P_j^*(X_1^*) \text{ and } n \leqslant m\Big\} = K_2,$$

so that by Theorem 6.1,

$$K_2 \geqslant K_3 \geqslant K_4 \geqslant K_2$$

and the normality of $k^n - X_s$ for $n \geqslant 1$ shows that $K_2 = K_{n+2}$ for all n.

To prove (ii) we note that since X_s is quasi-normal, we have from [3], p. 117,

$$|J(x)|_1^{**} \leqslant |x|_1 \leqslant C(X_s)\,|J(x)|_1^{**}$$

for all $x \in X$. Hence,

$$\begin{aligned}K_3 &= \sup\Big\{\Big|\sum_{k=1}^{n} F_k\Big|_1^{**} \Big/ \Big|\sum_{k=1}^{m} F_k\Big|_1^{**} : F_j \in A(k - X_s)_j,\ n \leqslant m\Big\}\\ &\geqslant \sup\Big\{\Big|\sum_{k=1}^{n} J(x_k)\Big|_1^{**} \Big/ \Big|\sum_{k=1}^{m} J(x_k)\Big|_1^{**} : x_j \in M_j,\ n \leqslant m\Big\}\\ &\geqslant \sup\Big\{\Big|\sum_{k=1}^{n} x_k\Big|_1 \Big/ C(X_s) \cdot \Big|\sum_{k=1}^{m} x_k\Big|_1 : x_j \in M_j,\ n \leqslant m\Big\}\\ &= K_1 / C(X_s).\end{aligned}$$

It follows from our Remark 3.2 that

$$1 \leqslant r(A(X_s)) \cdot K_1 \leqslant K_3 = K_2 \leqslant K_1$$

completing the proof of the Theorem.

Remark 6.4. In the special case when the decomposition $\{M_k\}$ reduces to a basis, that is each M_k is of dimension 1, conclusion (ii) of Theorem 6.3 has been proved by Singer [15], p. 126, although our proof is somewhat simpler. Singer also observes that $r(A(X_s)) > 0$ and that (in our notation) $K_1 \geqslant 1/r(A(X_s))$. He gives an example which shows that inequalities (ii) of Theorem 6.3 are best possible. In this example $K_2 > 1$ so that the decomposition for $A(X_s)$ is non-monotone. This shows that the canonical two-norm space may be normal (indeed $k^n - X_s$ is normal for each n) without the decomposition being monotone (cf. Remark 3.2). We also note that if the decomposition for X is monotone, then the decomposition for A $(k^n - X_s)$ is also monotone.

References

[1] A. Alexiewicz and Z. Semadeni, *Linear functionals on two-norm spaces*, Studia Math. 17 (1958), pp. 121–140.

[2] — *The two-norm spaces and their conjugate spaces*, ibidem 18 (1959), pp. 275–293.

[3] — *Some properties of two-norm spaces and a characterization of reflexivity of Banach spaces*, ibidem 19 (1960), pp. 115–132.

[4] J. B. Conway, *The inadequacy of sequences*, Amer. Math. Monthly 76 (1969), pp. 68–69.

[5] R. C. James, *A non-reflexive Banach space isometric with its second conjugate*, Proc. Nat. Acad. Sci. (USA) 37 (1951), pp. 174–177.

[6] J. T. Marti, *Introduction to the Theory of Bases*, New York 1969.

[7] W. Orlicz, *On the continuity of linear operations in Saks spaces with an application to the theory of summability*, Studia Math. 16 (1957), pp. 69–73.

[8] — and V. Ptak, *Some remarks on Saks spaces*, ibidem 16 (1957), pp. 56–58.

[9] S. Rothman, *Banach spaces via two-norm spaces*, Dissertation, The University of Wisconsin, Milwaukee 1970.

[10] W. H. Ruckle, *The infinite sum of closed subspaces of an F-space*, Duke Math. J. 31 (1964), pp. 543–554.

[11] S. Saks, *On some functionals*, Trans. Amer. Math. Soc. 35 (1933), pp. 549–556.

[12] — *Addition to the note on some functionals*, ibidem 35 (1933), pp. 965–970.

[13] B. L. Sanders, *Decompositions and reflexivity in Banach spaces*, Proc. Amer. Math. Soc. 16 (1965), pp. 204–208.

[14] I. Singer, *Basic sequences and reflexivity in Banach spaces*, Studia Math. 21 (1962), pp. 351–369.

[15] — *On the constants of basic sequences in Banach spaces*, ibidem 31 (1968), pp. 125–134.

[16] P. K. Subramanian and S. Rothman, *Two-norm spaces and decomposition of Banach spaces*, II (to appear).

[17] A. Wilansky, *Functional analysis*, New York 1964.

[18] — *Topics in functional analysis*, Lecture notes in Mathematics, Berlin – Heidelberg – New York 1967.

[19] — *The basis in Banach space*, Duke Math. J. 18 (1951), pp. 799–798.

DEPARTMENT OF MATHEMATICS
MISSOURI SOUTHERN COLLEGE
JOPLIN, MISSOURI, USA)

Received May 5, 1971 (330)

STUDIA MATHEMATICA, T. XLIII. (1972)

The domain of attraction of a normal distribution in a Hilbert space

by

M. KŁOSOWSKA (Łódź)

Abstract. Let H be a separable real Hilbert space. Denote by $\Pi^{(nd)}$ the domain of attraction of normal non-degenerate probability distributions in H. If $p \in \Pi^{(nd)}$, then

$$\int_H \|x\|^\delta p(dx) < +\infty \quad \text{for } 0 \leqslant \delta < 2.$$

Assign to a distribution p in H the family of S-operators S defined by the bilinear form

$$(S_X g, h) = \int_{\|x\|<X} (x, g)(x, h) p*^{-}p(dx) \quad \text{for every } g, h \in H.$$

In terms of operators S_X we give necessary and sufficient conditions in order that $p \in \Pi^{(nd)}$.

Introduction. The paper is an attempt to extent the known results of A. J. Khinchin and P. Lévy concerning the domain of attraction of a normal distribution on a straight line to Hilbert spaces (see [6] and [8]).

Section 1 of the paper contains the basic definitions and theorems of the theory of probability distributions in a Hilbert space.

Section 2 includes the theorems concerning the shift-convergence of a sequence of distributions $\mu_n = \prod_{k=1}^{k_n} \mu_{n,k}$ with $\mu_{n,k}$ uniformly asymptotically negligible to a normal distribution. These theorems follow from the results formulated in the papers by Varadhan [11] and Jajte [3]. In Section 3 we give theorems which are the basic aim of the paper, viz. we formulate some properties of distributions belonging to the domain of attraction of a normal distribution in a Hilbert space and also the necessary and sufficient conditions in order that a distribution belong to the domain of attraction of a normal distribution in a Hilber space.

1. Let H be a separable real Hilbert space with the inner product $(\cdot, \cdot)$ and the norm $\|\cdot\|$. Let $\mathfrak{M}$ denote the set of all probability distributions in H, i.e. the set of normed regular measures defined in a σ-field $\mathscr{B}$ of Borelian subsets of H. $\mathfrak{M}$ is a complete space with the Lévy–Prochorov