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p-trivial Banach spaces

by

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Abstract. A pair of Banach spaces X and Y is said to be *p-trivial* if every bounded linear operator $T: X \to Y$ is p-absolutely summing. Continuing a study initiated by Lindenstrauss and Pełczyński we study various implications of the statement "$\langle X, Y\rangle$ is p-trivial". We also study the dual notion of strongly p-trivial.

As a by-product of this study we obtain an apparently new characterization of the isomorphs of Hilbert space.

§ 0. INTRODUCTION

This work is based on the papers [17], [18] of Lindenstrauss, Pełczyński and Rosenthal. In particular we present a detailed study of Proposition 8.1 of [17]. We will frequently use results and ideas from the somewhat neglected paper [12] of Grothendieck. In addition we will make use of two of the most profound theorems in modern functional analysis: the ε-isometry theorem of Dvoretzky [6] and the modification of the principle of local reflexivity [18] found in the remarkable paper of Johnson, Rosenthal and Zippin [14]. Since many of the concepts used in this work are fairly recent, we list below the definitions and results we need.

Classical concepts. All spaces considered are Banach spaces. The word *operator* will mean a bounded linear transformation. We shall denote by $\mathscr{L}(E, F)$ the operators from E to F. By an isomorphism, we mean a one-to-one operator that is open. A *projection* P is a member of $\mathscr{L}(E, E)$ such that $P^2 = P$. If A is a subspace (= closed linear manifold) of E then A is *complemented* in E if there is a projection $P \in \mathscr{L}(E, E)$ with $P(E) = A$.

If $\{x_a\} \subset E$ then by $[x_a]$ we denote the closed linear span of $\{x_a\}$ in E.

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By a *biorthogonal system* (x_i, f_i) *in* E we mean sequences $(x_i) \subset E$, $(f_i) \subset E^*$ such that

$$f_i(x_j) = \delta_{ij}.$$

The expression $\sum_{i=1}^{\infty} f_i(x)x_i$ is the *formal expansion* of $x \in E$ with respect to the biorthogonal system (x_i, f_i). A (*Schauder*) *basis* for E is a biorthogonal system (x_i, f_i) such that the formal expansion of each $x \in E$ converges to x in the norm topology of E.

A sequence $(x_i) \subset E$ is a *basic sequence* if (x_i) is a basis for $[x_i]$. The *Grinblyum constant*, K, of a basic sequence (x_i) is defined by

$$K = \sup_n \Big\| \sum_{i=1}^{n} f_i(x)x_i \Big\|$$

where $x \in [x_i]$ and (f_i) is biorthogonal to (x_i).

We will have occasion to use the following easily proved fact.

(0.1) *If* (x_i) *is a basic sequence in* E *with coefficient functionals* $(g_i) \subset [x_i]^*$ *and if* f_i *is a norm preserving extension of* g_i *to* E *then*

$$\frac{1}{\|x_i\|} \leqslant \|f_i\| \leqslant \frac{2K}{\|x_i\|},$$

where K *is the Grinblyum constant.*

For $1 \leqslant p \leqslant \infty$ we denote by l_p the Banach space of scalar sequences $a = (a_i)$ with

$$\|a\| = \begin{cases} \Big(\sum_{i=1}^{\infty} |a_i|^p\Big)^{1/p} & \text{if } 1 \leqslant p < \infty, \\ \sup_i |a_i| & \text{if } \quad p = \infty. \end{cases}$$

By l_p^n we denote the space of n-tuples with the above norm. Also by c_0 we mean the closed subspace of l_∞ consisting of those sequences which tend to 0.

Let (x_i) be a sequence in a Banach space E, $1 \leqslant p \leqslant \infty$ and p' given by $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. Then

(i) (x_i) is *weakly p-summing*, written $(x_i) \in l_p[E]$, if for each $f \in E^*$, $(f(x_i)) \in l_p$.

With the norm $\varepsilon_p(x_i) = \sup_{\|f\| \leqslant 1} \Big(\sum_{i=1}^{\infty} |f(x_i)|^p\Big)^{1/p}$, $l_p[E]$ is a Banach space;

(ii) (x_i) is *p-summing*, written $(x_i) \in l_p(E)$, if $(\|x_i\|) \in l_p$.

With the norm $\alpha_p(x_i) = \big(\sum_{i=1}^{\infty} \|x_i\|^p\big)^{1/p}$, $l_p(E)$ is a Banach space; and

(iii) (x_i) is *strongly p-summing*, written $(x_i) \in l_p\langle E \rangle$, if for each $(f_i) \in l_{p'}[E^*]$, $(f_i(x_i)) \in l_1$.

With the norm $\sigma_p((x_i)) = \sup_{\varepsilon_{p'}(f_i) \leqslant 1} \sum f_i(x_i)$, $l_p\langle E \rangle$ is a Banach space.

Moreover, we clearly have the set theoretic inclusions

$$l_p\langle E \rangle \subset l_p(E) \subset l_p[E].$$

(Of course above for $p = \infty$ we understand $p' = 1$, for $p = 1$, $p' = \infty$ and ε_∞, α_∞, σ_∞ given by the appropriate supremums.)

The principal work for weakly p-summing and p-summing sequences is [12]; for strongly p-summing sequences, see [3].

p-absolutely summing operators. If γ is one of the norms $\varepsilon_p, \alpha_p, \sigma_p$ above we will abuse the notation and write for finite sets $\{x_i\}_{i=1}^N \subset E$, $\gamma(x_i)$. This is of course meaningful if we consider the sequence $(\hat{x}_i)$, $\hat{x}_i = x_i$ for $i \leqslant N$, $\hat{x}_i = 0$ for $i > N$ in the appropriate space above.

Let $T \in \mathscr{L}(E, F)$. Then

(i) T is *p-absolutely summing* if there is a constant C such that

$$\alpha_p(Tx_i) \leqslant C\varepsilon_p(x_i)$$

for all finite sets $\{x_i\}_{i=1}^N$ in E; and

(ii) T is *strongly p-summing* if there is a constant C such that

$$\sigma_p(Tx_i) \leqslant C\alpha_p(x_i)$$

for all finite sets $\{x_i\}_{i=1}^N$ in E.

It is clear from the above definitions that T is p-absolutely summing if and only if $T(l_p[E]) \subset l_p(F)$ and T is strongly p-summing if and only if $T(l_p(E)) \subset l_p\langle F \rangle$.

The p-absolutely summing operators have been extensively studied in [17] and [21]. The strongly p-summing operators were introduced in [3].

We denote the p-summing operators from E to F by $\Pi_p(E, F)$ and the strongly p-summing operators from E to F by $D_p(E, F)$. If $\pi_p(T)$ and $d_p(T)$ denote, respectively, the infimum of the constants C occuring in (i) and (ii) above, then with these respective norms $\Pi_p(E, F)$ and $D_p(E, F)$ are Banach spaces ([21] and [3]).

Finally we say that a pair of Banach space E and F is *p-trivial*, written $\langle E, F \rangle$ is p-trivial, if $\Pi_p(E, F) = \mathscr{L}(E, F)$, and *strongly p-trivial*, written analogously, if $D_p(E, F) = \mathscr{L}(E, F)$.

Motivation for this concept is [17]. Our 1-trivial spaces coincide with the "unconditionally trivial" spaces of [17].

We need the following facts:

(0.2) *If $\langle E, F\rangle$ is p-trivial (strongly p-trivial) then there is a constant M such that*

$$a_p(Tx_i) \leqslant M\|T\|\varepsilon_p(x_i)$$

$\left(\sigma_p(Tx_i) \leqslant M\|T\|a_p(x_i)\right)$ *for every* $T \in \mathscr{L}(E, F)$ *and finite set* $\{x_i\} \subset E$.

(0.2) is immediate from the above definitions and the open mapping theorem.

(0.3) *Let $1 \leqslant p < \infty$. Then $T \in \Pi_p(E, F)$ if and only if the adjoint $T' \in D_{p'}(F^*, E^*)$.*

Let $1 < p \leqslant \infty$. Then $T \in D_p(E, F)$ if and only if $T' \in \Pi_{p'}(F^*, E^*)$.

(0.3) is the main result of [3]. (Here, and for the remainder of the paper p' is determined by $\dfrac{1}{p} + \dfrac{1}{p'} = 1$).

The $\mathscr{L}_p$-spaces. Our main concern in the paper is the $\mathscr{L}_p$-spaces of [17].

If E and F are isomorphic Banach spaces, the *distance coefficient* of E *and* F, $d(E, F)$, is defined by

$$d(E, F) = \inf\|T\|\,\|T^{-1}\|$$

where the infimum is over all isomorphisms from E onto F.

Let $\lambda \geqslant 1$ and $1 \leqslant p \leqslant \infty$. A Banach space E is a $\mathscr{L}_{p,\lambda}$ space if for each finite dimensional subspace $F \subset E$ there is a finite dimensional subspace B with $F \subset B \subset E$ such that

$$d(B, l_p^n) \leqslant \lambda, \quad \text{where } n = \dim B,$$

the dimension of B.

A space E is an $\mathscr{L}_p$-space [17] if E is an $\mathscr{L}_{p,\lambda}$-space for some $\lambda \geqslant 1$. These spaces include and generalize the classical $L_p(S, \Sigma, \mu)$ spaces and $C(K)$-spaces. We frequently use the following results of [17] and [18].

(0.4) *The conjugate of an $\mathscr{L}_p$-space is an $\mathscr{L}_{p'}$-space.*

(0.5) *If E is an $\mathscr{L}_1$-space and F an $\mathscr{L}_2$-space then $\langle E, F\rangle$ is p-trivial for all $p \geqslant 1$.*

If E is an $\mathscr{L}_\infty$-space and F an $\mathscr{L}_q$-space $1 \leqslant q \leqslant 2$, then $\langle E, F\rangle$ is p-trivial for all $p \geqslant 2$.

(0.6) *For $1 \leqslant p < \infty$ an $\mathscr{L}_p$-space contains a complemented subspace isomorphic to l_p.*

Local reflexivity and the bounded approximation property. One of the most remarkable results of the past few years is the principle of local reflexivity of Rosenthal and Lindenstrauss [18] whose proof rests on a selection theorem of Klee. We use the version of [14]:

(0.7) (The principle of local reflexivity) *Let X be a Banach space (regarded as a subspace of X^{**}) and let U and F be finite dimensional subspaces of X^{**} and X^*, respectively, and let $\varepsilon > 0$. Then there is a one-to-one operator $T\colon U \to X$ with $Tx = x$ for $x \in X \cap U$, $f(Te) = e(f)$ for $e \in U$ and $f \in F$ and $\|T\|\,\|T^{-1}\| < 1+\varepsilon$.*

An immediate consequence of (0.7) is

(0.8) *Let X and Y be Banach spaces with $\dim Y < +\infty$. Let F be a finite dimensional subspace of X^*, let R be an operator from X^* to Y and $\varepsilon > 0$. Then there is a weak*-continuous operator S from X^* to Y such that*

(i) *R and S agree on F; and,*

(ii) $\|S\| \leqslant (1+\varepsilon)\|R\|$.

A Banach space X has the *bounded approximation property* (b.a.p.) if there is a constant C such that if B is a finite dimensional subspace of X there is a $T \in \mathscr{L}(X, X)$ with finite dimensional range such that $\|T\| \leqslant C$ and T restricted to B is the identity.

From a series of papers [15], [14], and [20] the relationship between various approximation properties and other structures (e.g. having a basis) are becoming clear.

While there are no known Banach spaces lacking the b.a.p. it is now known, combining the results above, that a separable E has b.a.p. if and only if E is a quotient of a space with a basis.

The Dvoretzky Theorem; $\mathscr{S}_p$ and $\mathscr{D}_p$-spaces. Perhaps the most profound result in the isomorphic theory of Banach spaces is the following result of Dvoretzky [6] concerning spherical sections of convex bodies in Banach spaces.

(0.9) *For each $\varepsilon > 0$ and each positive integer n, there exists a positive integer $n(\varepsilon)$ such that if E is any Banach space and $\dim E > n(\varepsilon)$, then there exists a subspace F of E such that*

$$d(F, l_2^n) \leqslant 1+\varepsilon.$$

In particular, in any infinite dimensional Banach space, there are finite dimensional subspaces of arbitrary large dimension, nearly isometric to Euclidean spaces.

This result motivates our next two definitions.

We say that a Banach space E is an $\mathscr{S}_{p,\lambda}$ *space* if there is a constant $\lambda > 0$ and sequences of operators $\{J_n\}$, $\{P_n\}$ such that

$$l_p^n \xrightarrow{J_n} E \xrightarrow{P_n} l_p^n,$$

where $P_n J_n$ is the identity on l_p^n and $\|P_n\|\,\|J_n\| \leqslant \lambda$.

In [24] the $\mathscr{S}_{2,\lambda}$ spaces were called sufficiently Euclidean.

Given $p \geqslant 1$ and $\lambda \geqslant 1$, a Banach space E is called a $\mathscr{D}_{p,\lambda}$*-space* if for each positive integer n there is a subspace U in E with $d(U, l_p^n) \leqslant \lambda$. A space E is a $(\mathscr{S}_p)\mathscr{D}_p$-space if it is a $(\mathscr{S}_{p,\lambda})\mathscr{D}_{p,\lambda}$-space for some $\lambda \geqslant 1$.

Some remarks concerning these classes of spaces are in order.

The $\mathscr{L}_p$-spaces are a true isomorphic class and a beautiful theory of these spaces is now emerging.

However, the spaces in class $\mathscr{S}_p$ or $\mathscr{D}_p$ have little structure placed on them. Indeed the Dvoretzky ε-isometry theorem says that every infinite dimensional Banach space is in class $\mathscr{D}_2$. It is quite probable that every Banach space E is an $\mathscr{S}_p$-space for some p. As a technical device, however, these spaces appear useful.

(0.10) (i) *For* $1 \leqslant p < \infty$ $\mathscr{L}_p \subset \mathscr{S}_p \subset \mathscr{D}_p$ *and all containments are proper*;

(ii) $\mathscr{L}_\infty \subset \mathscr{S}_\infty = \mathscr{D}_\infty$ *and the first containment is proper*;

(iii) $\mathscr{D}_\infty \subset \mathscr{D}_p$ *for all* $p \geqslant 1$;

(iv) $\mathscr{D}_p \subset \mathscr{D}_2$ *for* $1 \leqslant p < \infty$.

Now (iii) is true since for each n and $\varepsilon > 0$ there is an $m(n)$ and $E \subset l_\infty^{m(n)}$ such that $d(l_p^n, E) < 1+\varepsilon$; for (ii) $c_0 \oplus l_2$ is an $\mathscr{S}_\infty$ space but not an $\mathscr{L}_\infty$-space. For the equality if X is a $\mathscr{D}_{\infty,\lambda}$ space then for each n there is an $E_n \subset X$ and an isomorphism T from E_n onto l_∞^n with $\|T\|\,\|T^{-1}\| \leqslant \lambda$. By the Hahn–Banach theorem T has an extension $T^\wedge$ from X onto l_∞^n with $\|T^\wedge\| = \|T\|$. Let $Q = T^{-1}T^\wedge$. Then $Q: X \to E_n$ is a projection and $\|Q\| \leqslant \lambda$, i.e. $X \in \mathscr{S}_{\infty,\lambda}$.

For $1 \leqslant p < \infty$, l_∞ is a $\mathscr{D}_p$-space (by (iii)) but not an $\mathscr{S}_p$-space and $l_p \oplus c_0$ is in $\mathscr{S}_p$ and not $\mathscr{L}_p$. Statement (iv) is immediate from the discusstion on the preceeding page. Also, statement (i) follows from (0.6).

It follows from the principle of local reflexivity that E is in $\mathscr{S}_p$ if and only if $E^* \in \mathscr{S}_{p'}$. Most of the statements concerning $\mathscr{S}_2$-spaces in [24] are valid for $\mathscr{S}_p$-spaces. However, arbitrary Banach spaces need not have $\mathscr{S}_p$-subspaces for $p \neq 2$, contrary to [24]:

(0.11) *Every infinite dimensional Banach space E contains an infinite dimensional subspace* $E_0 \in \mathscr{S}_2$.

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§ 1. p-TRIVIAL SPACES

We first show that p-triviality is implied by seemingly weaker assumptions. We denote by $K(X, Y)$ the space of compact operators from X into Y.

1.1. REMARK. *Let $1 \leqslant p < \infty$, and let X and Y be Banach spaces. If either X or Y has the b.a.p., then $K(X, Y)$ is contained in $\Pi_p(X, Y)$ if and only if $\langle X, Y\rangle$ is p-trivial.*

Proof. Assume that X has the b.a.p. By hypothesis and the closed graph theorem, there is a $C > 0$ such that

$$\Pi_p(T) \leqslant C\|T\| \quad \text{for all } T \in K(X, Y).$$

Let $S \in \mathscr{L}(X, Y)$ and $x_1, \ldots, x_n$ be in X. Since X has the b.a.p. there is a T: $X \to Y$, T compact, $\|T\| \leqslant M$ and $Tx_j = x_j$ for $j = 1, \ldots, n$. Here the constant M depends only on X. Thus, $\alpha_p(Sx_j) = \alpha_p(STx_j) \leqslant C\|ST\|\varepsilon_p(x_j) \leqslant CM\|S\|\varepsilon_p(x_j)$ i.e. $S \in \Pi_p(X, Y)$.

A similar proof works if Y has the b.a.p.

Since the notion of a strongly p-summing operator involves only finite sums we have replacing α_p by σ_p and ε_p by α_p in 1.1, the following corollary.

1.2. COROLLARY. *Let $1 < p \leqslant \infty$ and let X or Y have the b.a.p. Then $K(X, Y) \subset D_p(X, Y)$ if and only if $\langle X, Y\rangle$ is strongly p-trivial.*

Since the space of all adjoint operators in $\mathscr{L}(Y^*, X^*)$ is closed the same argument used above together with (0.8) proves the following result.

1.3. THEOREM. *If every adjoint operator T': $Y^* \to X^*$ is p-absolutely summing (strongly p-summing) and Y^* has the b.a.p. then $\langle Y^*, X^*\rangle$ is p-trivial (strongly p-trivial).*

For the next corollary we adopt the following notation. For a Banach space X, $X_0 = X$ and $X_n = X_{n-1}^*$.

1.4. COROLLARY. *Suppose X_n and Y_n have the b.a.p. for each integer $n \geqslant 0$. The following are equivalent*:

(a) $\langle X_{2n}, Y_{2n}\rangle$ *is p-trivial (strongly p-trivial) for all $n \geqslant 0$; and,*

(b) $\langle Y_{2n+1}, X_{2n+1}\rangle$ *is strongly p'-trivial (p'-trivial) for all $n \geqslant 0$.*

The proof is immediate from (0.3), 1.2 and 1.3.

As an application of the above results we give an apparently new characterization of the $\mathscr{L}_2$-spaces (= isomorphs of Hilbert spaces).

Let $T \in \mathscr{L}(X, Y)$. We say that T can be *factored through a Banach space Z* if there are operators $T_0 \in \mathscr{L}(X, Z)$ and $T_1 \in \mathscr{L}(Z, Y)$ such that $T = T_1T_0$. An operator T is *Hilbertian* [17] if it can be factored through a Hilbert space H.

We recall the following result (Theorem 5.2, p. 293 of [17]).

1.5. THEOREM. *Let X be an $\mathscr{L}_p$-space with $2 \leqslant p \leqslant \infty$ and let Y be an $\mathscr{L}_r$-space with $1 \leqslant r \leqslant 2$. Then every $T \in \mathscr{L}(X, Y)$ is Hilbertian.*

In particular Hilbertian operators can have a $\mathscr{L}_\infty$-domain and a range totally incomparable [25] to a Hilbert space. For $\mathscr{L}_\infty$-ranges the situation is different. We first prove a result which is probably known. However, to our knowledge, the result does not appear in print.

1.6. THEOREM. *Suppose every $T \in \mathscr{L}(E, F)$ is Hilbertian and F is an $\mathscr{L}_\infty$-space. Then E is an $\mathscr{L}_2$-space. And, of course, conversely.*

Proof. If $T \in \mathscr{L}(E, F)$ then by hypothesis there is a Hilbert space H such that

$$\begin{tikzcd} E \arrow[rr, "T"] \arrow[dr, "T_0"'] & & F \\ & H \arrow[ur, "T_1"'] & \end{tikzcd}$$

commutes. Thus we have

$$\begin{tikzcd} F^* \arrow[rr, "T'"] \arrow[dr, "T_1'"'] & & E^* \\ & H \arrow[ur, "T_0'"'] & \end{tikzcd}$$

Since F^* is an $\mathscr{L}_1$-space [18] it follows from (0.5) that T_1' hence T' is absolutely summing. Thus by 1.3 $\langle F^*, E^*\rangle$ is 1-trivial. The proof now proceeds as in Theorem 4.2 of [17]. By Proposition 7.3 of [17] there is a projection P from F^* onto l_1. Let Z be a separable subspace of E^* and Q an operator from l_1 onto Z. Since $\langle F^*, E^*\rangle$ is 1-trivial, PQ is absolutely summing and so [21] is Hilbertian. This says that Z is a quotient of a Hilbert space. Thus by Lemma 3 of [16] E^* and hence E, is an $\mathscr{L}_2$-space. We remark that 1.3 applies since an $\mathscr{L}_1$-space has the b.a.p.

We now show that a considerably stronger result can be proved.

1.7. THEOREM. *Suppose every $T \in \mathscr{L}(E, Y)$ is Hilbertian and Y is a $\mathscr{D}_\infty$-space. Then E is an $\mathscr{L}_2$-space.*

Proof. Let $H(E, Y)$ denote the class of Hilbertian operators from E to Y. Under the norm $\varrho(T) = \inf\|A\|\,\|B\|$, where the infimum is taken over all factorizations (A, B) of T through a Hilbert space, $H(E, Y)$ is a Banach space. Thus by our hypothesis and the closed graph theorem there is a $C > 0$ such that

$$\varrho(T) \leqslant C\|T\| \quad \text{for all } T \in \mathscr{L}(E, Y).$$

Let E_0 be a subspace of E, $\dim E_0 < +\infty$. Then there is an n and a subspace F of l_∞^n such that $d(E_0, F) < 2$. Since Y is a $\mathscr{D}_{\infty,\lambda}$ space there is a subspace G in Y such that $d(G, l_\infty^n) \leqslant \lambda$ (for each n). Thus we may choose G and $F \subset G$ in Y and $T\colon E_0 \to F$ such that T is an isomorphism and $\|T\| = 1$, $\|T^{-1}\| \leqslant 2\lambda$.

Since $d(G, l_\infty^n) \leqslant \lambda$ and l_∞^n has the extension property, there is an operator $T^\sim\colon E \to G$ such that $\|T^\sim\| \leqslant \lambda\|T\| = \lambda$ and $T^\sim | E_0 = T$.

By hypothesis there is a factorization

$$\begin{tikzcd} E \arrow[rr, "T^\sim"] \arrow[dr, "A"'] & & F \\ & l_2(\Gamma) \arrow[ur, "B"'] & \end{tikzcd}$$

such that $\|A\| = 1$ and $\|B\| \leqslant C\|T^\sim\| \leqslant C\lambda$.

Let $T = A(E_0)$. Since $BA = T^\sim$, $B|Z$ is onto F. Let $D\colon Z \to E_0$ be defined by $D = T^{-1}B|Z$. Then D is an isomorphism of Z onto E_0 and $D^{-1} = A|E_0$. Thus $d(Z, E_0) \leqslant \|D\|\,\|D^{-1}\| \leqslant 2C\lambda^2$. Let $K = \dim E_0$. Since $Z \subset l_2(\Gamma)$ and $\dim Z = K$ we have $d(E_0, l_2^K) \leqslant 2C\lambda^2$. Since E_0 was an arbitrary finite dimensional subspace of E, E is an $\mathscr{L}_2$-space.

Of course 1.6 is subsumed by 1.7. However, the contrast in the vastly different proofs seems, to us, to give a clearer picture of the situation. A further comment is made concerning the constants appearing in 1.6 and 1.7 at the end of the paper.

§ 2. THE GENERALIZATION OF THE LINDENSTRAUSS–PEŁCZYŃSKI THEOREM

We first recall Proposition 8.1 of [17] which motivated this paper.

2.1. THEOREM. *Let E and F be infinite dimensional Banach spaces such that $\langle E, F\rangle$ is 1-trivial. Then,*

(a) *$\langle E, l_2\rangle$ is 1-trivial;*

(b) *For any unconditionally convergent series $\sum x_i$ in E, $\sum \|x_i\|^2 < +\infty$; and,*

(c) *For any $\mathscr{L}_\infty$-space G, $\langle G, E\rangle$ is 2-trivial.*

Our aim is to obtain the p-trivial and strongly p-trivial versions of 2.1.

The proof of 2.1 (a) depends on the fact that every infinite dimensional Banach space is a $\mathscr{D}_2$-space and that the notion of an absolutely summing operator depends only on the domain. Of course this latter statement is valid for p-absolutely summing operators. Thus the following is true.

2.2. THEOREM. *If $\langle E, F\rangle$ is p-trivial then $\langle E, l_2\rangle$ is p-trivial.*

Our next result is the analogue for strongly p-summing operators. More work is needed since the notion of a strongly p-summing operator depends on both the domain and range.

In the proof we denote the canonical operators from $E \to E^{**}$ and $E^* \to E^{***}$ by J and J_* respectively.

2.3. THEOREM. *Let $1 < p \leqslant +\infty$. If $\langle E, F\rangle$ is strongly p-trivial and F^* has the b.a.p. then $\langle l_2, F\rangle$ is strongly p-trivial.*

Proof. If $\langle E, F\rangle$ is strongly p-trivial then by (0.3) and 1.3 $\langle F^*, E^*\rangle$ is p'-trivial and so by 2.2 $\langle F^*, l_2\rangle$ is p'-trivial. Again by (0.3) and 1.3 $\langle l_2, F^{**}\rangle$ is strongly p-trivial. Let $(f_i) \in l_{p'}[F^*]$ $(x_i) \in l_p(l_2)$ and $T \in \mathscr{L}(l_2, F)$.

We want to show that $\sum_{i=1}^{\infty} \langle Tx_i; f_i\rangle$ converges.

Since $\langle Tx_i, f_i\rangle = \langle JTx_i, J_*f_i\rangle$ and JT is strongly p-summing, it suffices to show that $(J_*f_i) \in l_{p'}[F^{***}]$. But if $\varphi \in F^{****}$, $\varphi(J_*f_i) = J_*^*(\varphi)(f_i)$ and $J_*^*(\varphi) \in F^{**}$ and so $\sum |\varphi(J_*f_i)|^{p'} < +\infty$. Thus $\langle l_2, Y\rangle$ is strongly p-summing.

Since $\langle l_1, l_2\rangle$ is 2-trivial (indeed 1-trivial) $\langle l_2, l_\infty\rangle$ is, by 1.4, strongly 2-trivial. Since the identity $i: l_2 \to l_2$ is not strongly 2-summing (since its' adjoint is not 2-absolutely summing) $\langle l_2, l_2\rangle$ is not strongly 2-trivial. However, $l_\infty \notin \mathscr{S}_2$. Indeed the following is true.

2.4. THEOREM. *If $\langle E, F\rangle$ is strongly p-trivial and $F \in \mathscr{S}_2$ then $\langle E, l_2\rangle$ is strongly p-trivial.*

Proof. Suppose F is an $\mathscr{S}_{2,\lambda}$-space. Then there are operators $l_2^n \xrightarrow{J_n} F \xrightarrow{P_n} l_2^n$ such that $\|J_n\|\,\|P_n\| \leqslant \lambda$ and P_nJ_n is the identity. If $(x_i)_{i=1}^N \subset E$ and $T \in \mathscr{L}(E, l_2)$, let $E_0 = [Tx_i\colon i \leqslant N]$ and $P\colon l_2 \to E_0$ the canonical projection. Since F is an $\mathscr{S}_{2,\lambda}$-space there is a $\varphi \in \mathscr{L}(E_0, F)$ such that $\|\varphi\|\,\|\varphi^{-1}\| \leqslant A$. If $F_0 = \varphi(E_0)$ let $\psi \in \mathscr{L}(F_0, l_2^m)$ be such that $\|\psi\|\,\|\psi^{-1}\| \leqslant B$ where $m = \dim F_0$, and A and B are absolute constants. Then

$$Tx_i = \varphi^{-1}\psi^{-1}P_mJ_m\psi\varphi PTx_i \quad \text{for} \quad i = 1, \ldots, N.$$

By hypothesis $J_m\psi\varphi PT \in D_p(E, F)$ and so by (0.2) there is a constant M (independent of the x_i) such that

$$\sigma_p(J_m\psi\varphi pTx_i) \leqslant M\|J_m\|\,\|\psi\|\,\|\varphi\|\,\|T\|\,\alpha_p(x_i).$$

Thus

$$\sigma_p(Tx_i) \leqslant \|\varphi^{-1}\|\,\|\psi^{-1}\|\,\|P_m\|\,\sigma_p(J_m\psi\varphi PTx_i)$$

and so

$$\sigma_p(Tx_i) \leqslant AB\lambda M\|T\|\,\alpha_p(x_i)$$

i.e. T is strongly p-summing.

As a corollary to 2.4 we obtain a generalization of one of the main results of [24].

2.5. COROLLARY. *Let $E \in \mathscr{S}_2$ and F be infinite dimensional. Then $\langle E, F\rangle$ is not p-trivial for any $p \geqslant 1$.*

Proof. If $\langle E, F\rangle$ is p-trivial then $\langle E, l_2\rangle$ is p-trivial and so $\langle l_2, E^*\rangle$ is strongly p'-trivial. But $E^* \in \mathscr{S}_2$ and so by 2.4 $\langle l_2, l_2\rangle$ is strongly p'-trivial, which is a contradiction.

2.6. COROLLARY. *Let E and F be infinite dimensional spaces. Then there is an infinite dimensional subspace E_0 in E (F_0 in F) and a $T \in \mathscr{L}(E_0, F)$ [$S \in \mathscr{L}(E, F_0)$] such that $T \notin \Pi_p(E_0, F)$ [$S \notin D_p(E, F)$]. (In particular, $S' \notin \Pi_p(F_0^*, E^*)$ for any $p \geqslant 1$).*

Proof. By 2.5 we need only choose E_0 and F_0 to be $\mathscr{S}_2$-spaces. This is possible by (0.11).

Before giving our next result we recall the following fact: If Y is a finite dimensional subspace of l_q and $\varepsilon > 0$ then there is a subspace Z of l_q, $\dim Z = n < +\infty$, $Y \subset Z$, $d(Z, l_q^n) < 1+\varepsilon$ and a projection P of l_q onto Z with $\|P\| \leqslant 1+\varepsilon$. (If one replaces $1+\varepsilon$ by some suitable constant λ the same is true for any $\mathscr{L}_q$-space.)

2.7. THEOREM. *Let E be a Banach space and F a $\mathscr{D}_{q,\lambda}$-space for some $q > 1$. If $\langle E, F\rangle$ is p-trivial ($1 \leqslant p < \infty$) then $\langle E, l_q\rangle$ is p-trivial.*

Proof. Since $\langle E, F\rangle$ is p-trivial there is a $C > 0$ such that $\Pi_p(T) \leqslant C\|T\|$ for all $T \in \mathscr{L}(E, F)$. Let $S \in \mathscr{L}(E, l_q)$ and $x_1, \ldots, x_N$ be in E. Let Z be a subspace of l_q such that $\dim Z = n$, $d(Z, l_q^n) < 2$, $Z \supset [Tx_i\colon i \leqslant N]$ and such that there is a projection $P\colon l_q \to Z$ with $\|P\| \leqslant 2$. Let G be a subspace of F such that $d(G, l_q^n) \leqslant \lambda$. Then there is an isomorphism T from Z onto G with $\|T\| = 1$, $\|T^{-1}\| \leqslant 2\lambda$. Let $R \in \mathscr{L}(E, F)$ be defined by $= TPS$. Then $\|R\| \leqslant 2\|S\|$, $\Pi_p(R) \leqslant 2C\|S\|$. Thus if $R_0(S_0)$ denotes the restriction of $R(S)$ to $[x_i\colon i \leqslant N]$ then $\Pi_p(S_0) = \Pi_p(T^{-1}R_0) \leqslant \|T^{-1}\|\Pi_p(R) \leqslant 4C\lambda\|S\|$. Since $(x_i)_{i=1}^N$ was arbitrary $\Pi_p(S) \leqslant 4C\lambda\|S\|$ and hence $\langle E, l_q\rangle$ is p-trivial. Of course, 2.7 is a generalization of 2.1 (a).

2.8. COROLLARY. *Let E be a $\mathscr{S}_{q,\lambda}$-space and F a Banach space with F, F^* having the b.a.p. If $\langle E, F\rangle$ is strongly p-trivial ($1 < p \leqslant \infty$) then $\langle l_q, F\rangle$ is strongly p-trivial.*

Proof. If $\langle E, F\rangle$ is strongly p-trivial then by 1.3 $\langle F^*, E^*\rangle$ is p'-trivial. Clearly E^* is a $\mathscr{S}_{q',\lambda}$-space. Thus by 2.7 $\langle F^*, l_{q'}\rangle$ is p'-trivial and so $\langle l_q, F\rangle$ is strongly p-trivial.

The results of this section sheds some new light on a conjecture of Grothendieck. Recall that an operator $T \in \mathscr{L}(E, F)$ is *nuclear* if it can be represented in the form $Tx = \sum_{i=1}^{\infty} f_i(x)y_i$ with $(f_i) \subset E^*$, $(y_i) \subset F$ and $\sum \|f_i\| < +\infty$ and $\sup_n \|y_n\| < \infty$ [10]. Grothendieck [10] page 47, has conjectured that if every $T \in \mathscr{L}(E, F)$ is nuclear then $\min(\dim E, \dim F) < +\infty$.

It is known ([17] page 319), that to answer this problem of Grothendieck it is enough to show that under the above hypothesis F is an $\mathscr{L}_2$-space. With the following observation it follows from 2.4 that to answer this problem it is enough to show that F is an $\mathscr{S}_2$-space, a much weaker condition than the above.

2.9. Remark. *Let $T \in \mathscr{L}(E,F)$ be nuclear. Then $T \in \Pi_p(E,F)$ and $T \in D_p(E,F)$ for every p.*

Proof. The p-absolutely summing assertion is well-known. If $(x_i) \subset l_p(E)$, $(g_i) \subset l_{p'}[F^*]$ and $Tx = \sum_{j=1}^{\infty} f_j(x) y_j$ is a representation of T satisfying the above conditions, then

$$\sum_{i=1}^{\infty} |< Tx_i, g_i >| = \sum_{i=1}^{\infty} \Big| \sum_{j=1}^{\infty} f_j(x_i) g_i(y_j) \Big|$$
$$\leqslant \Big(\sum_{j=1}^{\infty} \|f_j\| \Big) \varepsilon_{p'}(g_i) \alpha_p(x_i) \sup_i \|y_i\| < +\infty.$$

Again the notion of a nuclear operator depends on both the domain and range. To overcome some of the difficulties in working with such operators, the notion of a fully nuclear operator was introduced. An operator $T \in \mathscr{L}(E,F)$ is *fully nuclear* [23] if the astricted operator T_a: $E \to T(E)$ is nuclear. Motivated by this we say that $T \in \mathscr{L}(E,F)$ is *fully p-summing* if the astriction T_a: $E \to T(E)$ is strongly p-summing. The proof of 2.9 shows that a fully nuclear operator is fully p-summing for any p. Denote the fully p-summing operators from E to F by $\mathfrak{F}_p(E,F)$. Clearly if $\mathscr{L}(E,F) = \mathfrak{F}_p(E,F)$ then $\mathscr{L}(E,F_0) = D_p(E,F_0)$ for any $F_0 \subset F$. Thus by 2.6 (choosing F_0 to be an $\mathscr{S}_2$-space) it follows that $\mathscr{L}(E,F) = \mathfrak{F}_p(E,F)$ if and only if $\min(\dim E, \dim F) < +\infty$. This together with the above remarks gives a new proof of [23].

§ 3. WEAKLY p-SUMMING SERIES AND p-TRIVIALITY

We now consider the implication 2.1(b). Using the same argument as [17] for 2.1(b) one can prove the following result.

3.0. Theorem. *If $\langle X, Y\rangle$ is p-trivial and $1 \leqslant p < 2$ then*

$$l_p[X] \subset l_{\frac{2p}{2-p}}(X).$$

Theorem 3.7 below shows that there is no comparable result for $p > 2$ and arbitrary Y. Of course the proof of 3.0 rests on the fact that every infinite dimensional Banach space is a $\mathscr{D}_2$-space. We show in 3.11 below that there is a true generalization of 2.1(b). An immediate corollary to 3.0 is the following result.

Corollary. *If $\langle X, Y\rangle$ is strongly p-trivial, $p > 2$ then*

$$l_{p'}[Y^*] \subset l_{\frac{2p'}{2-p'}}(Y^*).$$

Theorem 3.0 above shows that the result of Grothendieck [12] is the best possible. We now outline a known technique for constructing basic sequences. This is then used to generalize a result of Grothendieck [12].

Let P and Q be linear subspaces of a Banach space X. The *inclination* of P and Q is defined to be

$$I(P;Q) = \inf\{\|x+y\| \,|\, x \in P,\ \|x\| = 1,\ y \in Q\}.$$

Let (x_i) be a sequence of elements in a Banach space X and let $L(x_i)_{i=n}^m$ be the finite dimensional subspace spanned by $\{x_n, x_{n+1}, \ldots, x_m\}$. We define the *index* of the ordered n-tuple $(x_i)_{i=1}^n$ to be

$$\theta(x_i)_{i=1}^n = \min\{I(L(x_i)_{i=1}^p; L(x_i)_{i=p+1}^n) \,|\, 1 \leqslant p < n\}.$$

The *index* of the sequence (x_i) is defined by

$$\theta(x_i)_{i=1}^{\infty} = \inf\{\theta(x_i)_{i=1}^n \,|\, n \geqslant 1\}.$$

3.1. Proposition. *Let X be a Banach space and (x_i) a sequence in X. Then (x_i) is basic if and only if $\theta(x_i)_{i=1}^{\infty} > 0$.*

3.2. Theorem. *Given $\varepsilon > 0$ and a finite dimensional subspace P of an infinite dimensional Banach space X, there exists an infinite dimensional subspace Q of X such that*

$$I(P;Q) > 1 - \varepsilon.$$

3.3. Theorem. *Let $(x_i)_{i=i_{k-1}+1}^{i_k}$ be a basis for P_k, where $(i_k)_{k=1}^{\infty}$ is an increasing sequence of positive integers. If*

$$\theta(x_i)_{i_{k-1}+1}^{i_k} \geqslant \alpha > 0$$

for each k, and for any integers m, n

$$I(P_1 \oplus \ldots + P_m, P_{m+1} \oplus \ldots \oplus P_{m+n}) \geqslant \beta > 0,$$

then (x_i) is basic and

$$\theta(x_i)_{i=1}^{\infty} \geqslant \frac{\alpha\beta^2}{2+\alpha}.$$

These results are essentially due to Gurarii [13].

Next we observe some properties of sequences in infinite dimensional Banach spaces.

3.4. Lemma. *Let X be an infinite dimensional Banach space. Then for any positive integer n and any $\varepsilon > 0$ there is a sequence $(y_i)_{i=1}^n$ in X such that $\|y_i\| = 1$ and such that if $(\lambda_i)_{i=1}^n$ is any collection of real numbers,*

$$\Big\| \sum_{i=1}^{n} \lambda_i y_i \Big\| \leqslant (1+\varepsilon) \Big(\sum_{i=1}^{n} \lambda_i^2 \Big)^{1/2}$$

and

$$\theta(y_i)_{i=1}^n \geqslant \frac{1}{1+\varepsilon}.$$

Proof. Let n be a positive integer and $\varepsilon > 0$. Then by (0.9) there is a subspace U of X such that $d(U, l_2^n) < 1+\varepsilon$. Thus there is an isomorphism T from l_2^n onto U such that $\|T\| = 1$ and $\|T^{-1}\| \leqslant 1+\varepsilon$. If $(e_i)_{i=1}^n$ is the unit vector basis for l_2^n, let $x_i = T(e_i)$ for each i. Then

$$\|T^{-1}\|\cdot\|x_i\| = \|T^{-1}\|\cdot\|T(e_i)\| \geqslant \|T^{-1}T(e_i)\| = \|e_i\|.$$

This implies that $\|x_i\| \geqslant \frac{1}{1+\varepsilon}$. Let $y_i = \frac{x_i}{\|x_i\|}$ for each i, and consider $(\lambda_i)_{i=1}^n$. Then

$$\begin{aligned}\Big\|\sum_{i=1}^n \lambda_i y_i\Big\| &= \Big\|\sum_{i=1}^n \lambda_i \frac{x_i}{\|x_i\|}\Big\| = \Big\|\sum_{i=1}^n \frac{\lambda_i}{\|x_i\|} T(e_i)\Big\| \\ &= \Big\|T\Big(\sum_{i=1}^n \frac{\lambda_i e_i}{\|x_i\|}\Big)\Big\| \leqslant \|T\|\cdot\Big\|\sum_{i=1}^n \frac{\lambda_i e_i}{\|x_i\|}\Big\|_{l_2^n} \\ &= \Big\|\sum_{i=1}^n \frac{\lambda_i}{\|x_i\|} e_i\Big\|_{l_2^n} = \Big(\sum_{i=1}^n \Big(\frac{|\lambda_i|}{\|x_i\|}\Big)^2\Big)^{1/2} \\ &\leqslant \frac{1}{\Big(\frac{1}{1+\varepsilon}\Big)}\Big(\sum_{i=1}^n |\lambda_i|^2\Big)^{1/2} = (1+\varepsilon)\Big(\sum_{i=1}^n \lambda_i^2\Big)^{1/2}.\end{aligned}$$

Thus $\|\sum_{i=1}^n \lambda_i y_i\| \leqslant (1+\varepsilon)\,(\sum_{i=1}^n \lambda_i^2)^{1/2}$, and $(y_i)_{i=1}^n$ is the desired sequence if we show that $\theta(y_i)_{i=1}^n \geqslant \frac{1}{1+\varepsilon}$. We do this by showing that $\theta(x_i)_{i=1}^n \geqslant \frac{1}{1+\varepsilon}$.

Let $1 < p < n$, $w \in L(x_i)_{i=1}^p$, and $z \in L(x_i)_{i=p+1}^n$. Then $w = \sum_{i=1}^p a_i T(e_i)$, $z = \sum_{p+1}^n a_i T(e_i)$. and $\|z+w\| = \|T(\sum_{i=1}^n a_i e_i)\|$; hence

$$\begin{aligned}(1+\varepsilon)\|z+w\| &\geqslant \|T^{-1}\|\cdot\Big\|T\Big(\sum_{i=1}^n a_i e_i\Big)\Big\| \geqslant \Big\|\sum_{i=1}^n a_i e_i\Big\|_{l_2^n} \\ &\geqslant \Big\|\sum_{i=1}^p a_i e_i\Big\|_{l_2^n} = \|T\|\cdot\Big\|\sum_{i=1}^n a_i e_i\Big\|_{l_2^n} \geqslant \Big\|T\Big(\sum_{i=1}^p a_i e_i\Big)\Big\| = \|w\|.\end{aligned}$$

Thus

$$\|z+w\| \geqslant \frac{1}{1+\varepsilon}\|w\|.$$

Consequently, $\theta(x_i)_{i=1}^n \geqslant 1+\varepsilon$, and hence $\theta(y_i)_{i=1}^n \geqslant 1+\varepsilon$. This completes the proof.

3.5. Lemma. *Let (x_i) be a sequence in a Banach space X such that*

$$(*)\qquad \Big\|\sum_{i=1}^\infty \lambda_i x_i\Big\| \leqslant \Big(\sum_{i=1}^\infty (\lambda_i)^2\Big)^{1/2}(1+\varepsilon)$$

for every square summable sequence (λ_i). Then

$$\frac{1}{1+\varepsilon} \geqslant \sup\Big\{\Big(\sum_{i=1}^\infty |x^*(x_i)|^p\Big)^{1/p} \Big|\ x^* \in X^*, \|x^*\| \leqslant 1\Big\} \quad \textit{for all } p \geqslant 2.$$

Proof. Define a linear operator S from l_2 into X by $S((\lambda_i)) = \sum_{i=1}^\infty \lambda_i x_i$ for each $(\lambda_i) \in l_2$. Then by $(*)$

$$\|S((\lambda_i))\| = \Big\|\sum_{i=1}^\infty \lambda_i x_i\Big\| \leqslant (1+\varepsilon)\Big(\sum_{i=1}^\infty \lambda_i^2\Big)^{1/2} \leqslant (1+\varepsilon)\|(\lambda_i)\|_{l_2}.$$

Hence S is continuous and $\|S\| \leqslant 1+\varepsilon$. Now consider S^* from X^* into l_2. Then

$$(S^*(x^*))(\lambda_i) = x^*(S((\lambda_i))) = x^*\Big(\sum_{i=1}^\infty \lambda_i x_i\Big) = \sum_{i=1}^\infty \lambda_i x^*(x_i).$$

Since (λ_i) is arbitrary in l_2, $S^*(x^*) = (x^*(x_i))_{i=1}^\infty$.

$$\begin{aligned}(1+\varepsilon) \geqslant \|S\| &= \|S^*\| = \sup\{\|S^*(x^*)\|_{l_2} \mid \|x^*\| \leqslant 1\} \\ &= \sup\Big\{\Big(\sum_{i=1}^\infty |x^*(x_i)^2\Big)^{1/2} \Big|\ \|x^*\| \leqslant 1\Big\} \geqslant \sup\Big\{\sum_{i=1}^\infty |x^*(x_i)|^p)^{1/p} \Big|\ \|x^*\| \leqslant 1\Big\}.\end{aligned}$$

Hence $(1+\varepsilon) \geqslant \sup\{(\sum_{i=1}^\infty |x^*(x_i)|^p)^{1/p} \mid \|x^*\| \leqslant 1\}$ for all $p \geqslant 2$ and the proof is complete.

3.6. Corollary. *If X is an infinite dimensional Banach space and $\varepsilon > 0$ then there is a sequence $(y_i)_{i=1}^n$ in X such that $\|y_i\| = 1$, $\theta(y_i)_{i=1}^n \geqslant \frac{1}{1+\varepsilon}$, and*

$$(1+\varepsilon) \geqslant \sup\Big\{\Big(\sum_{i=1}^n |x^*(y_i)|^2\Big)^{1/2}:\ \|x^*\| \leqslant 1\Big\}.$$

3.7. Theorem. *If X is an infinite dimensional Banach space and (a_i) is an element of c_0 with $0 < a_i < 1$, then there is a basic sequence (x_i) in X such that $\|x_i\| = a_i$ for all i and*

$$1 \geqslant \sup\Big\{\Big(\sum_{i=1}^\infty |x^*(x_i)|^2\Big)^{1/2}:\ x^* \in X^*, \|x^*\| \leqslant 1\Big\}.$$

Proof. Let $\alpha = \frac{1}{2}(1-\sup_i(a_i))$. Then

(a) $$1-2\alpha = \sup_i(a_i)$$

and $1 > \alpha > 0$ since $a_i \to 0$ and $1 > a_i$. For each positive integer k select i_k, increasing with k, such that

(b) $$\text{if } i > i_k \text{ then } a_i \leqslant \frac{\alpha}{2^{k+1}}.$$

Let $\varepsilon = \dfrac{\alpha}{1-\alpha}$. Then $\varepsilon > 0$ and $1+\varepsilon = \dfrac{1}{(1-\alpha)}$. Since X is infinite dimensional, we can select $(y_i)_{i=1}^{i_1}$ such that $\|y_i\| = 1$,

$$\theta(y_i)_{i=1}^{i_1} \geqslant \frac{1}{1+\varepsilon} = 1-\alpha,$$

and $\dfrac{1}{1-\alpha} = 1+\varepsilon \geqslant \sup\{(\sum_{i=1}^{i_1}|x^*(y_i)|^2)^{1/2}:\ \|x^*\| \leqslant 1\}$ (this is possible by Corollary 3.6). Let $x_i = a_i y_i$ for $1 \leqslant i \leqslant i_1$. Then $\theta(x_i)_{i=1}^{i_1} \geqslant 1-\alpha$ and $\|x_i\| = a_i$. For convenience, let $A_1 = (z_i)_{i=1}^{\infty}$, where $z_i = x_i$ for $1 \leqslant i \leqslant i_1$ and $z_i = 0$ otherwise. Let $\varepsilon_2(A_1) = \sup\{(\sum_{i=1}^{\infty}|x^*(z_i)|^2)^{1/2}:\ \|x^*\| \leqslant 1\}$. Then

$$\begin{aligned}\varepsilon_2(A_1) &= \sup\Big\{\Big(\sum_{i=1}^{\infty}|x^*(z_i)|^2\Big)^{1/2}:\ \|x^*\| \leqslant 1\Big\}\\ &= \sup\Big\{\Big(\sum_{i=1}^{i_1}|x^*(x_i)|^2\Big)^{1/2}:\ \|x^*\| \leqslant 1\Big\}\\ &= \sup\Big\{\Big(\sum_{i=1}^{i_1}|x^*(a_i y_i)|^2\Big)^{1/2}:\ \|x^*\| \leqslant 1\Big\}\\ &\leqslant (\sup_i a_i)\Big(\sup\Big\{\Big(\sum_{i=1}^{i_1}|x^*(y_i)|^2\Big)^{1/2}:\ \|x^*\| \leqslant 1\Big\}\Big)\\ &\leqslant (1-2\alpha)\Big(\frac{1}{1-\alpha}\Big)\end{aligned}$$

(the last inequality is due to (a) and the selection of $(y_i)_{i=1}^{i_1}$). Thus $\varepsilon_2(A_1) \leqslant (1-2\alpha)\Big(\dfrac{1}{1-\alpha}\Big)$.

Now select a sequence (ε_i) with $0 < \varepsilon_i < 1$ and $\prod_{i=1}^{\infty}(1-\varepsilon_i) = \beta$ for some $\beta > 0$, and let $E_1 = [(x_i):\ i \leqslant i_1]$. Then by Theorem 3.2 there is an infinite dimensional subspace of X, say F_2, such that $I(E_1, F_2) > 1-\varepsilon_1$. Since F_2 is infinite dimensional there is, by Corollary 3.6 a collection $(y_i)_{i=i_1+1}^{i_2}$ in F_2 such that

$$\|y_i\| = 1, \quad \theta(y_i)_{i=i_1+1}^{i_2} \geqslant 1-\alpha,$$

and

$$\frac{1}{1-\alpha} \geqslant \sup\Big\{\Big(\sum_{i=i_1+1}^{i_2}|x^*(y_i)|^2\Big)^{1/2}:\ \|x^*\| \leqslant 1\Big\}.$$

Again letting $x_i = a_i y_i$, we get $\|x_i\| = a_i$, $\theta(x_i)_{i=i_1+1}^{i_2} \geqslant 1-\alpha$. If $A_2 = (z_i)_{i=1}^{\infty}$, where $z_i = x_i$ for $i_1+1 \leqslant i \leqslant i_2$, and $z_i = 0$ otherwise, then

$$\begin{aligned}\varepsilon_2(A_2) &= \sup\Big\{\Big(\sum_{i=1}^{\infty}|x^*(z_i)|^2\Big)^{1/2}:\ \|x^*\| \leqslant 1\Big\}\\ &= \sup\Big\{\Big(\sum_{i=i_1+1}^{i_2}|x^*(x_i)|^2\Big)^{1/2}:\ \|x^*\| \leqslant 1\Big\}\\ &= \sup\Big\{\Big(\sum_{i=i_1+1}^{i_2}|x^*(x_i)|^2\Big)^{1/2}:\ \|x^*\| \leqslant 1\Big\}\\ &\leqslant \Big(\sup_{i_1<i\leqslant i_2}(a_i)\Big)\Big(\sup\Big\{\Big(\sum_{i=i_1+1}^{i_2}|x^*(y_i)|^2\Big)^{1/2}:\ \|x^*\| \leqslant 1\Big\}\Big)\\ &\leqslant \Big(\frac{\alpha}{2^2}\Big)\Big(\frac{1}{1-\alpha}\Big)\end{aligned}$$

(the latter inequality is due to (b) and the selection of $(y_i)_{i=i_1+1}^{i_2}$). Thus $\varepsilon_2(A_2) \leqslant \dfrac{\alpha}{2^2}\Big(\dfrac{1}{1-\alpha}\Big)$. Let $E_2 = [(x_i):\ i \leqslant i_2]$ and choose an infinite dimensional subspace F_3 of X such that $I(E_2, F_3) \geqslant (1-\varepsilon_2)$. In the same manner as above we get a sequence $(x_i)_{i=i_2+1}^{i_3}$ in F_3 such that $\|x_i\| = a_i$, $\theta(x_i)_{i=i_2+1}^{i_3} \geqslant 1-\alpha$, and $\varepsilon_2(A_3) \leqslant \Big(\dfrac{\alpha}{2^3}\Big)\Big(\dfrac{1}{1-\alpha}\Big)$, where $A_3 = (z_i)_{i=1}^{\infty}$ with $z_i = x_i$ for $i_2+1 \leqslant i \leqslant i_3$ and $z_i = 0$ otherwise.

Continuing in this manner we get for each k an infinite dimensional subspace F_k and a collection $(x_i)_{i=i_{k-1}+1}^{i_k}$ in F_k such that $\|x_i\| = a_i$, $\theta(x_i)_{i=i_{k-1}+1}^{i_k} \geqslant 1-\alpha$, and $\varepsilon_2(A_k) \leqslant \Big(\dfrac{\alpha}{2^k}\Big)\Big(\dfrac{1}{1-\alpha}\Big)$, where $A_k = (z_i)_{i=1}^{\infty}$ with $z_i = x_i$ for $i_{k-1}+1 \leqslant i \leqslant i_k$ and $z_i = 0$ otherwise, and $I(E_{k-1}, F_k) \geqslant 1-\varepsilon_{k-1}$ where $E_{k-1} = [x_i:\ i \leqslant i_{k-1}]$.

Now,

$$\varepsilon_2\big((x_i)_{i=1}^{\infty}\big) = \sup\Big\{\Big(\sum_{i=1}^{\infty}|x^*(x_i)|^2\Big)^{1/2}:\ \|x^*\|\leqslant 1\Big\}\leqslant \sum_{i=1}^{\infty}\varepsilon_2(A_i)$$

$$= \varepsilon_2(A_1)+\sum_{j=2}^{\infty}\varepsilon_2(A_j)\leqslant\Big[(1-2a)\Big(\frac{1}{1-a}\Big)+\sum_{k=2}^{\infty}\frac{a}{2^k}\Big(\frac{1}{1-a}\Big)\Big]$$

$$=\Big(\frac{1}{1-a}\Big)\Big[1-2a+\sum_{k=2}^{\infty}\frac{a}{2^k}\Big]\leqslant\Big(\frac{1}{1-a}\Big)[1-a]=1.$$

Thus $1\geqslant \sup\{(\sum_{i=1}^{\infty}|x^*(x_i)|^2)^{1/2}:\ \|x^*\|\leqslant 1\}$ and (x_i) is the desired sequence. This completes the proof.

The above proof, combining ideas of Gurarii [13], Grothendieck [12] and Dvoretzky [6] is very useful for constructing operators of specific types.

To illustrate the technique we prove the following theorem.

3.8a. THEOREM. *Let X be a Banach space, $2\leqslant p<\infty$, and $\varepsilon>0$. If Y is an $\mathscr{L}_{2p+\varepsilon}$-space and $\langle X, Y\rangle$ is p-trivial then X is finite dimensional.*

Proof. Since every infinite dimensional $\mathscr{L}_q$-space contains a complemented copy of l_q for $1\leqslant q<\infty$ it suffices to prove the theorem for $Y=l_{2p+\varepsilon}$.

Thus choose $(a_i)\in l_{2p+\varepsilon}\setminus l_{2p}$ with $0<a_i<1$ (e.g. $\Big(\dfrac{1}{(1+n)^{1/2p}}\Big)_{n=1}^{\infty}$). Since $(a_i)\in c_0$, by Theorem 3.7 there is a basic sequence (x_i) such that $\|x_i\|=a_i$ for each i and $\sum_{i=1}^{\infty}|x^*(x_i)|^p<\infty$ for every $x^*\in X^*$. Let (f_i) be a sequence of norm-preserving extensions to all of X of the associated sequence of coefficient functionals. Then by (0.1)

$$\frac{1}{a_i}\leqslant\|f_i\|\leqslant\frac{2K}{a_i}.$$

Let $y_i=\dfrac{a_i}{\|f_i\|}e_i$ for each i, where (e_i) is the unit vector basis for $l_{2p+\varepsilon}$. Define $T_n(x)=\sum_{i=1}^{n}f_i(x)y_i$. Then for each $x\in X$,

$$\|T_n(x)\|_{l_{2p+\varepsilon}}=\Big\|\sum_{i=1}^{n}f_i(x)y_i\Big\|_{l_{2p+\varepsilon}}=\Big\|\sum_{i=1}^{n}\frac{f_i(x)}{\|f_i\|}a_ie_i\Big\|_{l_{2p+\varepsilon}}$$

$$=\Big(\sum_{i=1}^{n}\Big|\frac{f_i(x)a_i}{\|f_i\|}\Big|^{2p+\varepsilon}\Big)^{1/2p+\varepsilon}\leqslant\|x\|\Big(\sum_{i=1}^{n}|a_i|^{2p+\varepsilon}\Big)^{1/2p+}$$

$$\leqslant\|x\|\Big(\sum_{i=1}^{\infty}|a_i|^{2p+\varepsilon}\Big)^{1/2p+\varepsilon}.$$

Hence $\|T_n(x)\|_{l_{2p+\varepsilon}}\leqslant\|x\|\cdot\|(a_i)\|_{l_{2p+\varepsilon}}$, and (T_n) is pointwise bounded. If $n>m$ and $x\in X$, then

$$\|T_n(x)-T_m(x)\|_{l_{2p+\varepsilon}}=\Big\|\sum_{i=m}^{n}f_i(x)y_i\Big\|_{l_{2p+\varepsilon}}\leqslant\Big(\sum_{m+1}^{n}|a_i|^{2p+\varepsilon}\Big)^{1/2p+\varepsilon}.$$

Thus $(T(x_n))$ is Cauchy for each x since $(a_i)\in l_{2p+\varepsilon}$. Thus by the Banach–Steinhaus theorem $T(x)=\sum_{i=1}^{\infty}f_i(x)y_i$ is continuous. For $N\in\omega$,

$$\sum_{k=1}^{N}\|T(x_k)\|_{l_{2p+\varepsilon}}^p=\sum_{k=1}^{N}\Big\|\sum_{i=1}^{\infty}\frac{f_i(x_k)a_ie_i}{\|f_i\|}\Big\|_{l_{2p+\varepsilon}}^p$$

$$=\sum_{k=1}^{N}\Big\|\frac{a_ke_k}{\|f_k\|}\Big\|_{l_{2p+\varepsilon}}^p\geqslant\sum_{k=1}^{N}\frac{a_k^p}{\Big(\dfrac{2K}{a_k}\Big)^p}=\frac{1}{(2K)^p}\sum_{k=1}^{N}a_i^{2p}.$$

Now $\sum_{k=1}^{\infty}\|T(x_k)\|^p=+\infty$ since $(a_n)\notin l_{2p}$, and so T is not p-absolutely summing. This is a contradiction; hence X must be finite dimensional. This completes the proof.

A stronger theorem is true for $1\leqslant p\leqslant 2$. However, the method of proof used in 3.8a is no longer valid.

3.8b. THEOREM. *Let $q>2$ and $1\leqslant p\leqslant 2$. If Y is an $\mathscr{L}_q$-space and $\langle X, Y\rangle$ is p-trivial then X is finite dimensional.*

Proof. Since Y contains a complemented subspace isomorphic to l_q we may suppose $Y=l_q$. Since a p-summing operator is r-summing for $r\geqslant p$ [21] we may suppose $p=2$. Suppose X is infinite dimensional. By (0.9) we can then find for each integer n a subspace X_n of X, $\dim X_n=n$ and $d(X_n, l_2^n)<1+\varepsilon$. Let $x_1,\ldots,x_n$ correspond under this ε-isometry to the unit vector basis $u_1,\ldots,u_n$ of l_2^n and let $e_1,\ldots,e_n$ denote the unit vector basis of l_∞^n. Also let $f_1,\ldots,f_n$ denote Hahn–Banach extensions of the coefficient functionals of $x_1,\ldots,x_n$ to all of X and let (v_n) denote the unit vector basis of l_q. Choose $(\delta_i)\in l_q$ with $\sum_{i=1}^{\infty}|\delta_i|^q=1$ and $\sum_{i=1}^{\infty}|\delta_i|^2=+\infty$. Let $(\beta_i)\in l_2$ with $\sum\beta_i^2=1$. Consider

$$l_\infty^n\xrightarrow{R_n}l_2^n\xrightarrow{T_n}X\xrightarrow{S_n}l_q,$$

where $R_n\big((\lambda_i)\big)_{i=1}^n=\sum\lambda_i\beta_iu_i$, T_n is the ε-isometry of (0.9) and $S_n(x)=\sum_{i=1}^{n}f_i(x)v_i$. Since, by hypothesis, $\langle X, l_q\rangle$ is 2-trivial and by [17] $\langle l_\infty, l_2\rangle$ is 2-trivial there is a constant $C>0$ such that

$$\Pi_2(S_n)\leqslant C\|S_n\|\quad\text{and}\quad\Pi_2(R_n)\leqslant C\|R_n\|$$

for all n. Thus by Theorem 4 of [21],

$$\Pi_1(S_n T_n R_n) \leqslant \|T_n\| \Pi_2(S_n) \Pi_2(R_n) \leqslant (1+\varepsilon) C^2 = M.$$

Since the domain of $S_n T_n R_n$ is l_∞^n it is easily checked that $\Pi_1(S_n T_n R_n) = \sum_{i=1}^n |\delta_i \beta_i|$. Thus $\sum_{i=1}^n |\delta_i \beta_i| \leqslant M$ for all n and it follows that $(\delta_i) \in l_2$, contradicting our assumption. Thus X is finite dimensional.

Actually, by 2.7, Y need only be a $\mathscr{D}_{2p+\varepsilon}$ space in 3.8a. Since a $\mathscr{D}_\infty$-space is a $\mathscr{D}_p$-space for all $p \geqslant 1$ and a $\mathscr{S}_\infty$-space we obtain the following corollary:

3.9. Corollary. (a) *If X is a Banach space and Y is a $\mathscr{D}_\infty$-space then $\langle X, Y\rangle$ is p-trivial $(1 \leqslant p < \infty)$ if and only if X is finite dimensional.*

(b) *If X^* is a $\mathscr{S}_1$-space and Y^* has the b.a.p. then $\langle X, Y\rangle$ is strongly p-trivial $(1 < p \leqslant \infty)$ if and only if Y is finite dimensional.*

The proof of (a) is immediate from the above discussion and (b) follows from 2.8 and 1.3.

Using the argument of 3.8a with e.g. $a_n = (\ln(n+1))^{-1}$ one can constructively prove the following result.

3.10. Theorem. (a) *Let X be a Banach space and Y an infinite dimensional $\mathscr{L}_\infty$-space. If $1 \leqslant p < \infty$ and $\langle X, Y\rangle$ is p-trivial then X is finite dimensional.*

(b) *Let X be an infinite dimensional $\mathscr{S}_1$-space, Y a Banach space with the b.a.p. and $1 < p \leqslant \infty$. If $\langle X, Y\rangle$ is strongly p-trivial then Y is finite dimensional.*

We now prove the generalization of 2.1(b).

3.11. Theorem. *Let Y be a $\mathscr{D}_{pq}$-space, $1 \leqslant p < \infty$, $q > 1$ and X a Banach space. If $\langle X, Y\rangle$ is p-trivial then $l_p[X]$ is contained in $l_{pq'}(X)$.*

Proof. Since $\langle X, Y\rangle$ is p-trivial by Theorem 2.7 $\langle X, l_{pq}\rangle$ is p-trivial. Let (λ_i) be a sequence of real numbers such that $\sum_{i=1}^\infty |\lambda_i|^q = 1$ and let $(x_i) \in l_p[X]$. Define $T \in \mathscr{L}(X, l_{pq})$ by $T(x) = \sum_{i=1}^\infty (\lambda_i)^{1/p} (x_i^*(x)) e_i$ where each x_i^* is chosen in X^* in such a way that $\|x_i^*\| = 1$ and $x_i^*(x_i) = \|x_i\|$ and (e_i) is the unit vector basis in l_{pq}. Then

$$\|T(x)\|_{l_{pq}} \leqslant \Big(\sum_{i=1}^\infty |\lambda_i^{1/p}|^{pq} \|x_i^*\|^{pq}\Big)^{1/pq} \|x\| = \Big(\sum_{i=1}^\infty \lambda_i^q\Big)^{1/pq} \|x\| = \|x\|.$$

Hence $\|T\| \leqslant 1$ and

$$\|T(x_i)\|_{l_{pq}} = \Big\|\sum_{k=1}^\infty \lambda_k^{1/p} x_k^*(x_i) e_k\Big\|_{l_{pq}} \geqslant |\lambda_i|^{1/p} \|x_i\|$$

for each i. Hence

$$\Big(\sum_{i=1}^\infty \lambda_i \|x_i\|^p\Big)^{1/p} \leqslant \Big(\sum_{i=1}^\infty \|T(x_i)\|^p\Big)^{1/p} \leqslant \pi_p(T) \sup\Big\{\Big(\sum_{i=1}^\infty |x^*(x_i)|^p\Big)^{1/p}: \|x^*\| \leqslant 1\Big\}$$
$$\leqslant M \varepsilon_p(x_i) \|T\| \leqslant M \varepsilon_p(x_i).$$

Thus $\sum_{i=1}^\infty \lambda_i \|x_i\|^p < \infty$ for every sequence of real numbers (λ_i) in l_q with $\|(\lambda_i)\|_q = 1$. Thus $(\|x_i\|^p)_{p=1}^\infty \in (l_q)^* = l_{q'}$. Thus $\sum_{i=1}^\infty \|x_i\|^{pq'} < \infty$.

Letting $p = 1$, $q = 2$. We see that 2.1(b) is an immediate consequence of 3.11.

We now give the dual result.

3.12. Corollary. *Let $1 \leqslant p \leqslant \infty$, $q > 1$ and let X be a $\mathscr{S}_{(pq)'}$-space and Y^* have the b.a.p. Then, if $\langle X, Y\rangle$ is strongly p'-trivial then $l_p[Y^*]$ is contained in $l_{pq'}(Y^*)$.*

Unfortunately the hypotheses of 3.11 and 3.12 are rarely met.

3.13. Corollary. *If Y is a $\mathscr{D}_{pq}$-space $p \geqslant 2$, $q > 1$, X has the b.a.p. and $\langle X, Y\rangle$ is p-trivial then X is finite dimensional.*

Proof. By 3.7, if $\dim X = +\infty$ there is an $(x_i) \in l_p[X]$ with $\|x_i\| = \dfrac{1}{\ln(i+1)}$, i.e. $(x_i) \notin l_q(X)$ for any $q \geqslant 1$. This contradicts 3.11.

§ 4. REMARKS AND UNSOLVED PROBLEMS

In this section we make some remarks concerning the preceeding section and raise some unsolved problems.

Remark 1. *For $p \geqslant 2$ there is no analogue of 2.1(c).*

Indeed $\langle l_\infty, l_q\rangle$ is p-trivial for $1 \leqslant q \leqslant 2$ and any $p \geqslant 2$. However by 3.10 if $\langle X, l_\infty\rangle$ is γ-trivial for any γ then $\dim X < +\infty$.

We do not know if there is any analogue for $p < 2$. This question is closely related to the following problem.

Problem 1. ([17] p. 319.) *If $\langle X, Y\rangle$ is p-trivial for some fixed p, $1 < p < 2$ is $\langle X, Y\rangle$ 1-trivial?*

A somewhat weaker question is the following.

Problem 2. *If $\langle X, Y\rangle$ is p-trivial for some fixed p, $1 < p < 2$ is $\langle c_0, X\rangle$ 2-trivial?*

Of course there is a dual to 2.1(c) for strongly ∞-summing operators.

Remark 2. *If $\langle X, Y\rangle$ is strongly ∞-trivial then for any $\mathscr{L}_1$-space Z, $\langle Y, Z\rangle$ is strongly 2-trivial, whenever Y' has the b.a.p.*

Indeed if $\langle X, Y\rangle$ is strongly ∞-summing then $\langle Y', X'\rangle$ is 1-trivial and so $\langle Z', Y'\rangle$ is 2-trivial. This is true since Z' is an $\mathcal{L}_\infty$-space [18]. But then $\langle Y, Z\rangle$ is strongly 2-trivial.

There are obvious problems analogous to Problems 1 and 2 for strongly p-summing operators.

An affirmative answer to the following problem would settle several of the outstanding problems concerning p-trivial spaces.

PROBLEM 3. *Is every infinite dimensional Banach space either a $\mathcal{S}_1$, $\mathcal{S}_2$ or $\mathcal{S}_\infty$-space?*

In our next remark we summarize what is known about p-triviality and $\mathcal{L}_q$-spaces.

REMARK 3. *First recall that if X is an $\mathcal{L}_\infty$-space and Y and $\mathcal{L}_q$-space $1 \leqslant q \leqslant 2$ then $\langle X, Y\rangle$ is p-trivial for all $p \geqslant 2$. We show that this is false for $1 \leqslant p < 2$. Indeed if X is an $\mathcal{L}_\infty$-space and $1 \leqslant p < 2$ then if $\langle X, Y\rangle$ is p-trivial,* $\dim Y < +\infty$.

Indeed if $\langle X, Y\rangle$ is p-trivial, $\langle X, l_2\rangle$ is p-trivial. Since X is an $\mathcal{L}_\infty$-space, X^* is an $\mathcal{L}_1$-space [18] and so ([17] Prop. 7.3, p. 311) X^* has a complemented subspace isomorphic to l^1. But then X^{**} contains an isomorph of c_0.

Choose $(a_i) \in l_2 \setminus l_p$, $a_i > 0$ and define T by $T(x^{**}) = \sum_{i=1}^{\infty} x^{**}(f_i) a_i e_i$, where (f_i) corresponds to the unit vector basis of l_1 and (e_i) is the unit vector basis of l_2. Then $\|Tx^{**}\| \leqslant \|x^{**}\| \sup_i \|f_i\| \big(\sum_{i=1}^{\infty} a_i^2\big)^{1/2}$ and T is continuous. Define $S \in \mathcal{L}(X, l_2)$ by $Sx = \sum f_i(x) a_i e_i$ then clearly $S^{**} = T$. If (u_i) corresponds to the unit vector basis of c_0 in X^{**} then (u_i) is weakly p-summing. But $\|Tu_i\|^p = a_i^p$ and so $\sum_{i=1}^{\infty} \|Tu_i\|^p = +\infty$, i.e. T is not absolutely p-summing. By [21] S is not p-absolutely summing, contradicting our hypothesis.

If X is an $\mathcal{L}_p$-space $1 < p < \infty$ and $\langle X, Y\rangle$ is q-trivial for some $q \geqslant 1$ then $\dim Y < +\infty$.

Indeed and $\mathcal{L}_p$-space for $1 < p < \infty$ is an $\mathcal{S}_2$-space [24]. Thus by 2.5, $\dim Y < +\infty$.

The analogous results for $\mathcal{L}_1$-spaces and for the strongly p-trivial case is covered in the above or the unsolved problems.

Finally we mention the following striking result of S. Kwapień [26]: *Let E an $\mathcal{L}_\infty$-space and F an $\mathcal{L}_p$-space, $p > 2$. Then*

(i) *$\langle E, F\rangle$ is q-trivial for any $q > p$; and*

(ii) *$\langle E, F\rangle$ is not p-trivial.*

We end this paper with some remarks on Hilbertian operators.

Lindenstrauss and Pełczyński ([17] Prop. 5.2, p. 294) have shown that the property of being a Hilbertian operator is a local property of the domain.

More precisely, *an operator T from X to Y is Hilbertian if and only if there is a constant C such that for every finite dimensional subspace E of X there are operators $U_{T,E}\colon E \to l_2$ and $V_{T,E}\colon l_2 \to Y$ such that $V_{T,E} U_{T,E}$ is the restriction of T to E and $\|V_{T,E}\| \, \|U_{T,E}\| \leqslant C$.*

Motivated by this result we define the *Hilbertian Constant* $h = h(X, Y)$ of the Banach spaces X and Y by

$$h(X, Y) < \sup_{\|T\| \leqslant 1} \varrho(T),$$

where $\varrho(T)$ is the infimum of the constants C satisfying the above.

It follows from [17] that if X is an $\mathcal{L}_{\infty,\lambda}$ and Y an $\mathcal{L}_{p,\sigma}$-space $1 \leqslant p \leqslant 2$ then $h(X, Y) \leqslant \lambda\sigma K_G$, where K_G is the Grothendieck constant [17].

From the definition it is clear that if X or Y is isomorphic to a Hilbert space then $h(X, Y) \leqslant d(Z, H)$ where Z is the isomorph of Hilbert space H. We now prove that if every $T \in \mathcal{L}(X, Y)$ *is Hilbertian then* $h(X, Y) < +\infty$.

First observe that if every $T \in \mathcal{L}(X, Y)$ is Hilbertian the same is true for any complemented subspace X_0 of X, in particular for the subspaces of finite co-dimension in X. By this remark and the Lindenstrauss–Pełczyński theorem above if $h(X, Y) = +\infty$, we can construct a sequence of disjoint finite dimensional spaces (F_n) in X and operators $T_n\colon X \to Y$, $\|T_n\| = 1$ such that for any $U_n\colon F_n \to l_2$, $V_n\colon l_2 \to Y$ with $V_n U_n$ the restriction of T_n to F_n, $\|V_n\| \, \|U_n\| \geqslant 4^n \|I - P_{n-1}\|^{-1}$, where P_0 is the identity on X and P_K the projection onto X_K determined by $X = F_k \oplus X_k$. Define $T\colon X \to Y$ by $T = \sum_{n=1}^{\infty} \frac{1}{2^{n+1}\|P_{n-1}\|} T_n[I - P_{n-1}]$. Then $\|T\| \leqslant 1$ and since (F_n) are disjoint

$$T|_{F_n} = \frac{1}{2^{n+1}\|P_{n-1}\|} T_n[I - P_{n-1}] = \frac{1}{2^{n+1}\|P_{n-1}\|} V_n U_n [I - P_{n-1}].$$

But

$$\frac{1}{2^n \|P_{n-1}\|} \|U_n\| \, \|V_n\| \, \|I - P_{n-1}\| \geqslant 2^{n-1}$$

and this contradicts [17].

With this observation we can sharpen 1.6. *If X is a Banach space, Y an $\mathcal{L}_{\infty,\lambda}$-space and every $T \in \mathcal{L}(X, Y)$ is Hilbertian then X is an $\mathcal{L}_{2,\beta}$-space, where $\beta \leqslant h(X, Y)\lambda(1+\varepsilon)$, where $\varepsilon > 0$ is arbitrary.*

Indeed, if $F \subset X$, $\dim F < +\infty$ then there is an operator $S\colon F \to Y$ with $d\big(F, S(F)\big) < \lambda(1+\varepsilon)$. Since Y is an $\mathcal{L}_{\infty,\lambda}$-space and S has finite rank, there is an extension $\hat{S}$ of S to all of X with $\|\hat{S}\| \leqslant \lambda \|S\|$. We have, by hypothesis

$$\begin{tikzcd} F \arrow[rr, "\hat{S}|_F"] \arrow[dr, "U_S"'] & & Y \\ & l_2 \arrow[ur, "V_S"'] & \end{tikzcd}$$

$$\|U_S\| \, \|V_S\| \leqslant h(X, Y)[\lambda(1+\varepsilon)]^{-1}.$$

Now V_S must be onto $S(F)$ and so SF is isomorphic to a factor space of l_2 with a bound on the isomorphism no larger than $h(X, Y)\lambda(1+\varepsilon)$. Since this constant is independent of F, X is an $\mathscr{L}_{2,\beta}$-space with $\beta \leqslant h(X, Y)\lambda(1+\varepsilon)$.

References

[1] S. Banach, *Théorie des operations linéaires*, Warszawa 1932.

[2] — and S. Mazur, *Zur Theorie der Linearen Dimension*, Studia Math. 4 (1933), pp. 100–112.:

[3] J. Cohen, *p-summing operators and their adjoints*, Dissertation, University of Maryland, 1969.

[4] M. Day, *Normed linear spaces*, New York 1962.

[5] N. Dunford and J. Schwartz, *Linear operators*, Part I, New York 1958.

[6] A. Dvoretzky, *Some results on convex bodies in Banach spaces*, Proc. Symp. on Linear spaces, pp. 123–160, Jerusalem 1961.

[7] — and C. A. Rogers, *Absolute and unconditional convergence in normed linear spaces*, Proc. Nat. Acad. Sci. U.S.A. 36 (1950), pp. 192–197.

[8] B. R. Gelbaum, *Notes on Banach spaces and bases*, An. Acad. Brasil. Ci. 30 (1958), pp. 29–36.

[9] M. M. Grinblyum, *On the representation of a space of type B in the form of a direct sum of subspaces*, Dokl. Akad. Nauk SSSR (NS) 70 (1950), pp. 749–752.

[10] A. Grothendieck, *Produits tensoriels topologiques et spaces nucléaires*, Mem. Amer. Math. Soc. 16 (1955).

[11] — *Résumé de la theorie metrique des produits tensoriels topologiques*, Bol. Soc. Mat. São Paulo 8 (1956), pp. 1–79.

[12] — *Sur certaines classes de suites dans les espaces de Banach, le théoréme de Dvoretzsky–Rogers*, Bol. Soc. Mat. São Paulo 8 (1956), pp. 80; 110.

[13] V. I. Gurarii, *Inclination of subspaces and conditional bases in Banach spaces*, Dokl. Akad. Nauk SSSR 145 (1962), pp. 504–506.

[14] W. B. Johnson, H. P. Rosenthal and M. Zippin, *On bases, finite dimensional decompositions and weaker structures in Banach spaces*, Isreal J. Math. 9 (1971), pp. 488–506.

[15] J. Lindenstrauss, *On James' paper "Separable conjugate spaces"*, to appear in Israel J. Math.

[16] — *On the modulus of smoothness and divergent series in Banach spaces*, Michigan Math. J. 10 (1963), pp. 241–252.

[17] — and A. Pełczyński, *Absolutely summing operators in $\mathscr{L}_p$-spaces and their applications*, Studia Math. 29 (1968), pp. 275–327.

[18] — and H. P. Rosenthal, *The $\mathscr{L}_p$-spaces*, Israel J. Math. 7 (1969), pp. 325–349.

[19] A. Pełczyński, *A proof of the Eberlein–Smulian theorem by application of basic sequences*, Bull. Acad. Polon. Ser. Sci., Math., Astr., et Phys. 12 (1964), pp. 543–548.

[20] — and P. Wojtaszczyk, *Banach spaces with finite dimensional expansions of identity and universal bases of finite dimensional subspaces*, Studia Math., 40 (1971), pp. 91–108.

[21] A. Pietsch, *Absolut p-summierende Abbildungen in normierten Raumen*, Studia Math. 28 (1967), pp. 333–353.

[22] H. R. Pitt, *A note on bilinear forms*, J. London Math. Soc. 11 (1963), pp. 174–180.

[23] J. R. Retherford and C. Stegall, *Fully nuclear operators*, Bull. Amer. Math. Soc. 76 (1970), pp. 1077–1081.

[24] — *Fully nuclear and completely nuclear operators with applications to $\mathscr{L}_1$ and $\mathscr{L}_\infty$-spaces*, Trans Amer. Math. Soc. 163 (1972), pp. 457–492.

[25] H. P. Rosenthal, *On totally incomparable Banach spaces*, J. Func. Anal. 4 (1969), pp. 167–175.

[26] S. Kwapień, *On a theorem of L. Schwartz and its application to absolutely summing operators*, Studia Math. 38 (1970), pp. 193–201.

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