de g à F soit continue. Soit alors $\tilde{g}$ un prolongement continu de g/F à X tel que $\|\tilde{g}\| \leqslant \|g\|$. On a:

$$\left|\int g d\mu_n - \int g d\mu\right| \leqslant \left|\int g d\mu_n - \int \tilde{g} d\mu_n\right| + \left|\int \tilde{g} d\mu_n - \int \tilde{g} d\mu\right| + \left|\int \tilde{g} d\mu - \int g d\mu\right|.$$

Le troisième terme est, par hypothèse, plus petit que $\varepsilon\|g\|$; le premier plus petit que $2\varepsilon\|g\|$ à partir d'un certain rang d'après le lemme 1; quand au second il tend vers 0 quand n tend vers l'infini puisque $\{\mu_n\}$ tend faiblement vers μ. ▲

LEMME 3. *Soient X un F-espace compact, $\{\mu_n\}$ une suite de mesures de Radon positives tendant faiblement vers une mesure de Radon μ. L'application ψ de $C(X)$ dans l'espace vectoriel c des suites convergentes à valeurs dans $\boldsymbol{C}$ définie par $\psi(f) = \{\int f d\mu_n\}$ n'est pas surjective.*

Démonstration. Soit S_n la suite convergente définie par $S_n(p) = 0$ si $p > n$ et $S_n(p) = 1$ si $p \leqslant n$. Si l'application ψ qui est linéaire et continue était surjective, d'après le théorème de l'application ouverte, il existerait une suite $\{f_n\}$ d'éléments de $C(X)$ tels que:

(a) $\psi(f_n) = S_n$;

(b) $\|f_n\| \leqslant k$ pour tout $n \in \boldsymbol{N}$.

Soit alors g une valeur d'adhérence de $\{f_n\}$ dans $L^\infty(d\nu)$ où

$$\nu = \mu + \sum_{n=1}^{\infty} \frac{1}{2^n}\mu_n.$$

On vort que $\int g d\mu_n = 1$ pour tout $n \in \boldsymbol{N}$ et $\int g d\mu = 0$, ce qui est en contradiction avec le lemme 2. ▲

Bibliographie

[1] W. G. Bade et P. C. Curtis Jr, *Embedding theorems for commutative Banach algebras*, Pacific J. Math. (1966), pp. 391–409.

[2] A. Bernard, *Caractérisations de certaines parties d'un espace compact muni d'un espace vectoriel ou d'une algèbre de fonctions continues*, Ann. Inst. Fourier, 17 (1968) Fasc. 2, pp. 359–382.

[3] — *Algèbres quotients d'algèbres uniformes*, C. R. Acad. Sci. 272, Série A (1971), pp. 1101" 1104.

[4] J. Détraz, *Sous algébres d'algébres de Banach*, ibidem 264 (1967), pp. 187–189.

[5] A. Dufresnoy, *Compacts d'interpolation métrisables de certaines algèbres de fonctions*, ibidem 269 Série A (1969) pp. 1144–1146.

[6] A. Grothendieck, *Sur les applications linéaires faiblement compactes d'espaces de type $C(K)$*, Canad. J. Math. Vol. 3 (1953), pp. 129–173.

[7] K. Hoffman, *Banach spaces of analytic functions*, Englewood Cliffs, 1962.

[8] — *Bounded analytic functions and Gleason parts*, Ann. Math. 86 (1967), pp. 74–111.

[9] I. J. Schark, *The maximal ideals in an algebra of bounded analytic functions*, J. Math. Mech. (1961), pp. 735–746.

Received August 25, 1971 (413)

STUDIA MATHEMATICA, T. XLIII. (1972)

Hereditarily periodic distributions

by

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Abstract. A periodic distribution f of q real variables $\xi_1, \ldots, \xi_q$ is called *hereditarily periodic*, iff there is a periodic distribution g such that $f = \dfrac{\partial^q g}{\partial\xi_1 \ldots \partial\xi_q}$. Among all g satisfying this equality there is always exactly one which is hereditarily periodic. Hereditarily periodic distributions can be also characterized by their Fourier coefficients or by the integrals over their periods. Every periodic distribution is a sum of hereditarily periodic distributions of some of variables $\xi_1, \ldots, \xi_q$. An estimation of Fourier coefficients is given. Also a concept of a smooth integral is introduced as a substitute for the integral from a fixed to a variable point, which cannot be used in the case of distributions.

Introduction. In this paper we are concerned with distributions in the q-dimensional Euclidean space, which admit their values in a fixed Banach space $\mathfrak{X}$. The q-dimensional Euclidean space is denoted by $\boldsymbol{R}^q$ and its points by $x = (\xi_1, \ldots, \xi_q)$. Moreover, we shall use the following notation: $x+y = (\xi_1+\eta_1, \ldots, \xi_q+\eta_q)$, $x-y = (\xi_1-\eta_1, \ldots, \xi_q-\eta_q)$, $\lambda x = (\lambda\xi_1, \ldots, \lambda\xi_q)$, $x \cdot y = \xi_1\eta_1 + \ldots + \xi_q\eta_q$, $xy = (\xi_1\eta_1, \ldots, \xi_q\eta_q)$, $|x| = \sqrt{\xi_1^2 + \ldots + \xi_q^2}$, where $y = (\eta_1, \ldots, \eta_q)$, and λ is a real number.

By *hereditarily periodic distributions* we understand periodic distributions which are derivatives of periodic distributions.

In this paper we characterize the hereditarily periodic distributions by a few properties, namely:

A periodic distribution is hereditarily periodic, if for every $p \in \boldsymbol{B}^q$ with at least one vanishing coordinate, the corresponding Fourier coefficient is equal to 0 (v. Theorem 12).

A periodic distribution is hereditarily periodic, iff $\int_0^1 f(x)\, dx^{e_i} = 0$ for $i = 1, \ldots, q$ (v. Theorem 15).

The real valued functions of the class L^p ($p \geqslant 1$) form a particular class of distributions. They are hereditarily periodic, iff they satisfy a certain minimality condition (v. Theorem 11).

Every periodic distribution f whose values are in $\mathfrak{X}$ can be expanded into a Fourier series

$$f = \sum_{p \in \boldsymbol{B}^q} c_p E_p,$$

where B^q denotes the set of all integral points of R^q and $E_p(x) = \exp(2\pi ip\cdot x)$ (v. [4]).

L. Schwartz (v. [9]) considered Fourier series of real and complex valued distributions, defined on the thore T^q, defined as the set of abstract classes $[x]$, $[x] = \{y \in R^q\colon\ y-x \in R^q\}$. For coefficients c_p of a distribution f defined on T^q L. Schwartz has given the following a symptotical estimation:

$$\lim_{|p|\to\infty} \frac{c_p}{(1+|p|^2)^k} = 0,$$

where k is a number which depends on the expanded distribution. Since a distribution defined on T^q can be considered, as a matter of fact, as a periodic distribution, the equality holds also for periodic distributions.

For the Fourier coefficients c_p of a periodic distribution f defined in R^q which admits its value in $\mathscr{X}$, J. Mikusiński gave, in [4], an estimation of the form

$$|c_p| \leqslant M(1+2\pi\overline{p})^k,$$

where M is a number which depends on f and k (the meaning of quantities $\overline{p}$, k will be explained in the sequel).

In the paper [11] an estimation of the number M is given.

The introduction of the notion of hereditarily periodic distributions enables us to give a further improvement of the estimation. The concept of a smooth integral of a distribution, which is a particular case of a primitive distribution, is a basic tool in investigating hereditarily periodic distributions.

This paper consists of the following sections.

1. Terminology and notation,
2. Smooth integral of order k of a distribution f,
3. Hereditarily periodic distributions
4. An extremal condition for hereditarily periodic distributions,
5. An estimation of Fourier coefficients of periodic distributions.

1. Terminology and notation. In what follows, we shall use, as far as possible, the notation from ETD (v. [8]). The set of all non-negative integral points of R^q will be denoted by P^q; e_i denotes the point, whose i-th coordinate is 1 and all the remaining ones are 0. If a point x has the coordinates $\xi_1, \ldots, \xi_q$, then the point $x+e_i\varkappa_i$ differs from x only by the i-th coordinate which is, equal $\xi_i+\varkappa_i$. It will also be convenient to use the notation: $e = (1, \ldots, 1)$ and $a^k = a^{\varkappa_1+\ldots+\varkappa_q}$, where a is a number and $k = (_1,\varkappa \ldots, \varkappa_q)$.

Let $a = (a_1, \ldots, a_q)$, $b = (\beta_1, \ldots, \beta_q)$. The set of the points $x \in R^q$ such that $a_j < \xi_j < \beta_j$ $(j = 1, \ldots, q)$ will be called a *q-dimensional open interval* and denoted by $a < x < b$ or (a, b). Infinite values for a_j and β_j are admitted. If a_j and β_j are finite, then set of points $x \in R^q$, whose coordinates satisfy inequality $a_j \leqslant \xi_j \leqslant \beta_j$ $(j = 1, \ldots, q)$ will be called a *q-dimensional closed interval* and denoted by $a \leqslant x \leqslant b$ or $[a, b]$.

We say that a function defined in R^q is *smooth* if it is continuous in R^q as well as its partial derivatives of any order. The class of all smooth functions with real values and bounded support will be denoted by $\mathscr{D}(R^q)$ or $\mathscr{D}$.

If $\varphi(x)$ is a smooth function and $k = (\varkappa_1, \ldots, \varkappa_q)$ is a system of non-negative integers, i.e. $k \in P^q$, then by its derivative of order k we shall understand the function

$$\varphi^{(k)}(x) = \frac{\partial^{\varkappa_1+\ldots+\varkappa_q}}{\partial^{\varkappa_1}\xi_1\ldots\partial^{\varkappa_q}\xi_q}\varphi(\xi_1, \ldots, \xi_q).$$

A function f is of class C^m in $[a, b]$ $(m = (\mu_1, \ldots, \mu_q) \in P^q)$, if all its derivatives of order $\leqslant m$ exist and are continuous functions in $[a, b]$.

It is known that if a function f is of class C^m and two mixed derivatives of order m differ only by the ordering of differentiation, then the both derivatives are equal (v. [7]). In other words, in the class C^m there exists only one derivative of the order m.

The symbol

$$f_n(x) \rightrightarrows f(x) \quad \text{in } I$$

means that the sequence of functions f_n converges to f, uniformly in I.

Let $\mathcal{O}$ be an open set in the q-dimensional Euclidean space. A sequence $\varphi_n(x)$ of smooth functions is said to be *fundamental* in $\mathcal{O}$, iff for every interval I inside $\mathcal{O}$ there exists $k \in P^q$ and smooth functions $\Phi_n(x)$ such that

$$\Phi_n^{(k)}(x) = \varphi_n(x) \quad \text{and} \quad \Phi_n(x) \rightrightarrows \quad \text{in } I.$$

We say that two sequences $\varphi_n(x)$ and $\psi_n(x)$, fundamental in $\mathcal{O}$, are equivalent in $\mathcal{O}$ and we write

$$\varphi_n(x) \sim \psi_n(x),$$

iff the interlaced sequence

$$\varphi_1(x), \psi_1(x), \varphi_2(x), \psi_2(x), \ldots$$

is fundamental.

The class of all sequences equivalent with $\varphi_n(x)$ is denoted by $[\varphi_n(x)]$ and called a distribution (defined in $\mathcal{O}$) (v. [8]).

We say that a distribution f is of order $k = (\varkappa_1, \ldots, \varkappa_q) \in P^q$, if there exists a continuous function F such that $F^{(k)} = f$.

Remark. L. Schwartz considered the order of a distribution with respect to a measure, namely, the distribution f is of order $k \in P^q$, if there exists a measure μ, such that $\mu^{(k)} = f$.

Denote by $A\big(\varphi(x), \psi(x), \ldots\big)$ an operation on a finite number of smooth functions $\varphi(x), \psi(x), \ldots$ Such an operation is called *regular*, whenever it has the following property (v. [2]):

If sequences $\varphi_n(x), \psi_n(x), \ldots$ are fundamental, so is

$$A\big(\varphi_n(x), \psi_n(x), \ldots\big).$$

Every regular operation defined on smooth functions extends automatically onto distributions $f(x) = [\varphi_n(x)]$, $g(x) = [\psi_n(x)], \ldots$ by the formula

$$A\big(f(x), g(x), \ldots\big) = \big[A\big(\varphi_n(x), \psi_n(x), \ldots\big)\big].$$

This extension is always unique, i.e. independent of the choice of fundamental sequences $\varphi_n(x), \psi_n(x), \ldots$ Multiplication by a number, addition, subtraction, translation, differentation and multiplication of a distribution by a real valued smooth function are examples of regular operations (v. [2]). The result of an application of a regular operation on a smooth function is either a smooth function or an element of $\mathcal{X}$. An element of a Banach space can be considered as a smooth function in a zero-dimensional space. This convention simplifies the formulation of many facts. Thus we can say that the result of a regular operation on a smooth function is always a smooth function. The construction extending the space of smooth functions to the distribution space can be also performed in a zero-dimensional space. In this case the fundamental sequences are simply sequences of elements of $\mathcal{X}$ which are convergent in the usual sense. The exstension turns out to be trivial because, in the case of the dimension zero the distributions nothing but elements of $\mathcal{X}$.

By a *delta-sequence* in $\boldsymbol{R}^q$ we understand every sequence of smooth functions δ_n with the following properties:

1° There is a sequence of positive numbers α_n, tending to 0, such that $\delta_n(x) = 0$ for $|x| \geqslant \alpha_n$, $n \in N$;

2° $\int\limits_{\boldsymbol{R}^q} \delta_n(x)\,dx = 1$ for $n \in \boldsymbol{N}$;

3° For every $k \in \boldsymbol{P}^q$ there is a positive integer M_k such that $\alpha^k \int\limits_{\boldsymbol{R}^q} |\delta_n^{(k)}(x)|\,dx \leqslant M_k$ for $n \in \boldsymbol{N}$.

2. Smooth integral of order k of a distribution f. Let $a = (\alpha_1, \ldots, \alpha_q)$, $b = (\beta_1, \ldots, \beta_q)$ and let $c = (\gamma_1, \ldots, \gamma_q)$ be a fixed point in the open interval $I = (a, b)$ Let λ be a smooth function of a single real variable such that $\int\limits_{\boldsymbol{R}^1} \lambda = 1$ and let the carrier S of $\Lambda(x) = \lambda(\xi_1 - \gamma_1) \ldots \lambda(\xi_q - \gamma_q)$ be inside I, i.e. $\bar{S} \subset I$. We introduce the operation

$$\int\limits_{c_\lambda}^{x} f(t)\,dt^{e_j} = \int\limits_{\alpha_j}^{\beta_j} \lambda(\eta - \gamma_j)\,d\eta \int\limits_{\eta}^{\xi_j} f(x - e_j\xi_j + e_j\tau)\,d\tau. \tag{1.1}$$

In the case when f is a function of a single variable (1.1) reduces to the form

$$\int\limits_{c_\lambda}^{x} f(t)\,dt = \int\limits_{a}^{\beta} \lambda(\eta - \gamma)\,d\eta \int\limits_{\eta}^{x} f(\tau)\,d\tau.$$

If f is a continuous function of x in I, then also integral (1.1) is a continuous function F of the variable x in I. It is easy to see that $F^{(e_j)} = f$ in I, thus F is a primitive function of order e_j of f. If c and λ are fixed, then integral (1.1) determines an operation on f which depends only on j. Let us denote this operation by $\mathcal{L}^{e_j}$. By the Fubini theorem, we evidently have

$$\mathcal{L}^{e_j}\mathcal{L}^{e_i} f = \mathcal{L}^{e_i}\mathcal{L}^{e_j} f$$

for any positive integers i, j, less than q. For arbitrary $k = (\varkappa_1, \ldots, \varkappa_q) \in \boldsymbol{P}^q$, we define $\mathcal{L}^k$ by induction, on letting

$$\mathcal{L}^0 f = f, \quad \mathcal{L}^{k+e_j} f = \mathcal{L}^{e_j}\mathcal{L}^k f.$$

Besides $F = \mathcal{L}^k f$ we shall also use tne more suggestive notation

$$F(x) = \int\limits_{c_\lambda}^{x} f(t)\,dt^k. \tag{1.2}$$

If f is a continuous function, then also integral (1.2) is a continuous function and, moreover, it is a primitive function of order k of f, i.e. $F^{(k)} = f$. A primitive function which is of form (1.2) will be called a smooth integral of order k of f (see [5]). It can be easily shown that not every primitive function F of order k is a smooth integral. E.g., if $f = 0$ in I, then formula (1.2) gives $F = 0$ in I, but this is not the only primitive function of order k. Another example: if f is a polynomial of degree $m = (\mu_1, \ldots, \mu_q)$, then the function F, given by formula (1.2), is a polynomial of degree $m + k$. Evidently, there are many primitive functions which are not polynomials. E.g., in the case $q = 2$, the function $F(x) = \xi_1\xi_2 + \exp\xi_1 + \sin\xi_2$ is not a polynomial, but it is a primitive function of order $k = (1, 1)$ of the polynomial $f(x) = 1$. From this example it follows that the class of primitive functions is larger than the set of the smooth integrals. If f is a function of the class C^m, then the primitive function F, given by formula (1.2), is of class C^{m+k}. In order to prove this property it suffices to prove it first for $k = e_j$, which is rather trivial, and then apply the induction.

We shall show, that integral (1.2) is a regular operation for ever fixed λ.

In the proof above we make the use of the following

LEMMA 1. *For any closed interval $I_0 \subset I$ such that $\bar{S} \subset I_0$ and for every function G of class C^m in I_0, we have*

$$\int_{c_\Lambda}^{x} G^{m}(t)\,dt^{e_j} = \left(G(x) - \frac{(-\xi_j)^{\mu_j-1}}{(\mu_j-1)!}\int_{\alpha_j}^{\beta_j} \Lambda^{(\mu_j-1)}(\xi_j-\gamma_j)\,G(x)\,d\xi_j\right)^{(m-e_j)}. \tag{1.3}$$

Proof. We have

$$\int_{\eta}^{\xi_j} G^{(m)}(x-e_j\xi_j+e_j\tau)\,d\tau = G^{(m-e_j)}(x) - G^{(m-e_j)}(x-e_j\xi_j+\eta e_j).$$

Since $\int_{R^1}\Lambda = 1$, thus we obtain hence

$$\int_{c_\Lambda}^{x} G^{(m)}(t)\,dt^{e_j} = G^{(m-e_j)}(x) - \int_{\alpha_j}^{\beta_j} \Lambda(\eta-\gamma_j)\,G^{(m-e_j)}(x-e_j\xi_j+\eta e_j)\,d\eta.$$

Under the sign of integral, we can replace η by ξ_j, and write the integral in the form

$$\int_{\alpha_j}^{\beta_j} \Lambda(\xi_j-\gamma_j)\,G^{(m-e_j)}(x)\,d\xi_j.$$

On integrating then per parts, we obtain easily (1.3).

In order to prove the regularity of operation (1.2) it suffices to remark first that (1.2) is an iteration of a finite number of particular operations of type (1.1) and then to prove that each of them is regular. In turn, in order to prove that operation (1.1) is regular we can use Lemma 1.

Let $\bar{I}_0 \subset I$, $\bar{S} \subset I_0$. If a sequence of smooth functions f_n is fundamental in I, then there are smooth functions G_n in I_0 suh that $G_n^{(m)} = f_n$ in I_0 for some order m and G_n converges uniformly in I_0, as $n \to \infty$. Substituting G_n in the place of G in formula (1.3), we see that the expression in the parantheses converges uniformly, as $n \to \infty$. This proves that the sequence

$$\int_{c_\Lambda}^{x} f_n(t)\,dt^{e_j} \tag{1.4}$$

is fundamental in I_0. Since I_0 can be chosen arbitrary close to I, this implies that the sequence is fundamental in I. Thus operation (1.2) is regular.

Since operation (1.2) is regular, following definition of a *smooth integral of order k* of a distribution f follows:

$$\int_{c_\Lambda}^{x} f(t)\,dt^{k} = \left[\int_{c_\Lambda}^{x} f_n(t)\,dt^{k}\right],$$

where f_n is fundamental sequence for f, i.e. $f = [f_n]$.

For any distributions f and g, the following formulae hold:

$$\int_{c_\Lambda}^{x} \big(f(t)+g(t)\big)\,dt^{k} = \int_{c_\Lambda}^{x} f(t)\,dt^{k} + \int_{c_\Lambda}^{x} g(t)\,dt^{k},$$

$$\int_{c_\Lambda}^{x} \lambda f(t)\,dt^{k} = \lambda\int_{c_\Lambda}^{x} f(t)\,dt^{k}, \qquad \lambda \in \boldsymbol{R}^1.$$

3. Hereditarily periodic distributions. A periodic distribution f in $\boldsymbol{R}^q$ will be called *hereditarily periodic*, iff it is the derivative of a periodic distribution, i.e., if there exists a periodic distribution g, such that $g' = f$ (the sign $'$ means that a partial derivative is taken q times, once with respect to each variable).

Not every periodic distribution is hereditarily periodic (see for example Remark 1, p. 264). The class of all hereditarily periodic distributions will be denoted by $\mathscr{H}$.

THEOREM 1. *For every hereditarily periodic distribution h and for every $k \in \boldsymbol{P}^q$ there exists a unique hereditarily periodic distribution G such that $G^{(k)} = h$.*

If suffices to prove

THEOREM 1'. *If h is a hereditarily periodic distribution, then for every $j = 1, \ldots, q$ there exists a unique hereditarily periodic distribution g such that $g^{(e_j)} = h$.*

Proof. Let s be a smooth function of a single real variable such that $s(\xi) = 0$ for $\xi < -1$, $s(\xi) = 1$ for $\xi > 1$. Let $h = f'$, where f is a periodic distribution. Let j be fixed and let $f_1(x) = f(x)\cdot s(\xi_j)$ and $f_2(x) = f(x)\cdot[1-s(\xi_j)]$. Then $f = f_1+f_2$, where $f_1 = 0$ for $\xi_j < -1$ and $f_2 = 0$ for $\xi_j > 1$. Hence there are distributions F_1 and F_2 such that $F_1^{(e_j)} = f_1$ and $F_2^{(e_j)} = f_2$, $F_1 = 0$ for $\xi_j < -1$ and $F_2 = 0$ for $\xi_j > 1$.

If $i \neq j$, then $[F_1(x+e_i)-F_1(x)]^{(e_j)} = f_1(x+e_i)-f_1(x) = [f(x+e_i)-f(x)]s(\xi_j) = 0$ for every x, because f is periodic. Thus $F_1(x+e_i)-F_1(x)$ is constant with respect to ξ_j. Since $F_1(x+e_i)-F_1(x) = 0$ for $\xi_j < -1$, we have $F_1(x+e_i)-F_1(x) = 0$ everywhere. Similarly, $F_2(x+e_i)-F_2(x) = 0$ holds everywhere.

Let $F = F_1+F_2$. Then $F(x+e_i)-F(x) = 0$ for $i \neq j$ everywhere. Moreover, $F^{(e_j)} = f_1+f_2 = f$.

Let $G(x) = (1+\xi_j)F(x)-\xi_jF(x+e_j)$. Then the following equality is true

$$G(x+e_i)-G(x) = 0 \quad \text{for } i \neq j. \tag{2.1}$$

We also have

$$\begin{aligned} G(x+e_j)-G(x) &= (1+\xi_j)\,[2F(x+e_j)-F(x+2e_j)-F(x)] \\ &= -(1+\xi_j)\,[K(x+e_j)-K(x)], \end{aligned} \tag{2.2}$$

where

$$K(x) = F(x+e_j) - F(x). \tag{2.3}$$

The equality $K^{(e_j)}(x) = f(x+e_j) - f(x) = 0$ implies that (2.3) does not depend on ξ_j. Thus $G(x+e_j) - G(x) = 0$. This means that $G(x)$ is periodic. Since $f(x+e_j) - f(x) = 0$, we have $G^{(e_j)}(x) = F(x) - F(x+e_j) + f(x)$. Thus $[G^{(e_j)}]' = F'(x) - F'(x+e_j) + h(x) = h(x)$, since $F(x) - F(x+e_j)$ does not depend on ξ_j.

Let $g = G'$. Then $g \in \mathscr{H}$ and $g^{(e_j)} = h$. We shall show that g is the only distribution with these properties.

In fact, suppose that there are two such distributions g_1 and g_2. Then the distribution $g_0 = g_1 - g_2$ is hereditarily periodic and we have $g_0^{(e_j)} = 0$, i.e., g_0 is constant with respect to ξ_j. Therefore there exists a periodic distribution f, such that $f' = g_0$. Also $f_1 = f^{(e-e_j)}$ is a periodic. Moreover, from $f_1^{(e_j)} = g_0$ and from the fact that g_0 is constant with respect to ξ_j it follows that $f_1 = \xi_j g_0 + r$, where r does not depend on ξ_j. Thus $f_1(x+e_j) - f_1(x) = g_0(x)$. Since f_1 is a periodic distribution, thus $g_0(x) = 0$.

The proof of the Theorem 1′ is finished.

It is easy to check that the following Theorems are true:

THEOREM 2. *If f is a hereditarily periodic function of class L^1, then $\int_0^1 f(t)\,dt^{e_i} = 0$ for $i = 1, \ldots, q$.*

THEOREM 3. *If f is a hereditarily periodic distribution and $\varphi \in \mathscr{D}$, then the convolution $\varphi * f$ is also a hereditarily periodic distribution.*

We shall still prove

THEOREM 4. *If a function f of class $L^1_{[0,e]}$ is hereditarily periodic, $\varphi \in \mathscr{D}(\boldsymbol{R}^1)$, $\int_{\boldsymbol{R}^1} \varphi = 1$, $\sqcap$ is the characteristic function of the 1-dimensional interval $[0,1]$ and $\lambda = \sqcap * \varphi$, then*

$$\int_{\boldsymbol{R}^1} \lambda(\xi_i) f(x)\,d\xi_i = 0.$$

Proof. We have, by the Fubini Theorem

$$\begin{aligned}\int_{\boldsymbol{R}^1} \lambda(\xi_i) f(x)\,d\xi_i &= \int_{\boldsymbol{R}^1} f(x)\,d\xi_i \int_{\boldsymbol{R}^1} \varphi(\tau_i) \sqcap(\xi_i - \tau_i)\,d\tau_i \\ &= \int_{\boldsymbol{R}^1} \varphi(\tau_i)\,d\tau_i \int_{\boldsymbol{R}^1} f(x) \sqcap(\xi_i - \tau_i)\,d\xi_i \\ &= \int_{\boldsymbol{R}^1} \varphi(\tau_i)\,d\tau_i \int_{\boldsymbol{R}^1} f(x + e_i\tau_i) \sqcap(\xi_i)\,d\xi_i \\ &= \int_{\boldsymbol{R}^1} \varphi(\tau_i)\,d\tau_i \int_0^1 f(x + e_i\tau_i)\,d\xi_i.\end{aligned}$$

From the periodicity of f it follows that

$$\int_0^1 f(x + e_i\tau_i)\,d\xi_i = \int_0^1 f(x)\,d\xi_i.$$

But $\int_0^1 f(x)\,d\xi_i = 0$, because $f \in \mathscr{H}$ (v. Theorem 2). Hence

$$\int_{\boldsymbol{R}^1} \lambda(\xi_i) f(x)\,d\xi_i = 0,$$

which was our assertion.

THEOREM 5. *If a distribution $f \in \mathscr{H}$ and $\lambda = \sqcap * \varphi$, where $\sqcap$ is the characteristic function of the 1-dimensional interval $[0,1]$, $\varphi \in \mathscr{D}(\boldsymbol{R}^1)$ and $\int_{\boldsymbol{R}^1} \varphi = 1$, then the smooth integral*

$$F(x) = \int_{c_\lambda}^{x} f(t)\,dt^k$$

is a hereditarily periodic distribution in $\boldsymbol{R}^q$ which does not depend on the choice of c and φ.

Proof. Let $c = (\gamma_1, \ldots, \gamma_q)$ be a fixed point in the q-dimensional open interval I, and let φ be a smooth function of a single real variable, of bounded carrier, such that $\int_{\boldsymbol{R}^1} \varphi = 1$ and $\lambda = \sqcap * \varphi$. Besides, let the carrier S of function

$$\Lambda = \lambda(\xi_1 - \gamma_1) \ldots \lambda(\xi_q - \gamma_q)$$

be inside I. Evidently $\lambda \in \mathscr{D}(\boldsymbol{R}^1)$ and $1 * \lambda = (1 * \sqcap) * \varphi = 1 * \varphi = 1$ (see [4]).

Let $\{f_n\}$ be a fundamental sequence of $f \in \mathscr{H}$, i.e., $f = [f_n]$. Then the smooth functions f_n belong to $\mathscr{H}$. From the definition of the smooth integral of order k we get the equality

$$\int_{c_\lambda}^{x} f(t)\,dt^k = \Big[\int_{c_\lambda}^{x} f_n(t)\,dt^k\Big], \tag{2.4}$$

where, according to the notation introduced in Section 1, the expression between brackets denotes the distribution defined by the fundamental sequence inside.

In order to prove that (2.4) is a hereditarily periodic distribution it suffices to remark that (2.4) is an iteration of a finite number of particular operations

$$\int_{c_\lambda}^{x} f(t)\,dt^{e_j} = \Big[\int_{c_\lambda}^{x} f_n(t)\,dt^{e_j}\Big] \tag{2.5}$$

and then prove that (2.5) is a hereditarily periodic distribution.

It is known that for every element of the sequence $\{f_n\}$ there exists exactly one function $G_n \in \mathscr{H}$ such that $G_n^{(e_j)} = f_n$ (v. Theorem 1'). Thus

$$\int_{c_\Lambda}^{x} f_n(t)\,dt^{e_j} = \int_{a_j}^{\beta_j} \Lambda(\eta-\gamma_j)\,d\eta \int_{\eta}^{\xi_j} f_n(x-e_j\xi_j+e_j\tau_j)\,d\tau_j$$
$$= G_n(x) - \int_{a_j}^{\beta_j} \Lambda(\eta-\gamma_j) G_n(x-e_j\xi_j+e_j\eta)\,d\eta .$$

By Theorem 4 it is easy to see that

$$\int_{a_j}^{\beta_j} \Lambda(\eta-\gamma_j) G_n(x-e_j\xi_j+e_j\eta)\,d\eta = 0 .$$

The sequence G_n is fundamental, because it is equal to the sequence of integrals $\int_{c_\Lambda}^{x} f_n(t)\,dt^{e_j}$. Since the sequence G_n is a fundamental sequence of hereditarily periodic smooth functions, it defines a hereditarily periodic distribution, i.e.,

$$\int_{c_\Lambda}^{x} f(t)\,dt^{e_j} = [G_n(x)] = G(x),$$

where $G \in \mathscr{H}$, $G^{(e_j)} = f$. Thus the proof is complete.

Every periodic distribution is a tempered distribution, i.e., there are $k \in \boldsymbol{P}^q$ and a continuous function F of polynomial growth such that $F^{(k)} = f$ (v. [4]). By Theorem 1, for $k \in \boldsymbol{P}^q$, there exists a unique distribution $G \in \mathscr{H}$, such that $G^{(k)} = f$. It is necessary to prove that it is a continuous function. In other words we have to prove the following

THEOREM 6'. *If $f \in \mathscr{H}$ is the derivative of some order k of a continuous function F, then there exists a $G \in \mathscr{H}$, such that $G^{(k)} = f$, which is also a continuous function.*

We shall show a more general theorem, namely:

THEOREM 6. *If $f \in \mathscr{H}$ is a derivative of some order k of a function $F \in C^p$, then the distribution $G \in \mathscr{H}$ such that $G^{(k)} = f$, is of class C^p.*

Before giving the proof of Theorem 6 we shall prove two Lemmas.

LEMMA 2. *For every fixed $\varphi \in \mathscr{D}$ the integral*

$$\int_{\boldsymbol{R}^1} \varphi(\xi_j) g(x)\,d\xi_j \tag{2.6}$$

is a regular operation on g.

Proof. Let f_n be any fundamental sequence of smooth functions in I. Then, for any given interval I_0 inside I, there is a sequence of smooth functions F_n, uniformly convergent in I_0, such that $F_n^{(k)} = f_n$ for some order $k = (\varkappa_1, \ldots, \varkappa_q)$. We have

$$\int_{a_j}^{\beta_j} \varphi(\xi_j) f_n(x)\,d\xi_j = \int_{a_j}^{\beta_j} \varphi(\xi_j) F_n^{(k)}(x)\,d\xi_j \tag{2.7}$$
$$= (-1)^{\varkappa_j}\Big(\int_{a_j}^{\beta_j} \varphi^{(\varkappa_j)}(\xi_j) F_n(x)\,d\xi_j\Big)^{(k-\varkappa_j e_j)}.$$

Now, the sequence in the big parantheses is a sequence of smooth functions (constant in ξ_j) which converges in I_0. This proves that the sequence on the left-hand side of (2.7) is fundamental. Since I_0 is arbitrary, this sequence is fundamental in the whole interval I. Thus operation (2.6) is regular.

LEMMA 3. *For any distribution g and every $m \in \boldsymbol{P}^q$ the equality*

$$\int_{c_\Lambda}^{x} g^{(m)}(t)\,dt^{e_j} = \Big(g(x) - \frac{(-\xi_j)^{\mu_j-1}}{(\mu_j-1)!} \int_{a_j}^{\beta_j} \Lambda^{(\mu_j-1)}(\xi_j-\gamma_j) g(x)\,d\xi_j\Big)^{(m-e_j)} \tag{2.8}$$

holds for $j = 1, \ldots, q$.

Proof. Formula (2.8) was proved for functions g from class C^m (v. Lemma 1). However, it is correct for any distribution G. In order to see it, it suffices to show that the integral

$$\int_{a_j}^{\beta_j} \Lambda^{(\mu_j-1)}(\xi_j-\gamma_j) g(x)\,d\xi_j$$

is a regular operation on g (v. Lemma 2).

Proof of Theorem 6. It follows from Theorem 5 that the smooth integral

$$F(x) = \int_{c_\Lambda}^{x} f(t)\,dt^k, \tag{2.9}$$

where $\Lambda = \text{п} * \varphi$, $\varphi \in \mathscr{D}$ and $\int_{\boldsymbol{R}^1} \varphi = 1$ is a hereditarily periodic distribution. By Theorem 1, there exists a unique distribution $G \in \mathscr{H}$ which, given any fixed $k \in \boldsymbol{P}^q$, satisfies condition $G^{(k)} = f$. This means, that $F = G$ and that the integral is defined uniquely in the class $\mathscr{H}$. We shall show that integral (2.9) is a function of class C^p whenever the distribution f is a derivative of order k of a function $F \in C^p$. It suffices to show that, ifhe distribution f is the derivative of some order $m = (\mu_1, \ldots, \mu_q)$ of a function $g \in C^p$, then the distribution

$$\Phi(x) = \int_{c_\Lambda}^{x} f(t)\,dt^{e_j}$$

is a derivative of order $(m-e_j)$ of another function of class C^p. Applying formula (2.8) in the case, when the distribution f is a derivative of order m of function g of class C^p, we obtain

$$\int_{c_\Lambda}^{x} f(t)\,dt^{e_j} = \int_{c_\Lambda}^{x} g^{(m)}(t)\,dt^{e_j} = \big(T(x)\big)^{(m-e_j)},$$

where

$$T(x) = g(x) - \frac{(-\xi_j)^{\mu_j-1}}{(\mu_j-1)!}\int_{a_j}^{\beta_j} \Lambda^{(\mu_j-1)}(\xi_j-\gamma_j)\,g(x)\,d\xi_j$$

and $T(x)$ is a function of class C^p.

We get the assertion by induction.

THEOREM 7. *If $f\in\mathscr{H}$ is a derivative of some order k of a function $F\in L^p_{[a,b]}$ $(p\geqslant 1)$, then the distribution $G\in\mathscr{H}$ such that $G^{(k)}=f$ is a function of class $L^p_{[a,b]}$.*

The proof of this Theorem is similar to the proof of Theorem 6. We only need to apply the Minkowski inequality (to see that the difference of functions of class L^p is again a function of class L^p) and the following

LEMMA 4. *If a function g defined in $\boldsymbol{R}^q$ is of class $L^p_{[a,b]}$ $(p\geqslant 1)$, and, for a fixed i, φ is a smooth function of a single real variable whose carrier is inside 1-dimensional interval $[a_i,\beta_i]$, then the function*

$$\int_{\boldsymbol{R}^1}\varphi(\xi_i)\,g(x)\,d\xi_i$$

is of class $L^p_{[a,b]}$ (this function is constant with respect to ξ_i).

Proof. By the Hölder inequality, we have

$$\Big|\int_{\boldsymbol{R}^1}\varphi(\xi_i)g(x)\,d\xi_i\Big|^p \leqslant \Big(\int_{a_i}^{\beta_i}|\varphi(\xi_i)g(x)|\,d\xi_i\Big)^p$$

$$\leqslant \Big[\Big(\int_{a_i}^{\beta_i}|\varphi(\xi_i)|^q\,d\xi_i\Big)^{\frac{1}{q}}\cdot\Big(\int_{a_i}^{\beta_i}|g(x)|^p\,d\xi_i\Big)^{\frac{1}{p}}\Big]^p.$$

Since $\varphi\in\mathscr{D}$, there exists a number M, such that $|\varphi|\leqslant M$. Therefore

$$\Big|\int_{\boldsymbol{R}^1}\varphi(\xi_i)g(x)\,d\xi_i\Big|^p\leqslant M^p\cdot K\cdot\int_{a_i}^{\beta_i}|g(x)|^p\,d\xi_i. \tag{2.10}$$

Let $\bar a=(a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_q)$, $\bar b=(\beta_1,\ldots,\beta_{i-1},\beta_{i+1},\ldots,\beta_q)$. By the assumption and by the Fubini Theorem, the function on the right-hand side of inequality (2.10) is integrable in the $(q-1)$-dimensional interval $[\bar a,\bar b]$; this means that

$$\Big|\int_{\boldsymbol{R}^1}\varphi(\xi_i)g(x)\,d\xi_i\Big|^p$$

is an integrable function in $[\bar a,\bar b]$. Hence it is easy to see that the integral

$$\int_a^b\Big|\int_{\boldsymbol{R}^1}\varphi(\xi_i)g(x)\,d\xi_i\Big|^p dx$$

exists. The proof of Lemma 4 is finished.

In what follows we shall use the notation $x^k=\xi_1^{\varkappa_1}\ldots\xi_q^{\varkappa_q}$, where $k=(\varkappa_1,\ldots,\varkappa_q)$. Let $T^q\subset\boldsymbol{P}^q$ denote the set of all vectors whose coordinates are 0 or 1. Let T_i^q denote the set of all permutations consisting of i 1's and $q-i$ 0's. It is easy to see that $T^q=\bigcup_{1\leqslant i\leqslant q}T_i^q$. Elements of T^q will be used to denote exponents at powers of x or indexes as well. For example, if $k=(0,1,1,0)\in T_2^4$ and $x=(\xi_1,\ldots,\xi_4)$, then $x^k=\xi_1^0\cdot\xi_2^1\cdot\xi_3^1\cdot\xi_4^0=\xi_2\cdot\xi_3$. We say that a distribution f is *constant with respect to* x^k $(k\in T^q)$, if f is constant with respect to the coordinates ξ_i for which $\varkappa_i=1$. In particular, a function which is constant with respect to x^e is constant in the usual sense. For symmetry, we also can speake of a function which is constant with respect to x^0; this does not impose any condition at all, i.e., the function can be not constant with respect to any coordinate.

A periodic distribution f is said to be *hereditarily periodic with respect to* x^k $(k\in T^q)$, iff there exists a periodic distribution F such that $F^{(k)}=f$.

Let Λ be a smooth function of a single real variable such that $\int_{\boldsymbol{R}^1}\Lambda=1$, and let the carrier S of function $\Lambda(x)=\Lambda(\xi_1-\gamma_1)\ldots\Lambda(\xi_q-\gamma_q)$ be inside I. Finally, let $c=(\gamma_1,\ldots,\gamma_q)$ be an arbitrary fixed point in the interval I.

We introduce the operation

$$f(x-kx+kc_\Lambda)=\int_{\boldsymbol{R}^q}\big(\Lambda(\eta-c)\big)^k f(x-kx+k\eta)\,d\eta^k, \tag{2.11}$$

where $k\in T^q$, $kx=(\varkappa_1\xi_1,\ldots,\varkappa_q\xi_q)$, $k\eta=(\varkappa_1\eta_1,\ldots,\varkappa_q\eta_q)$ and $\big(\Lambda(\eta-c)\big)^k=\big(\Lambda(\eta_1-\gamma_1)\big)^{\varkappa_1}\ldots\big(\Lambda(\eta_q-\gamma_q)\big)^{\varkappa_q}$.

In the case, when $k=e_i$,

$$f(x-e_ix+e_ic_\Lambda)=\int_{\boldsymbol{R}^1}\Lambda(\eta_i-\gamma_i)f(x-e_ix+e_i\eta)\,d\eta_i. \tag{2.12}$$

Operation (2.11) is a substitute of fixing variables. It is always feasible for any distribution f, because the right-hand side of (2.11) is a composition of a finite number of regular operations of form (2.6) (v. Lemma 6), thus it is itself regular. This operation gives us a possibility to transfer onto distributions many classical proofs in which a fixation of variables occurs.

The *smooth integral* $\int_{c_\Lambda}^{x}f(t)\,dt$ can be defined as the composition of the regular operation $\int_{v}^{x}f(t)\,dt$ and of an operation of type (2.11). Since this definition is equivalent to the previous one, we get a new proof of the regularity of the smooth integral.

LEMMA 5. *If $f(x)$ is a periodic distribution (or a function of class C^m, or a function of class $L^p_{[a,b]}$, $p \geqslant 1$), then $f(x-xk+kc_A)$ is a periodic distribution (or a function of class C^m, or a function of class $L^p_{[a,b]}$, $p \geqslant 1$) constant with respect to x^k.*

Proof. It can be easily seen, that the left-hand side of equality (2.11) is an composition of a finite number of integrals of the form

$$\int_{R^1} \Lambda(\eta_i-\gamma_i) f(x-e_ix+e_i\eta)\,d\eta_i.$$

This implies, that if f is a periodic distribution (or a function of class C^m), then $f(x-kx+kc_A)$ is a periodic distribution (or a function of class C^m) constant with respect to x^k.

If $f \in L^p_{[a,b]}$, then applying Lemma 4 a suitable number of times, we easily see that $f(x-kx+kc_A) \in L^p_{[a,b]}$. This completes the proof of Lemma 5.

Using the definition of the smooth integral and definition (2.11), it is easy to see that for any distribution f the equality

$$\int_{c_A}^{x} f^{(e_i)}(t)\,dt^{e_i} = f(x)-f(x-e_ix+e_ic_A) \tag{2.13}$$

holds. Denoting the right-hand side of (2.13) by $\mathop{\triangle_i}\limits_{\gamma_{iA}}^{\xi_i} f(x)$ we can write

$$\int_{c_A}^{x} f^{(e_i)}(t)\,dt^{e_i} = \mathop{\triangle_i}\limits_{\gamma_{iA}}^{\xi_i} f(x). \tag{2.14}$$

Evidently,

$$\mathop{\triangle_i}\limits_{\gamma_{iA}}^{\xi_i} \mathop{\triangle_j}\limits_{\gamma_{jA}}^{\xi_j} f(x) = \mathop{\triangle_j}\limits_{\gamma_{jA}}^{\xi_j} \mathop{\triangle_i}\limits_{\gamma_{iA}}^{\xi_i} f(x).$$

The operation $\mathop{\triangle_i}\limits_{\gamma_{iA}}^{\xi_i} f(x)$ is, of course, regular.

Remark. In the classical case the symbol $\mathop{\triangle_i}\limits_{\gamma_i}^{h_i} f(x)$ is called a *difference operator* (v. [10]) and it means simply $f(x-e_ix+e_ih_i)-f(x-e_ix+e_i\gamma_i)$.

LEMMA 6. *For any distribution f in $\boldsymbol{R}^q$, the equality*

$$\big(\mathop{\triangle_i}\limits_{\gamma_{iA}}^{\xi_i} f(x)\big)^{(e_j)} = \mathop{\triangle_i}\limits_{\gamma_{iA}}^{\xi_i} f^{(e_j)}(x) \quad \text{for } i \neq j \tag{2.15}$$

holds.

Proof. The operation $\mathop{\triangle_i}\limits_{\gamma_{iA}}^{\xi_i}$ and the differentiation are regular operations, thus their composition is also a regular operation. This means that it suffices to show that equality (2.15) holds for smooth functions, which is simple verification. Thus the proof of Lemma 6 is finished.

By Lemma 6 and definition of the smooth integral of a distribution f, it follows that

$$\int_{c_A}^{x} f'(t)\,dt = \mathop{\triangle_1}\limits_{\gamma_{1A}}^{\xi_1} \cdots \mathop{\triangle_q}\limits_{\gamma_{qA}}^{\xi_q} f(x). \tag{2.16}$$

Denoting the right-hand side of equality (2.16) by $\mathop{\triangle}\limits_{c_A}^{x} f(x)$, we can write

$$\int_{c_A}^{x} f'(t)\,dt = \mathop{\triangle}\limits_{c_A}^{x} f(x). \tag{2.17}$$

It is known that if $f \in C$ $(m = (\mu_1, \ldots, \mu_q))$, $f^{(m)} = 0$, then the f can be represented in the form

$$f(x) = \sum_{i=0}^{\mu_1-1} \xi_1^i f_{1i}(x) + \ldots + \sum_{i=0}^{\mu_q-1} \xi_q^i f_{qi}(x) \qquad (x = (\xi_1, \ldots, \xi_q)), \tag{2.18}$$

where $f_{ji} \in C$ and f_{ji} are constant in ξ_j. (If $\mu_j = 0$ for some j, then the corresponding sum in (2.18) is to be replaced by 0.) The proof (2.18) can be found, e.g., in the book by J. Mikusiński and R. Sikorski, *The elementary theory of distributions*, p. 45. In that book, there is an analogous theorem for the case, when f is a distribution and the derivative $f^{(m)}$ is understood in the general sense. A similar theorem for locally integrable functions can be found in paper [12].

If f is a periodic distribution, representation (2.18) holds also with periodic distributions on the right-hand side. However, this fact does not follow directly from the general theorem. An analogous remark is also actual, when f is a function of class C^m or a function of class L^p. (In what follows, the class $L^p_{[a,b]}$ will be denoted by L^p.)

We are going to give a direct proof for these classes, but on restricting ourselves to the case $m = (1, \ldots, 1)$. Then, instead of $f^{(m)}$ we shall write f'.

It is easy to prove that

$$(x-c)^e = \sum_{k \in T^q} (-1)^k x^{e-k} c^k,$$

where, according to the adopted notation, we have $(x-c)^e = (\xi_1-\gamma_1) \ldots \ldots (\xi_q-\gamma_q)$, $x^{e-k} = \xi_1^{1-\varkappa_1} \ldots \xi_q^{1-\varkappa_q}$ and $c^k = \gamma_1^{\varkappa_1} \ldots \gamma_q^{\varkappa_q}$.

Similarly,

$$\mathop{\triangle}\limits_{c}^{x} g = \sum_{k \in T^q} (-1)^k g\big((e-k)x+kc\big),$$

or more generally

$$\mathop{\triangle}\limits_{c_A}^{x} g = \sum_{k \in T^q} (-1)^k g\big((e-k)x+kc_A\big). \tag{2.19}$$

Both the formulae holds, in particular, if g is a smooth function. Since in (2.19) there are only regular operations, the formula extends automatically onto arbitrary distributions.

Using formula (2.19) we can now easily prove

THEOREM 8. *If f is a distribution defined in $\boldsymbol{R}^q$ and if $f'(x) = 0$, then*

$$f(x) = -\sum_{k\in T_0^q} (-1)^k f(x-kx+kc_\Lambda), \tag{2.20}$$

where $T_0^q = T^q \setminus \{(0, \ldots, 0)\}$.

Moreover, if f is a periodic distribution (or a function of class C^m, or a function of class L^p), then the members of the sum (2.20) are periodic distributions (or functions of class C^m, or functions of class L^p).

Proof. If $f' = 0$, then $\overset{x}{\underset{c_\Lambda}{\triangle}} f = 0$ and, in view of formula (2.19), we have

$$\sum_{k\in T^q} (-1)^k f(x-kx+kc_\Lambda) = 0,$$

since $(e-k)x = x-kx$. But this is equivalent to equality (2.20). The fact that the functions on the right-hand side are in required class follows from Lemma 5.

Applying Theorem 8 (and Minkowski inequality for functions of class L^p), we obtain the following

COROLLARY 1. *If f is a distribution defined in $\boldsymbol{R}^q$ and if $f'(x) = 0$, then*

$$f(x) = \sum_{1\leqslant i\leqslant q} f_i(x), \tag{2.21}$$

where f_i are distributions constant with respect to ξ_i.

Moreover, if f is a periodic distribution (or a function of class C^m, or a function of class L^p, $p \geqslant 1$), then the members of the sum (2.21) can be chosen so as to be periodic distributions (or functions of class C^m, or functions of class L^p).

THEOREM 9. *Every periodic distribution f in $\boldsymbol{R}^q$ is of the form*

$$f(x) = \sum_{k\in T^q} f_k(x), \tag{2.22}$$

where T^q is the set of all vectors whose each coordinate is equal 0 or 1, f_k are periodic distributions, constant with respect to x^k, and hereditarily periodic with respect to x^{e-k}.

Proof. Let f be a periodic distribution of q-variables. Then $g = f'$ is a hereditarily periodic distribution. By Theorem 5, the distribution of the form

$$F(x) = \int_{c_\Lambda}^{x} g(t)\,dt, \quad \text{where } \Lambda = \Pi * \varphi,\ \varphi\in\mathscr{D}(\boldsymbol{R}^1),\ \int_{\boldsymbol{R}^1}\varphi = 1$$

is hereditarily periodic. Since $F' = f'$, thus $(F-f)' = 0$. By Corollary 1, we can write

$$f-F = \sum_{1\leqslant i\leqslant q} f_i,$$

where f_i are periodic distributions, constant with respect to ξ_i, $i = 1, \ldots, q$. Thus

$$f = F + \sum_{1\leqslant i\leqslant q} f_i,$$

where $F \in \mathscr{H}$ in $\boldsymbol{R}^q$ (i.e., F is a hereditarily periodic distribution of q-variables), f_i are periodic distributions, constant with respect to ξ_i.

Let $g_i = f_i^{(e-e_i)}$ $(i = 1, \ldots, q)$. Then g_i are hereditarily periodic distributions with respect to x^{e-e_i} and, by Theorem 5, the distributions

$$G_i(x) = \int_{c_\Lambda}^{x} g_i(t)\,dt^{e-e_i} \qquad (i = 1, \ldots, q,\ \Lambda = \Pi*\varphi,\ \varphi\in\mathscr{D}(\boldsymbol{R}^1),\ \int_{\boldsymbol{R}^1}\varphi = 1)$$

are also hereditarily periodic with respect to x^{e-e_i}, i.e., there are periodic distributions F_i, constant with respect to ξ_i, such that $F_i^{(e-e_i)} = G_i$. Moreover, $(f_i-G_i)^{(e-e_i)} = 0$ for $i = 1, \ldots, q$. Hence

$$f_i = G_i + \sum_{\substack{1\leqslant j\leqslant q\\ i\neq j}} f_{ij},$$

where f_{ij} are periodic distributions, constant with respect to $x^{e_i+e_j}$. Thus

$$f = F + \sum_{1\leqslant i\leqslant q} G_i + \sum_{\substack{1\leqslant i\leqslant q\\ i\neq j}} \sum_{1\leqslant j\leqslant q} f_{ij}.$$

This is equivalent to the expression

$$f = F + \sum_{k\in T_1^q} G_k + \sum_{k\in T_2^q} f_k,$$

where $F \in \mathscr{H}$ in $\boldsymbol{R}^q$, G_k are periodic distributions, constant with respect to x^k, and hereditarily periodic with respect to x^{e-k}, $k \in T_1^q$, f_k are periodic distributions, constant with respect to x^k, $k \in T_2^q$.

Repeating this procedure $q-2$ times we obtain

$$f = F + \sum_{k\in T_1^q} G_k + \sum_{k\in T_2^q} G_k + \ldots + \sum_{k\in T_q^q} G_k,$$

where $F \in \mathscr{H}$, G_k are periodic distributions, constant with respect to x^k, and hereditarily periodic with respect to x^{e-k}. In particular, the functions G_k in the last sum are constant with respect to all coordinates. The above expression is equivalent to (2.22).

Using the above method, with proper modifications, we get

THEOREM 10. *Every periodic function of class L^p, $p \geqslant 1$ (or of class C^m) is of the form*

$$f = \sum_{k \in T^q} f_k, \tag{2.23}$$

where f_k are periodic functions of class L^p (or of class C^m) constant with respect to x^k and hereditarily periodic with respect to x^{e-k}. More exactly, there are periodic functions F_k of class L^p (or of class C^m) constant with respect to x^k such that $F_k^{(e-k)} = f_k$.

4. An extremal condition for hereditarily periodic distributions. Given a $k \in \boldsymbol{P}^q$, let K be a subclass of L^2 such that $f, g \in K \Rightarrow f^{(k)} = g^{(k)}$ and $(f \in K, g \notin K) \Rightarrow f^{(k)} \neq g^{(k)}$, where the derivative is understood in the distributional sense. Our actual problem is to minimize the integral $\int_0^e f^2$ for $f \in K$. It turns out that in K there is a function f that minimizes the integral $\int_0^e f^2$. Such a function is hereditarily periodic.

THEOREM 11. (a) *If a function $f \in K$ is hereditarily periodic, then for every function $h \in K$ holds the following inequality*

$$\int_0^e f^2 \leqslant \int_0^e h^2. \tag{2.24}$$

(b) *If $k \geqslant 1$ and a function $f \in K$ satisfies inequality* (2.24) *for every $h \in K$, then f is hereditarily periodic.*

Proof. (a) Let f be a hereditarily periodic function of class K, and let h be a periodic function of the same class. Thus $f^{(k)} = h^{(k)}$. Let $k = (\varkappa_1, \ldots, \varkappa_q)$. Evidently k can be represented in the form: $k = 1 + k_1 - k_2$, where $k_1, k_2 \in \boldsymbol{P}^q$, $k_1 k_2 = 0$, $k_1 = (\varkappa_{11}, \ldots, \varkappa_{1q})$, $k_2 = (\varkappa_{21}, \ldots, \varkappa_{2q})$, $\varkappa_{1i} \geqslant 0$ and $0 \leqslant \varkappa_{2i} \leqslant 1$ for $i = 1, \ldots, q$. Obviously, $(f^{(k)})^{(k_2)} = (h^{(k)})^{(k_2)}$ and this means that

$$(f')^{(k_1)} = (h')^{(k_1)}. \tag{2.25}$$

Since $f', h' \in \mathscr{H}$ as derivatives of periodic functions, we have $f' = h'$, by Theorem 1 and by equality (2.25). Thus $(f-h)' = 0$. Hence, by Corollary 1, we obtain

$$h - f = \sum_{1 \leqslant i \leqslant q} g_i,$$

where g_i are periodic functions of class L^2 constant with respect to ξ_i $(i = 1, \ldots, q)$. Applying Theorem 10 for g_i and then applying Minkowski's inequality, we get

$$h = f + \sum_{k \in T_0^q} f_k,$$

where $T_0^q = T^q \setminus \{(0, \ldots, 0)\}$, f_k are periodic functions of class L^2 constant with respect to x^k and hereditarily periodic with respect to x^{e-k}.

Without loss of generality we can assume that $\int_0^e f_k^2 \neq 0$ for $k \in T_0^q$. The function

$$\varphi = \int_0^e \Big(f + \sum_{k \in T_0^q} c_k f_k\Big)^2$$

is a non-negative polynomial of variables c_k, thus evidently

$$\frac{\partial \varphi}{\partial c_l} = 2 \int_0^e \Big(f + \sum_{k \in T_0^q} c_k f_k\Big) \cdot f_l,$$

where $l \in T_0^q$. Since $f \in \mathscr{H}$ in $\boldsymbol{R}^q$, f_k are hereditarily periodic functions with respect to x^{e-k}, $k \in T_0^q$, we have, by Theorem 2,

$$\int_0^1 f(x)\, d\xi_i = 0 \text{ for } i = 1, \ldots, q, \qquad \int_0^e f_k(x)\, dx^{(e-k)e_i} = 0 \text{ for } k \in T_i^q,$$

$i = 1, \ldots, q-1$. Applying these equalities it is easy to see that $\partial\varphi/\partial c_k = 0$ when $c_k \cdot \int_0^e f_k^2(x)\, dx = 0$, i.e., when $c_k = 0$ for $k \in T_0^q$.

The function φ can reach its extremes values in those points only, where all partial derivatives vanish. There is only one such point and all its coordinates c_k are 0. Since the value of the φ tends to ∞ when $c_k \to \infty$, the function reaches its minimum when all c_k are zero. In particular, the value of φ at $c_k = 0$ is not greater than the value of φ at $c_k = 1$, and this fact can be written as $\int_0^e f^2 \leqslant \int_0^e h^2$. The inequality holds for all periodic functions of the class L^2 such that $h^{(k)} = f^{(k)}$, $k \in \boldsymbol{P}^q$.

(b) Let $k \geqslant 1$ and let a periodic function f satisfy the following inequality

$$\int_0^e f^2 \leqslant \int_0^e h^2 \tag{2.26}$$

for every periodic function $h \in K$. We shall show that $f \in \mathscr{H}$, i.e., f is a hereditarily periodic function of q-variables. If it is not the case, then, by

Theorem 10, the function f is of the form

$$f = G + \sum_{k \in T_0^q} f_k,$$

where f_k are periodic functions of class L^2 constant with respect to x^k and hereditarily periodic with respect to x^{e-k}; $G \in \mathscr{H}$ and is a function of class L^2 of q-variables such that $f^{(k)} = G^{(k)}$ for $k \geqslant 1$. Applying the first part of this theorem we obtain the inequality

$$\int_0^e G^2 \leqslant \int_0^e f^2,$$

which is contradictory with the assumption (v. inequality (2.26)).

Remark 1. If $0 \leqslant k \leqslant 1$, it may happen that, in the class K there is no hereditarily periodic function f, even if inequality (2.24) is fulfilled. In fact, let $k = e_2$ and let K be the class of the functions containing

$$F(x, y) = \exp 2\pi ip(x+y) + \cos y.$$

All the functions of this class are of the form

$$G(x, y) = \exp 2\pi ip(x+y) + \cos y + f(x),$$

where $f(x)$ is an arbitrary periodic function. It can be easily seen that in the class K there is no hereditarily periodic function. Clearly $F^{(e_2)} = G^{(e_2}$ and since $F(x, y)$ is a hereditarily periodic with respect to y we get, by Theorem 11 (a),

$$\int_0^{2\pi} F^2(x, y)\, dy \leqslant \int_0^{2\pi} G^2(x, y)\, dy$$

for every fixed x. Hence

$$\int_0^{2\pi}\int_0^{2\pi} F^2(x, y)\, dy\, dx \leqslant \int_0^{2\pi}\int_0^{2\pi} G^2(x, y)\, dy\, dx.$$

Thus inequality (2.24) is satisfied and but the class K does not contain any hereditarily periodic function.

Remark 2. For any fixed $k \in P^q$ we consider the equivalence relation $h_1 \sim h_2$ defined by $h_1^{(k)} = h_2^{(k)}$. Let V be an equivalence class consisting of functions of L^2 (V can contain non-periodic functions as well). We give here an example of a class V which contains the hereditarily periodic function but inequality (2.24) is not fulfilled. Let V be the equivalence class of the hereditarily periodic function $f(x) = \sin x$ with $k = 2$. It can be easily seen that this function does not fulfil (2.24) for every function of the class V. It suffices to take $g(x) = \sin x + 3\pi^{-2}x - 3\pi^{-1}$ which belongs

to V ($k = 2$) and to note that the following inequality

$$\int_0^{2\pi} (\sin x + 3\pi^{-2}x - 3\pi^{-1})^2\, dx \leqslant \int_0^{2\pi} \sin^2 x\, dx$$

holds.

5. An Estimation of Fourier coefficients of periodic distributions. In this section we shall estimate the Fourier coefficients of hereditarily periodic distributions, determine Fourier coefficients of arbitrary periodic distributions and characterize the hereditarily periodic distributions by their Fourier coefficients.

1. It is known that every periodic distribution f whose values are in $\mathscr{X}$ can be expanded into a Fourier series (see p. 245)

$$f = \sum_{p \in B^q} c_p E_p.$$

The expansion is unique and its coefficients are given by the convolution

$$c_p = fE_{-p} * \text{п}, \tag{3.1}$$

where п is the characteristic function of the q-dimensional interval $[0, e]$.

However, some comments are needed. The left-hand side of the equality represents an element of a Banach space. The right-hand side of this equality as a convolution represents a constant function, whose values, at every point, are c_p. Thus, this equality implies in fact an identification of constant functions with elements of the space. This identification does not lead to a contradiction, because the subspace of constant functions is isomorphic with the given space.

The expression $fE_{-p} * \text{п}$ may be considered as a regular operation on f, because multiplication by smooth function E_{-p} is a regular operation, convolution by a distribution of a bounded carrier is a regular operation and a composition of two regular operations is again a regular operation.

Thus, if f_n is a fundamental sequence for f, we have

$$c_p = \lim_{n\to\infty} f_n E_{-p} * \text{п}.$$

If $f \in \mathscr{H}$, i.e., $F' = f$, where F is periodic, then the sequence $f_n = f * \delta_n$ is composed of hereditarily periodic smooth functions, because $f_n \rightrightarrows (F * \delta_n)'$ and $F * \delta_n$ are periodic. By Theorem 6′, it follows that f is a derivative of order k of a hereditarily continuous function G, thus $f_n = G^{(k)} * \delta_n = (G * \delta_n)^{(k)} = G_n^{(k)}$, where G_n are hereditarily smooth functions and $G_n \rightrightarrows G$. We may write

$$f_n E_{-p} * \text{п} = \int_0^e f_n E_{-p},$$

since f_n are smooth functions and the integral is meaning full. Thus

$$f_n E_{-p} * \sqcap = \int_0^e G_n^{(k)} E_{-p}.$$

Integrating by parts k times, we obtain

$$f_n E_{-p} * \sqcap = (-2\pi i p)^k \int_0^e G_n E_{-p},$$

where $(-2\pi i p)^k = (-2\pi i p_1)^{\varkappa_1} \dots (-2\pi i p_q)^{\varkappa_q}$, because all derivatives $G_n^{(m)}$ belongs to $\mathscr{H}$ (are hereditarily periodic). Since $G_n \rightrightarrows G$, we obtain the formula

$$c_p = (-2\pi i p)^k \int_0^e G E_{-p}. \tag{3.2}$$

By (3.2), we get the following estimation for Fourier coefficients of $f \in \mathscr{H}$:

$$|c_p| \leqslant (2\pi \overline{p})^k \int_0^e |G|, \tag{3.3}$$

where G is a primitive hereditarily periodic continuous function of order k for given distribution f, $\overline{p} = (|p_1|, \dots, |p_q|)$ for $p = (p_1, \dots, p_q)$.

When k is fixed our estimation is the best one amongst all estimations of the form $(2\pi\overline{p})^k A$, where A is constant. In fact, let $f = E_p$, then $G = (2\pi i p)^{-k} E_p$ $(k \in \boldsymbol{P}^q)$ is a function such that $G^{(k)} = E_p$. Evidently

$$c_p = (-2\pi i p)^k \int_0^e (2\pi i p)^{-k} E_p E_{-p} = (-1)^k,$$

and hence $|c_p| = 1$. By inequality (3.3) we obtain

$$|c_p| \leqslant (2\pi\overline{p})^k \int_0^e (2\pi\overline{p})^{-k} E_{-p} E_p = 1,$$

this means that the estimation cannot be improved.

Remark 1. The operation $\int_0^e f$ is an irregular operation. Namely, the sequences of integrals $\int_0^e f_n$ may converge to different limits depending on fundamental sequence f_n (for a given distribution) or they may be divergent. However, if f is a periodic distribution, then all sequences $\int_0^e f_n$ are always convergent to the same limit, independent of the choice of the regular sequence f_n. In fact, we see that

$$\int_0^e f_n(x)\,dx = \int_0^e f_n(x+t) \sqcap(x)\,dx = \int_{\boldsymbol{R}^q} f_n(x+t) \sqcap(t)\,dt$$
$$= \int_{\boldsymbol{R}^q} f_n(\tau - t)\, \overline{\sqcap}(t)\,dt = (f_n * \overline{\sqcap})(\tau),$$

where $\sqcap(x)$ is the characteristic function of the q-dimensional interval $[0, e]$, $\overline{\sqcap}(x) = \sqcap(-x)$. It is easy to prove that if f_n is a regular sequence, then $f_n^* \overline{\sqcap}$ is a fundamental sequence. Hence it follows already that the limit of the sequence $\int_0^e f_n$ exists and does not depend on the choice of the regular sequence f_n.

Hence it follows that the integral $\int_0^e f$ is always defined (as an irregular operation) for any periodic distribution f.

We shall show that $c_p = \int_0^e f E_{-p}$. In fact:

$$c_p = \lim_{n\to\infty} f_n E_{-p} * \sqcap = \lim_{n\to\infty} \int_0^e f_n E_{-p} = \int_0^e f E_{-p}.$$

2. Using the method of determining of the Fourier coefficients of a hereditarily periodic distribution and the Theorem 9 we can now express the Fourier coefficients of an arbitrary periodic distribution by the coefficients of hereditarily periodic distributions.

The following theorem is crucial in this section.

THEOREM 12. *A periodic distribution is hereditarily periodic, iff for every $p \in \boldsymbol{B}^q$ with at least one vanishing coordinate, we have $c_p = 0$.*

Proof. Let $f \in \mathscr{H}$. Then exists a periodic distribution $F = \sum_{p\in\boldsymbol{B}^q} \overline{c}_p E_p$ such that $F' = f$. Hence $f = \sum_{p\in\boldsymbol{B}^q} c_p E_p$, where $c_p = (2\pi i p)^e \cdot \overline{c}_p$. If at least one of the coordinates of p is 0, then $p^e = 0$ and hence $c_p = 0$.

Let U be a set of all integral points of $\boldsymbol{R}^q$, whose each coordinate are different from 0, and let $f = \sum_{p\in U} c_p E_p$. Since $p^e \neq 0$ for $p \in U$, we may write

$$F = \sum_{p\in U} \frac{c_p}{(2\pi i p)^e} E_p,$$

where F is evidently a periodic distribution. It is easy to see that $F' = f$. Thus the proof of the Theorem 12 is finished.

Let f be a periodic distribution of q-variables. By Theorem 9 f is of the form

$$f = \sum_{k\in T^q} f_k, \tag{3.4}$$

where $T^q \subset \boldsymbol{P}^q$ denotes a set of all vectors whose each coordinate is equal 0 or 1, f_k are periodic distributions constant with respect to x^k and hereditarily periodic with respect to x^{e-k} for $k \in T^q$.

According to formula (3.1), the Fourier coefficients of a periodic distribution f are of the form

$$c_p = fE_{-p} * \text{п}.$$

In the case when f is of the form (3.4), the Fourier coefficients of this distribution can be expressed by the coefficients of f_k. Namely, we have

THEOREM 13. *For any fixed $p \in \boldsymbol{B}^q$ and every periodic distribution f, we have*

$$fE_{-p} * \text{п} = f_k E_{-p} * \text{п}$$

when k is the element of T^q such that $\varkappa_i = 0$ iff $p_i \neq 0$.

Proof. By (3.4) we have

$$fE_{-p} * \text{п} = \sum_{k \in T^q} f_k E_{-p} * \text{п}. \tag{3.5}$$

Since f_k is hereditarily periodic with respect to x^{e-k}, it follows, by Theorem 12, that its Fourier coefficient $f_k E_{-p} * \text{п}$ vanishes, whenever a coordinate p_i of p such that $\varkappa_i = 0$ vanishes. Since f_k is constant with respect to x^k, it follows that $f_k E_{-p} * \text{п}$ vanishes, whenever $\varkappa_i = 1$ and $p_i \neq 0$. In fact

$$\begin{aligned} f_k E_{-p} * \text{п} &= \lim_{n\to\infty} \int_0^e (f_k * \delta_n)(x) E_{-p}(x)\, dx \\ &= \lim_{n\to\infty} \int_0^e (f_k * \delta_n)(x)\, dx^{e-e_i} \int_0^1 E_{-p}(x)\, dx^{e_i} = 0, \end{aligned}$$

because $\int_0^1 E_{-p}(x)\, dx^{e_i} = 0$ and $f_k * \delta_n$ is constant with respect to ξ_i. Subsequently, all terms on the right-hand side of (3.5) vanish except for a single one in which k satisfies the hypothesis of Theorem.

Let f be a smooth function of q-variables and let g be a smooth function of a single variable having a bounded support. We adopt the notation

$$f \overset{i}{*} g = \int_{\boldsymbol{R}^1} f(x - e_i \tau) g(\tau)\, d\tau. \tag{3.6}$$

For a convolution defined by equality (3.6) the following lemma holds:

LEMMA 7. *If f is a smooth function of q-variables, g is a smooth function of a single variable ξ_i having a bounded support, then*

$$(f \overset{i}{*} g)^{(k)} = f^{(k)} \overset{i}{*} g, \qquad (f \overset{i}{*} g)^{(\mu_i e_i)} = f \overset{i}{*} g^{(\mu_i e_i)}. \tag{3.7}$$

The proof of this Lemma is quite easy, and will be omitted here.

DEFINITION S. Let f be a distribution of q-variables, g a distribution of a single variable ξ_i and let $f_n = f * \delta_n$, $g_n = g * \tilde{\delta}_n$ be their regular sequences. We say that the i-th partial convolution of f and g exists, iff, for every regular sequences f_n, g_n, the corresponding convolutions $f_n \overset{i}{*} g_n$ represent a fundamental sequence and the integral

$$\int_{\boldsymbol{R}^1} |f_n^{(l)}(x - e_i \tau_i) g_n^{(ke_i)}(\xi_i)|\, d\xi_i$$

exists for every $k, l \in \boldsymbol{P}^q$. The distribution determined by that fundamental sequence is, by definition, the *i-th partial convolution* of f and g, and is denoted by $f \overset{i}{*} g$.

THEOREM 14. *If f is a distribution of q-variables and g is a distribution of a single variable having a bounded support, then*

$$(f \overset{i}{*} g)^{(k)} = f^{(k)} \overset{i}{*} g,$$

where both convolutions are understood in the sense of Definition S.

Remark. It is easy to see that the convolution appearing in this theorem is not defined by regular operations.

Proof. The proof of Theorem 14 will be based, among others, on Lemma 7.

Let f_n be an arbitrary fundamental sequence for f and g_n let be an arbitrary fundamental sequence for g such that the supports of g_n are commonly bounded, i.e., $g_n = 0$ for $|\xi_i| > \alpha > 0$, for all n.

Let I be an arbitrary interval in $\boldsymbol{R}^q$ and $I' = [a, b]$ let be such an interval in $\boldsymbol{R}^q$, that $I \subset I'_{-\alpha}$, where $I'_{-\alpha} = [a - e_i \alpha, b - e_i \beta]$. Since f_n is a fundamental sequence, there are an order $k \in \boldsymbol{P}^q$ and smooth functions F_n, F such that $F_n^{(k)} = f_n$, $F^{(k)} = f$ in I' and $F_n \rightrightarrows F$ in I'. The sequence g_n is a fundamental sequence of a smooth functions having their supports commonly bounded, thus there are an order $\mu \in \boldsymbol{P}^1$ and smooth functions G_n, G such that $G_n^{(\mu e_i)} = g_n$, $G^{(\mu e_i)} = g$ and $G_n \rightrightarrows G$. Hence we obtain

$$\begin{aligned} |F_n \overset{i}{*} G_n - F \overset{i}{*} G| &\leqslant |(F_n - F) \overset{i}{*} G_n| + |F \overset{i}{*} (G_n - G)| \\ &\leqslant \int_{\boldsymbol{R}^1} |(F_n - F)(x - e_i \tau_i)|\, |G_n(\tau_i)|\, d\tau_i + \int_{\boldsymbol{R}^1} |F(x - e_i \tau_i)|\, |(G_n - G)(\tau_i)|\, d\tau_i \\ &\leqslant \varepsilon_n \int_{\boldsymbol{R}^1} |G_n(\tau_i)|\, d\tau_i + \eta_n \int_{-\alpha}^{\alpha} |F(x - e_i \tau_i)|\, d\tau_i; \end{aligned}$$

this means that $F_n \overset{i}{*} G_n \rightrightarrows F \overset{i}{*} G$ in I. By Lemma 7 we have $f_n \overset{i}{*} g_n = F_n^{(k)} \overset{i}{*} G_n^{(\mu e_i)} = (F_n \overset{i}{*} G_n)^{(k + \mu e_i)}$ in I. This means that the sequence $f_n \overset{i}{*} g_n$ is fundamental in I. It is easy to see that the integral

$$\int_{\boldsymbol{R}^1} |f_n^{(l)}(x - e_i \tau_i) g_n^{(m e_i)}(\tau_i)|\, d\tau_i$$

exists for every $l, m \in \boldsymbol{P}^q$.

Since, in particular, the sequences $f_n = f * \delta_n$ and $g_n = g * \tilde{\delta}_n$ satisfy conditions which have been assumed at the beginning of this proof, thus the convolution $f \overset{i}{*} g$ exists in the sense of Definition S. It is easy to verify that also the convolutions $f^{(k)} \overset{i}{*} g$ and $(f \overset{i}{*} g)^{(k)}$ exist in the sense of Definition S. Since $f_n^{(k)} \overset{i}{*} g_n = (f_n \overset{i}{*} g_n)^{(k)}$, this proves Theorem 14.

Remark 1. In the proof of Theorem 14 we have obtained a little more than the theorem asserts, because the fundamental sequences f_n and g_n need not be regular.

Remark 2. If f is a periodic distribution of q-variables, f_n its fundamental sequence consisting of periodic smooth functions, then the sequence of integrals $\int_0^e f_n(t)\,dt_i$ is always convergent to the same limit, independent of the choice of f_n. In fact

$$\int_0^e f_n(t)\,dt_i = \int_{\boldsymbol{R}^1} f_n(x - e_i\tau)\,\overline{\Pi}(\tau)\,d\tau = f_n \overset{i}{*} \overline{\Pi},$$

where Π is the characteristic function of the 1-dimensional interval $[0,1]$, $\overline{\Pi}(\xi_i) = \Pi(-\xi_i)$. By Theorem 14 it is easy to verify that if f_n is a fundamental sequence, then $f_n \overset{i}{*} \overline{\Pi}$ is also a fundamental sequence. Hence it already follows that the limit of the sequence $\int_0^e f_n(t)\,dt_i$ exists and does not depend on the choice of the fundamental sequence f_n. Therefore the integral $\int_0^1 f(t)\,dt_i$ is always defined (as an irregular operation) for periodic distributions and is equal to the distributional limit of a sequence of integrals $\int_0^1 f_n(t)\,dt_i$.

Moreover, the integral $\int_0^e f(t)\,dt$ is an iteration of a finite number of integrals $\int_0^1 f(t)\,dt_i$, $i = 1, \ldots, q$, thus

$$\int_0^e f(t)\,dt = \int_0^1 dt_1 \ldots \int_0^1 f(t)\,dt_q \tag{3.8}$$

when f is a periodic distribution.

By Theorem 12 and Remark 2 we obtain the following criterion for being a hereditarily periodic distribution:

THEOREM 15. *A periodic distribution f is hereditarily periodic, iff*

$$\int_0^1 f(x)\,dx^{e_i} = 0 \quad \textit{for } i = 1, \ldots, q. \tag{3.9}$$

Proof. Let f be a periodic distribution and let equality (3.9) hold. Because the Fourier coefficients of a periodic distribution are of the form

$$c_p = fE_{-p} * \Pi = \int_0^e fE_{-p},$$

we get $c_p = 0$, provided at least one of coordinates of the vector p is zero, in view of equality (3.8). Thus, by Theorem 12, the periodic distribution f must be hereditarily periodic.

Let $f \in \mathscr{H}$ and f_n be its fundamental sequence. By Theorem 1, for every function f_n, there is a hereditarily periodic function G_n such that $G_n^{(e_i)} = f_n$, and hence

$$\int_0^1 f(t)\,dt_i = \lim_{n\to\infty} \int_0^1 f_n(t)\,dt_i$$
$$= \lim_{n\to\infty} [G_n(t - e_i t + e_i 1) - G_n(t - e_i t + e_i 0)] = 0.$$

It is easy to see that the above equality is true for $i = 1, \ldots, q$.

Remark 3. It may be noted that, if $f, g \in \mathscr{H}$, then also the convolution

$$\int_0^e g(\tau) f(t - \tau)\,d\tau \tag{3.10}$$

belongs to $\mathscr{H}$. This is another type of convolution, different from that given in Definition S. The integral in (3.10) is defined by the fundamental sequence $\int_0^e \varphi_n(\tau)\psi_n(t-\tau)\,d\tau$, where φ_n and ψ_n are fundamental sequences of periodic smooth functions corresponding to g and f. (It is true that, if φ_n and ψ_n are fundamental sequence of smooth periodic functions in $\boldsymbol{R}^q$, then also the sequence $\int_0^e \varphi_n(\tau)\cdot\psi_n(t-\tau)\,d\tau$ is fundamental in $\boldsymbol{R}^q$.) It is easy to verify that the set $\mathscr{H}$ constitute a ring with ordinary addition and convolution (3.10), as multiplication.

References

[1] J. Dieudonné, *Foundations of modern analysis*, 1960.

[2] J. Mikusiński, *Irregular operations on distributions*, Studia Math. 20 (1961), pp. 163–169.

[3] — *Sequential theory of the convolution of distributions*, ibidem 29 (1968), pp. 151–160.

[4] — *On the convergence of sequences of periodic distributions*, ibidem 31 (1968), pp. 1–14.

[5] — *Lectures on the constructive theory of distributions*, Dept. of Math. Univ. of Florida Gainesville, 1969.
[6] — *An introduction to the theory of the Lebesgue and Bochner integrals*, Gainesville Florida, 1964.
[7] — *O ciągłych pochodnych funkcji wielu zmiennych*, Comment. Math. Prace Mat. 1 (1962), pp. 55–58.
[8] — and R. Sikorski, *The elementary theory of distributions*, Dissertationes Math. Rozprawy Mat. 12 (1957), 25 (1961).
[9] L. Schwartz, *Théorie des distributions*, (1950), (1951).
[10] W. Sierpiński, *Działania nieskończone*, Warszawa 1947.
[11] K. Skórnik, *An estimation of Fourier coefficients of periodic distributions*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys. 16 (1968), pp. 581–585.
[12] — *Postać funkcji lokalnie całkowalnej, której m-ta pochodna lokalna znika prawie wszędzie*, Zesz. Nauk. WSP, Katowice (1966), pp. 127–152.

Received October 5, 1971

STUDIA MATHEMATICA, T. XLIII. (1972)

Correction to the paper "Random functionals on $K\{M_p\}$ spaces" (Studia Math., 39 (1971), pp. 233–240)

by

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In the paper in question, the proof of Lemma 4 is incomplete. More specifically when $f(\omega, \cdot)$ is approximated by continuous functions $f_n(\omega, \cdot)$ it was tacitly assumed that $f_n(\cdot, t)$ is measurable for all n and t. This, however, is not a consequence of the approximation theorem.

Lemma 4 is used in the proof of Theorem 2 to establish the measurability of the functions $h_a(\cdot, t)$. However, we can construct a separate proof of the measurability using Theorem 1.

Let Γ be as in the proof of Theorem 1 and $x \in \Gamma$, where $x = (0, \ldots, M_p C_{[0,t]}, \ldots, 0)$, $C_{[0,t]}$ being the characteristic function of $[0, t]$. From part (b) of the proof of Theorem 1

$$L^*(\omega, x) = \int_0^t M_p(s) f_a(\omega, s)\, ds = h_a(\omega, t)$$

but $L^*(\cdot, x)$ is measurable and hence so is $h_a(\cdot, t)$.

Although we have not remedied the defect in the proof of Lemma 4, Urbanik [1], pp. 569, seems to use the result so it may be known.

We also note that line 23, page 237 should read

$$\Omega = \bigcup_{N=1}^{\infty} \bigcap_{\varphi \in K} A_N(\varphi).$$

References

[1] K. Urbanik, *Generalized stochastic processes with independent values*, Proc. Fourth Berkeley Symp. II, pp. 569–580.

Received March 21, 1972