so each $\tilde{A}_i = \widetilde{\tilde{A}/\ker\| \|_i}$ is a topologically nilpotent Banach algebra. From ([5], Theorem 10.10) it follows that A is the projective limit of $\tilde{A}_i$. ∎

5.3. COROLLARY. *An m-convex B_0-algebra A is topologically nilpotent if and only if there exists a system of pseudonorms giving the topology in A such that every $\tilde{A}_i$ is a topologically nilpotent Banach algebra.*

5.4. Remark. For a given system of pseudonorms in A, $\tilde{A}_i$ need not be topologically nilpotent. Indeed, take the Cartesian product $\prod_{i=1}^{\infty} \tilde{A}_i$, where $\tilde{A}_i = C(0,1)$ from Example 2.3.

Put

$$|x|_i = \|x_1\| + \ldots + \|x_i\| + \int_0^1 |x_{i+1}(t)|\,dt \quad \text{for } x = (x_1, x_2, \ldots) \in A.$$

One may verify, that for each i, $\widetilde{A/\ker| \;|_i}$ is not a topologically nilpotent algebra, but A is a topologically nilpotent algebra.

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Diagonal nuclear operators

by

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Abstract. Let E and F be Banach spaces with total biorthogonal sequences (x_n, f_n) and (y_n, g_n) respectively. An operator $T\colon E \to F$ is called diagonal if $g_i(Tx_j) = 0$ for $i \neq j$. The diagonal of a linear operator T is the scalar sequence $(g_i(Tx_i))$. A sequence space representation $\mathscr{S}(E, F)$ for the diagonals of the nuclear operators is given and a necessary and sufficient condition is obtained for $\mathscr{S}(E, F)$ to be the diagonal nuclear operators. In particular this is the case when (x_n, f_n) is an unconditional shrinking basis for E and (y_n, g_n) is an unconditional basis for F. As another application of this result, it is shown that if the coordinate vectors form an unconditional basis for the BK-space E then the vectors from E give precisely the diagonal nuclear operators from l_1 into E.

1. Introduction. Let E and F be Banach spaces with total biorthogonal sequences (x_n, f_n) and (y_n, g_n) respectively. If T is a linear operator from E to F then by the diagonal of T we mean the sequence $\delta(T) = \big(g_i(Tx_i)\big)_{i=1}^{\infty}$. An operator T from E to F is called diagonal if $g_i(Tx_j) = 0$ for $i \neq j$. The purpose of this paper is to determine the diagonal nuclear operators between certain Banach spaces. In Section 3 we present a simple proof that the diagonal nuclear operators on a space with an unconditional basis are l_1 and we obtain a sequence space representation for the diagonals of the nuclear operators in the case where (x_n, f_n) and (y_n, g_n) are complete biorthogonal sequences. This sequence space $\mathscr{S}(E, F)$ is a generalization of the series space studied by Ruckle in [4]. In Section 4 we show that if E' or F has the approximation property then a necessary and sufficient condition for $\mathscr{S}(E, F)$ to be the diagonal nuclear operators is that the diagonal of every continuous linear operator from E' to F' be well defined as a linear operator from E' to F'. In particular if E has an unconditional shrinking basis and F has an unconditional basis then the diagonal nuclear operators are determined.

After completing this work, the authors became aware of the results of Ruckle in [5]. There is overlap between Ruckle's work and the results that appear in our preliminary section.

2. Notation and terminology. If (x_n, f_n) is a total biorthogonal sequence for the Banach space E then E can be identified with the linear space of all sequence $(f_i(x))$ under the correspondence $x \leftrightarrow (f_i(x))$. With the norm $\|(f_i(x))\| = \|x\|$ this space is a BK-space isometric to E. Under this correspondence x_i corresponds to $e_i = (\delta_{ij})_{j=1}^{\infty}$ and f_i corresponds to E_i, the ith coordinate functional. In light of this identification we will restrict our attention to E, F BK-spaces. A BK-space is said to have AD if φ, the span of the e_i's, is dense in E. Note that (x_n, f_n) is a complete biorthogonal sequence for the Banach space E if and only if under the above correspondence E is a BK-space with AD.

For E a BK-space let E^δ denote the space of all sequences $(f(e_i))$ as f ranges over E^*; with norm $\|(f(e_i))\| = \|f\|$ E^δ is a BK-space. If E has AD then E^δ is isometric to E^* under the map $f \leftrightarrow (f(e_i))$. In any case $E^\delta = E^{\circ\delta}$ where E° denotes the closure of φ in E.

2.1. Notation.

a) Let ω denote the space of all scalar sequences; with the topology of coordinate wise convergence ω is an FK-space.

b) Let cs denote the space of all sequences $x \in \omega$ such that $\sum_i x(i)$ converges.

c) Let $x, y \in \omega$ be such that $xy \in cs$; the sum $\sum_i x(i)y(i)$ is denoted by (x, y).

d) For $A \subseteq \omega$

$$A^\varphi = \{t \in \varphi\colon |(t, u)| \leqslant 1,\ u \in A\}.$$

e) For $A \subseteq \varphi$

$$A^\omega = \{t \in \omega\colon |(t, u)| \leqslant 1,\ u \in A\}.$$

f) For E a normed space

$$D_1(E) = \{x \in E\colon \|x\| \leqslant 1\}.$$

2.2. Definition. Let E and F be BK-spaces. A sequence $t \in \omega$ is called a *multiplier* from E *to* F if $tx \in F$ for each $x \in E$. The linear space of all multipliers from E to F is denoted by $M(E, F)$.

In the case $E = F$, $M(E, F)$ is denoted by $M(E)$ and is called the *multiplier algebra of* E. Detailed discussions on multiplier algebras can be found in [1], [3] and [4].

The proof of the following is similar to 3.3 of [3] and 7.1 of [4].

2.3. Proposition. *Let E and F be BK-spaces.*

a) *$M(E, F)$ is a BK-space with the norm* $\|t\| = \sup\limits_{\|x\|\leqslant 1} \|tx\|$.

b) *If E and F have AD then*

$$M(E, F) = \bigcup_{n=1}^{\infty} n(A_1 A_2^\omega)^\omega,$$

where $A_1 = D_1(E) \cap \varphi$ and $A_2 = D_1(F) \cap \varphi$.

The space $M(E, F)$ can be identified with the continuous diagonal operators from E to F under the correspondence $t \leftrightarrow T_t$ where $T_t(x) = tx$.

If A is a coordinatewise bounded subset of ω then there is a smallest BK-space containing A as a bounded subset. We will denote this space by $S(A)$ and it can be characterized as follows:

$$S(A) = \Big\{\sum\nolimits_i \lambda(i) x_i\colon\ \lambda \in l_1,\ x_i \in A \text{ for each } i\Big\}$$

with norm

$$\|x\|_A = \inf\Big\{\|\lambda\|_{l_1}\colon\ x = \sum\nolimits_i \lambda(i) x_i,\ x_i \in A\Big\}.$$

This is equivalent to the formulation of $S(A)$ given by Ruckle in [4] and so we omit the argument that $S(A)$ is a BK-space. If A and B are subsets of ω and each absorbs the other we write $A \sim B$. Note that if $A \sim B$ then $S(A) = S(B)$.

2.4. Definition. Let E, F be BK-spaces then

$$\mathscr{S}(E, F) = S(A_2 A_1^\omega) = \Big\{\sum\nolimits_i \lambda(i) s_i t_i \mid\ \lambda \in l_1,\ s_i \in A_2,\ t_i \in A_1^\omega\Big\}.$$

Note that for $E = F$ $\mathscr{S}(E, F) = \mathscr{S}(E)$ is the series space of E discussed by Ruckle in [4].

2.5. Proposition. *Let E, F be BK-spaces then $\mathscr{S}(E, F)$ is a BK-space with norm*

$$\|x\| = \inf \sum\nolimits_i |\lambda(i)|\, \|s_i\|\, \|t_i\|,$$

where the infimum is taken over all representations for x.

If F has AD then $\mathscr{S}(E, F)$ can be shown to consist of all sequences $\sum_i \lambda(i) s_i t_i$, where $\lambda \in l_1$, $s_i \in D_1(F)$ and $t_i \in A_1^\omega$.

3. Preliminary results. The following is a routine generalization of 6.2 of [4].

3.1. Proposition. *Let E and F be BK-spaces with AD then $\mathscr{S}(E, F)$ consists of all sequences of the form $\delta(T)$ as T ranges over $N(E, F)$, the nuclear operators from E to F. Furthermore, δ is a continuous linear operator from $N(E, F)$ onto $\mathscr{S}(E, F)$.*

3.2. Proposition. *Let E and F be BK-spaces with AD and let $B = D_1\big(M(E, F)\big)$ then $\mathscr{S}(F, E) = S(B^\varphi)$.*

Proof. By 2.3 $B \sim (A_1 A_2^\omega)^\omega$ thus by ([1], Proposition 3.5) $B^\varphi \sim (A_1 A_2^\omega)^{\omega\varphi} \sim K(A_1 A_2^\omega)$, where K denotes the absolutely convex hull. Therefore $S(B^\varphi) = S(A_1 A_2^\omega) = \mathscr{S}(F, E)$.

As a consequence of the above, we give a particularly simple proof that on a Banach space with an unconditional basis the nuclear operators diagonal with respect to this basis are precisely the l_1 sequences.

3.3. Theorem. *If (e_i, E_i) is an unconditional basis for the BK-space E then*

$$\mathscr{S}(E) = DN(E) = l_1,$$

where $DN(E)$ denotes the diagonal nuclear operators on E.

Proof. If (e_i, E_i) is an unconditional basis for E then by ([3], Theorem 5.7) $M(E) = m$ and so by Proposition 3.2 and ([4], Proposition 4.1) $\mathscr{S}(E) = l_1$. Thus by 3.1 $DN(E) \subseteq l_1$. Let $x \in l_1$ and define $T_x \colon E \to E$ by

$$T_x(y) = \sum_i x(i) E_i(y) e_i.$$

Then T_x is diagonal and is nuclear since $\sum_i |x(i)|\,\|E_i\|\,\|e_i\| \leqslant \sup_i \|E_i\|\,\|e_i\| \times \times \sum_i |x(i)| = M\|x\|_{l_1}$, where $M = \sup_i \|E_i\|\,\|e_i\| < \infty$ since (e_i, E_i) is a basis.

For $E = F$ a Banach space the following is an improvement of Proposition 3.5 of [3].

3.4. Proposition. *If E, F are BK-spaces with AD then* $M(E, F) = M(F^\delta, E^\delta)$.

Proof. Let $t \in M(E, F)$, $s \in F^\delta$ and $u \in A_1$ then $|(ts, u)| = |(s, tu)| = \|t\|_M \cdot |(s, \frac{1}{\|t\|_M} tu)| \leqslant \|t\|_M \|s\|_{F^\delta}$. Therefore $ts \in (\|t\|_M \|s\|_{F^\delta}) A_1^\omega \subset E^\delta$ so $t \in M(F^\delta, E^\delta)$.

Conversely let $t \in M(F^\delta, E^\delta)$, $u \in A_1$, and $v \in A_2^\omega$ then $|(tu, v)| = |(u, tv)| = \|t\|_M |(u, \frac{1}{\|t\|_M} tv)| \leqslant \|t\|_M$ and thus $tu \in \|t\|_M A_2^{\omega\varphi} \sim A_2$ ([1], Proposition 3.5). Therefore $T_t \colon (\varphi, \|\ \|_E) \to F$ is continuous and so has a continuous extension $T^\wedge$ to all of E. Since coordinates are continuous on E and F, we have that $T^\wedge(x) = tx$ for all $x \in E$ and thus $t \in M(E, F)$.

Remark. Let E be a BK-space with AD and let F be any BK-space. If $t \in M(E, F)$ then $T_t \colon E \to F$ is continuous and maps φ into φ thus $T_t(E) \subseteq F^\circ$. It follows that $M(E, F) = M(E, F^\circ)$. Since $F^\delta = F^{\circ\delta}$ we may drop the hypothesis from 3.4 that F has AD.

4. Main result. In this section, we obtain necessary and sufficient conditions for $\mathscr{S}(E, F)$ to be precisely the diagonal nuclear operators from E to F. We assume throughout that either E' or F has the approximation property. We define $D \colon L(E, F) \to L(E, \omega)$ by $D(T)(x) = \delta(T)x$.

4.1. Theorem. *Let E, F be BK-spaces with AD then D maps $N(E, F)$ into $N(E, F)$ if and only if D maps $L(E^\delta, F^\delta)$ into $L(E^\delta, F^\delta)$.*

Proof. Assume $D \colon L(E^\delta, F^\delta) \to L(E^\delta, F^\delta)$ and let $B = D_1(M(E^\delta, F^\delta))$ then by 3.4 $B \sim D_1(M(F, E))$. For each $t \in B^\varphi$ define $T_t = \sum_i t(i) E_i \otimes e_i$. Since $t \in \varphi$, $T_t \in N(E, F)$. We now show that $\{T_t / t \in B^\varphi\}$ is bounded in $N(E, F)$. In fact

$$\begin{aligned} \nu(T_t) &= \operatorname{Sup}\{|Q(T_t)|\ \ Q \in N(E, F)', \|Q\| \leqslant 1\} \\ &= \operatorname{Sup}\{|(t, \delta(P))|\ \ P \in L(E^\delta, F^\delta), \|P\| \leqslant 1\} \\ &\leqslant k \operatorname{Sup}\{|(t, s)|\ \ s \in M(E^\delta, F^\delta), \|s\| \leqslant 1\} \\ &\leqslant k, \end{aligned}$$

where $k = \|D\|$ (D is bounded on $L(E^\delta, F^\delta)$ as it has closed graph). Here we have used the fact that $E^\delta \otimes_\gamma^\wedge F$ can be identified with $N(E, F)$ and that in turn $N(E, F)'$ can be identified with $L(E^\delta, F^\delta)$ ([2], Proposition 2, § 1, #1). It is in these identifications that we have used the fact that E' or F has the approximation property.

Now take $u \in S(B^\varphi)$, $u = \Sigma \lambda(i) t_i$, where $\lambda \in l_1$, $t_i \in B^\varphi$. Let $T_u = \sum \lambda(i) T_{t_i}$, where T_{t_i} is defined as above. Since $\sum \lambda(i) T_{t_i}$ converges absolutely in $N(E, F)$, $T_u \in N(E, F)$ and clearly T_u is diagonal. Thus each sequence in $S(B^\varphi)$ corresponds to a diagonal nuclear operator from E to F but $S(B^\varphi) = \mathscr{S}(E, F)$ and by 3.1 $\mathscr{S}(E, F)$ consists of the diagonals of the nuclear operators from E to F.

Conversely suppose that D maps $N(E, F)$ into $N(E, F)$. Let $\|D\| = k_1$. Then as above if $t \in B^\varphi$

$$\operatorname{Sup}\{|(t, \delta(P))|\colon P \in L(E^\delta, F^\delta), \|P\| \leqslant 1\} = \nu(T_t).$$

It follows that for $P \in L(E^\delta, F^\delta)$ with $\|P\| \leqslant 1$

$$\operatorname{Sup}\{|(t, \delta(P))|\colon t \in B^\varphi\} \leqslant k_1.$$

Thus $\delta(P) \in k_1 B^{\varphi\omega} = k_1 B \subset M(E^\delta, F^\delta)$.

4.2. Corollary. *Let E, F be BK-spaces with AD then D maps $L(E^\delta, F^\delta)$ into $L(E^\delta, F^\delta)$ iff $\mathscr{S}(E, F) = DN(E, F)$.*

5. Some applications. If E has a shrinking unconditional basis and F has an unconditional basis then by ([6], Theorem 2.1) D maps $L(E^\delta, F^\delta)$ into $L(E^\delta, F^\delta)$ since E^δ is solid and $F^\delta = F^\alpha$ is perfect. It follows by 4.2 that $\mathscr{S}(E, F) = DN(E, F)$. In particular if E, F have unconditional bases and are reflexive then $\mathscr{S}(E, F) = DN(E, F)$.

The following example shows how one can determine the diagonal nuclear operators from E to F by using Proposition 2.3 and Corollary 4.2:

Let $1 < q < p < \infty$, $E = l_p$ and $F = l_q$ then we directly calculate $M(l_p, l_q)$ using Proposition 2.3 getting $M(l_p, l_q) = l_g$ where $g = \dfrac{pq}{p-q}$. Let $B = D_1(l_g)$ then $B^\varphi = D_1(l_h) \cap \varphi$, where h is the conjugate number of g. Thus by Proposition 3.2 and 4.2 $DN(l_q, l_p) = \mathscr{S}(l_q, l_p) = S(B^\varphi) = l_h$. A method for obtaining $DN(E, F)$ for E and F l_p spaces was given by Tong in [6].

As another example suppose E has unconditional basis (e_i, E_i) then by 2.3 $M(E, l_1) = \bigcup_{n=1}^{\infty} n(A_1 A_2^\omega)^\omega$. Since E has (e_i, E_i) as an unconditional basis and $A_2^\omega = D_1(m)$ by [3, Theorem 5.7] $A_1 A_2^\omega \sim D_1(E) \cap \varphi$. Thus $(A_1 A_2^\omega)^\omega \sim D_1(E^\delta) \sim A_1^\omega$ and so by ([1], Proposition 3.6) $D_1(E^\delta)^\varphi \sim A_1^{\omega\varphi} \sim A_1$. It follows that $\mathscr{S}(l_1, E) = DN(l_1, E) = E$.

Similarly if (e_i, E_i) is a basis for E then one reasons as follows to conclude $\mathscr{S}(cs, E) = E$. Let $A_2 = D_1(E) \cap \varphi$ and $A_1 = D_1(cs) \cap \varphi$ then

$A_1^\omega = D_1(bv)$. Since (e_i, E_i) is a basis for E it follows that E is bv-invariant [3] and so $\mathscr{S}(cs, E) = S(A_2 A_1^\omega) = S(A_2) = E$. In this situation one may have $DN(cs, E) \subsetneqq E$. For example let $E = c_0$ then the condition that D map $L(cs^\delta, c_0^\delta)$ into $L(cs^\delta, c_0^\delta)$ is not satisfied [to see this consider the natural isomorphism of $cs^\delta = bv$ into $c_0^\delta = l_1$ given by $Tx = (x_1, x_2 - x_1, x_3 - x_2, \ldots)$] and so by Corollary 4.2 $DN(cs, c_0) \subsetneqq c_0$.

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Helson sets and simultaneous extensions to Fourier transforms

by

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Abstract. A tensor algebra proof of this result is given: if K is an infinite Helson subset of an LCA group G, then there does not exist a continuous linear map $E\colon C(K) = A(K) \to A(G)$ such that $Ef(k) = f(k)$, for all $k \in K$.

1. A compact subset K of a LCA group G is a *Helson* set [5] if every $f \in C(K)$ may be extended to a Fourier transform $F \in A(G)$. We have this result:

THEOREM. *Let K be an infinite Helson subset of a LCA group G. Then there does not exist a continuous linear map $E\colon C(K) \to A(G)$ such that $Ef(k) = f(k)$ for all $k \in K$.*

More general results of this form have been proved: see [2], [8], [10], [11]. The fact that the existence of the map E of the theorem implies that $EC(K)$ is complemented in $A(G)$ implies (when G is the circle group and $A(G) \cong l^1$) that weak sequential convergence and norm convergence in $C(K)$ are equivalent (see [7], p. 431]). In this note we give a simple proof of the theorem, using tensor algebras. It is not too hard to see that a technique of Katznelson and McGehee [6] may be used, along with our proof, to show that if $K \subseteq \boldsymbol{R}$ is a convergent sequence, then there is no continuous linear map $E\colon A(K) \to A(\boldsymbol{R})$.

A Helson subset K of the circle group has the property that every $f \in C(K)$ has an extension to an absolutely convergent Taylor series (this was due to Wik; (see [5], p. 145])). The theorem shows immediately that a result [9] of Pełczyński for the disc algebra fails for absolutely convergent Taylor series.

We shall write $PM(G)$ for $A(G)^*$, and $M(X)$ for the set of regular Borel measures on a locally compact space X.

DEFINITION. If R and S are Banach spaces, $R \otimes S$ will denote the closure of $R \times S$ in the projective norm (see [3], [13]). If $R = C(X)$ and $S = C(Y)$, we set $V(X \times Y) = R \otimes S = C(X) \otimes C(Y)$.

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