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Almost fixed points of semigroups of non-expansive mappings

by

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Abstract. Let S be a topological semigroup. We show in this paper that whenever S acts non-expansively on a compact convex subset K of a locally convex space, then there exist $x \in K$ and φ in the closure of S in K^K such that $\varphi(s(x)) = x$ for all $s \in S$. Our method of proof yields an extremely short proof of a similar theorem of B. J. Pettis for affine equicontinuous maps.

Let E be a separated locally convex linear topological space and K be a non-empty compact convex subset of E. Let S be a semigroup of mappings from K into K and $\bar{S}$ be the closure of S in K^K. It is known and easy to see that if S is equicontinuous, then $\bar{S}$ is a compact topological semigroup with jointly continuous multiplication.

Recently, Pettis [5] obtained the following generalisation of Kakutani fixed point theorem [3], p. 45:

THEOREM (Pettis [6]). *If S is equicontinuous and affine, then there exist $x \in K$ and $\varphi \in \bar{S}$ such that $\varphi(\psi(x)) = x$ for all $\psi \in \bar{S}$* (1).

Let Q be a family of continuous seminorms on E which determines the topology of E. Then S is *non-expansive* (with respect to Q) if $p(s(x) - s(y) \leqslant p(x-y))$ for all $p \in Q$, $x, y \in K$ and $s \in S$.

It is the purpose of this note to prove an analogue of Pettis's result for semigroups of non-expansive mappings. In fact our technique yields a simpler proof of Pettis' Theorem.

MAIN THEOREM. *If S is non-expansive, then there exists $x \in K$ and $\varphi \in \bar{S}$ such that $\varphi(\psi(x)) = x$ for all $\psi \in \bar{S}$.*

Proof. Let J be a minimal right ideal in $\bar{S}$ (which exists and is necessarily closed by compactness of $\bar{S}$ and a simple application of Zorn's Lemma). Clearly J has no proper right ideal. Regarding J as a semigroup of non-expansive mappings from K to K, and applying Mitchell's fixed point theorem [5] (see also [4], Corollary 1) we can obtain a point x in K which is fixed under each element in J. This proves our theorem.

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(1) Actually, Pettis [6] showed that there exist $x \in K$, and $\varphi \in \bar{S}$ such that $\varphi(\psi(x)) = \varphi(x)$ for all $\psi \in \bar{S}$. But it can be shown that this conclusion and the one we stated here are equivalent.

An alternate proof of Pettis' Theorem. Let J be a minimal right ideal in $\bar{S}$. Since each element in J is affine, it follows from Rosen's Theorem [7], Theorem 1, and Day's fixed point theorem [1], Theorem 3, that K must contain a fixed point x for J.

Remark. If S has finite intersection property for right ideals (which is the case when S is commutative or when S is a group) and equicontinuous, then $\bar{S}$ has a unique minimal right ideal J. It follows that $\varphi(J) = J$ for all $\varphi \in \bar{S}$. Hence in this case, the element $x \in K$ chosen in the proof of our main theorem (and that of Pettis) is even a common fixed point for $\bar{S}$, and consequently for S (see [4] and [5]).

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Intertwining operators

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Abstract. This paper deals with operators which intertwine semi-spectral measures and subnormal operator representations of function algebras. It is shown thar such operators extend uniquely to those which intertwine dilations of measures or extensions of representations respectively. The function algebras in question are approximating in modulus. The extension mapping preserves several properties of operators.

The present paper deals with intertwining operators for representations of function algebras and for semi-spectral measures. We are interested mostly in subnormal representations and their intertwining operators. The principal question is when one can extend the intertwining operator so that the extension intertwines the extensions of representations. That this is not always possible is shown by an example given in [6]. However, if the function algebra satisfies a certain approximation property first defined by Glicksberg in [9], then intertwining operators for subnormal representations extend to interwining ones for minimal $*$-extensions of representations. But intertwining operators for $*$-representations are suitably decomposable. Consequently, we are able to describe some analytical properties of intertwining operators for subnormal representations at least in separable case.

We use the methods of dilation theory. As to this theory we refer to [1], [2], [13] and [23]. For references in $*$-representations of C^*-algebras see [3], for references in function algebras [8].

1. Let Z be a compact Hausdorff space. The uniformly closed subalgebra $A \subset C(Z)$ is called a *function algebra* on Z, if $1 \in A$ and the functions in A separate the points of Z. $\|\cdot\|$ is the sup-norm in $C(Z)$.

Suppose we are given two non-trivial complex Hilbert spaces S' and S''. The space of all linear bounded operator $X\colon S' \to S''$ is denoted by $L(S', S'')$. We write $L(S') = L(S', S')$. $I_{S'}, I_{S''}$ stand for the identity operators in S' and S'' respectively.

The algebra homomorphism $T\colon A \to L(S)$ (S — a complex Hilbert space) of the function algebra on Z is called a *representation* of A if:

$$T(1) = I_S, \tag{1.1}$$

$$\|T(u)\| \leqslant \|u\| \quad \text{for } u \in A. \tag{1.2}$$