STUDIA MATHEMATICA, T. XLIII. (1972)

On ω^*-basic sequences and their applications to the study of Banach spaces

by

W. B. JOHNSON and H. P. ROSENTHAL* (Berkeley, Ca.)

Abstract. ω^*-basic sequences in conjugate Banach spaces are investigated and existence theorems are obtained for them, analogous to the Bessaga–Pełczyński existence theorem for basic sequences. Immediate consequences are that every separable Banach space has a quotient with a basis and every separable conjugate space contains a boundedly complete basic sequence. Other examples of immediate applications of ω^*-basic sequences are that if X^* has a subspace with a separable dual, X has a quotient space with a boundedly complete basis, and if X^{**} is separable, then both X and X^* have reflexive subspaces. ω^*-basic sequences and previously known theorems are also applied to show, for separable X, that if X^* contains a subspace isomorphic to l^1, (respectively L_1), then X has a quotient isomorphic to c_0 (respectively, $C[0,1]$); the technique for obtaining the existence theorems is used to show that every separable X has a subspace Y such that Y and X/Y both have finite-dimensional decompositions.

I. Introduction. In this paper we give a proof of a theorem stated by Milman [12]:

If X^{**} is separable then both X and X^* have reflexive subspaces. (Throughout this paper X, Y, and Z refer to infinite dimensional Banach spaces. "Subspace" means "infinite dimensional closed linear submanifold.") Our Theorem IV.2 yields immediately that if X^{**} is separable, then both X and X^* are somewhat reflexive. (A space Y is called somewhat reflexive (see [5]) provided every subspace of Y contains a reflexive subspace). This result has also been obtained by Davis and Singer [2] under the additional hypothesis that X^{**} satisfies the approximation property. They also have given a counter example to a key lemma in [12], and thus the argument in [12] is incorrect.

The techniques developed to prove the above theorem stated by Milman led to new results concerning bases in Banach spaces and their quotients. Some of these results have consequences which may be stated without reference to bases.

* The research for the second-named author was partially supported by NSF-GP 12997.

For example, in Section IV we prove

THEOREM IV.1. (i) *If X is separable then X has a quotient space with a basis.*

(ii) *If X is separable and there is a subspace Y of X^* such that Y is isomorphic to a subspace of a separable conjugate space, then X has a quotient space with a shrinking basis; moreover this basis may be chosen with its biorthogonal functionals lying in Y, hence Y has a weak* closed subspace with a boundedly complete basis.*

(iii) *If there is a subspace Y of X^* such that Y^* is separable then X has a quotient space which has a boundedly complete basis. Consequently X^* contains a subspace isomorphic to a second conjugate space.*

(i) solves the separable version of a problem of Pełczyński's [13]: Does every Banach space have a quotient space which has a basis? (ii) yields that every subspace of a separable conjugate space contains a weak* closed subspace, and also that it contains a boundedly complete basic sequence, solving a problem raised in [3]. (iii) yields that if X^* has a subspace with a separable dual, X has a quotient space isomorphic to a separable conjugate space; the converse statement is obvious. (By quotient space of X, we mean a space of the form X/Z where Z is subspace of X with infinite codimension in X.)

In Theorem IV.3, we show that for separable X, if X^* contains a subspace isomorphic to l^1, X has a quotient space isomorphic to c_0, while if X^* has a subspace isomorphic to $L_1[0,1]$, X has a quotient space isomorphic to $C[0,1]$. (This extends a result of Pełczyński's [15].)

The concept underlying the proofs of the above results is that of *ω^*-basic sequence* (see the definition in Section II). Our main lemma, Theorem III.1, is the ω^*-basic analogue to the theorem of Bessaga and Pełczyński [1] on the existence of basic sequences.

Our final result, Theorem IV.4, shows that if X is separable then there is a $Y \subset X$ such that Y and X/Y both have finite dimensional decompositions (the relevant definitions are given immediately preceding the statement of IV.4). Although this result is not a direct application of ω^*-basic sequences, its proof has some common features with the proof of Theorem III.1.

II. Notation, definitions, and preliminary results. As mentioned in the introduction, X, Y, Z, etc., refer to infinite dimensional Banach spaces and "subspace" (resp. "quotient space") means "closed, infinite dimensional linear submanifold," (resp. "quotient manifold"). "Operator" means "bounded linear operator" and "isomorphism" means "linear homeomorphism." If T is an operator on X and $E \subset X$, then $T|E$ denotes the restriction of T to E. If $x \in X$ and $A \subset X$, $d(x, A) = \inf\{\|x-a\|: a \in A\}$. For $A \subset X$, $B \subset X$, $A+B$ denotes $\{a+b: a \in A, b \in B\}$.

If $A \subset X$, $A^\perp$ is the annihilator of A in X^*, i.e., $A^\perp = \{x^* \in X^*: x^*(a) = 0$ for every $a \in A\}$. If $A \subset X^*$, $A^\top$ is the annihilator of A in X, i.e., $A^\top = \{x \in X: a(x) = 0$ for every $a \in A\}$. For $A \subset X$, $\bar{A}$ is the norm closure of A in X and for $A \subset X^*$, $\tilde{A}$ is the weak* closure of A in X^*. Thus if A is a linear manifold in X^*, then $\tilde{A} = A^{\top\perp}$.

Sequences are denoted by using parentheses and $[x_n]$ denotes the norm-closed linear span of the sequence (x_n). If $(x_n) \subset X$, $x \in X$, we write $x_n \to x$ (respectively, $x_n \xrightarrow{\omega} x$) if (x_n) converges to x in the norm (respectively, weak) topology on X. If $(x_n^*) \subset X^*$, $x^* \in X^*$, we write $x_n^* \xrightarrow{\omega^*} x^*$ when (x_n^*) converges to x^* in the weak* topology on X^*.

We now recall some familiar facts about bases which will be used in the sequel without further reference. A sequence (x_n, x_n^*) with $(x_n) \subset X$, $(x_n^*) \subset X^*$ is called *biorthogonal* provided $x_m^*(x_n) = \delta_{mn}$ for all $m, n = 1, 2, \ldots$ A sequence $(x_n) \subset X$ is a *basis* for X provided that for each $x \in X$ there is a unique sequence $(x_n^*(x))$ of scalars for which $\sum_{i=1}^{n} x_i^*(x)x_i \to x$. The functionals x_n^* are necessarily linear and continuous, (x_n, x_n^*) is biorthogonal, and (x_n^*) forms a basis for $[x_n^*]$. A basis (x_n) for X is called *shrinking* provided the functionals (x_n^*) biorthogonal to (x_n) form a basis for X^*. A basis (x_n) is called *boundedly complete* provided that $\left(\sum_{i=1}^{n} a_i x_i\right)$ is convergent whenever it is bounded. We will use repeatedly two results due to R. C. James: A Banach space with a basis is reflexive if and only if the basis is shrinking and boundedly complete. If (x_n) is a basis with biorthogonal functionals (x_n^*), then (x_n) is shrinking (respectively, boundedly complete) if and only if (x_n^*) is a boundedly complete (respectively, shrinking) basis for $[x_n^*]$.

A sequence $(x_n) \subset X$ is called *basic* provided that (x_n) forms a basis for $[x_n]$. A basic sequence (x_n) is called *shrinking* (respectively, boundedly complete) provided that (x_n) is a shrinking (respectively, boundedly complete) basis for $[x_n]$. Two basic sequences (x_n) and (y_n) are called *equivalent* provided that for any sequence (a_n) of scalars, $\sum_{n=1}^{\infty} a_n x_n$ converges if and only if $\sum_{n=1}^{\infty} a_n y_n$ converges. It follows from the closed graph theorem that (x_n) is equivalent to (y_n) if and only if there is an isomorphism T from $[x_n]$ onto $[y_n]$ such that $Tx_n = y_n$ for $n = 1, 2, \ldots$ If (x_n) and (y_n) are bases with associated biorthogonal functionals (x_n^*) and (y_n^*), respectively, then (x_n) is equivalent to (y_n) if and only if (x_n^*) is equivalent to (y_n^*).

We refer the reader to [18] for proofs of the above and for a comprehensive study of bases and basic sequences in Banach spaces.

We now come to the concept which underlies most of our results:

DEFINITION II.1. A sequence $(y_n) \subset X^*$ is called *ω^*-basic* provided that there is a sequence $(x_n) \subset X$ so that (x_n, y_n) is biorthogonal and for each $y \in \widetilde{[y_n]}$, $\sum_{i=1}^{n} y(x_i) y_i \xrightarrow{\omega^*} y$. (Note that although (x_n) is not uniquely determined by (y_n), if (x_n, y_n) and (z_n, y_n) are both biorthogonal then for each $y \in \widetilde{[y_n]}$, $y(x_n) = y(z_n)$ for $n = 1, 2, \ldots$).

If $(y_n) \subset X^*$, then $\widetilde{[y_n]}$ can be identified with $(X/(y_n)^{\top})^*$ via the mapping T^*, where $T\colon X \to X/(y_n)^{\top}$ is the quotient mapping, because T^* is an isometry and a weak* isomorphism. From this it follows that (y_n) is ω^*-basic if and only if $(T^{*-1}y_n)$ is a ω^* Schauder basis for $(X/(y_n)^{\top})^*$ according to Singer's definition in [17] (see also [18], p. 144 ff.).

Proposition 1 summarizes some of the elementary properties of ω^*-basic sequences:

PROPOSITION II.1. *Suppose that $(y_n) \subset X^*$ and let $T\colon X \to X/(y_n)^{\top}$ be the quotient map. Then*

(a) *(y_n) is ω^*-basic if and only if $X/(y_n)^{\top}$ has a basis (x_n) with associated biorthogonal functionals (x_n^*) such that $T^*(x_n^*) = y_n$ for $n = 1, 2, \ldots$ Thus if (y_n) is ω^*-basic, then (y_n) is basic.*

(b) *The following are equivalent:*

(i) *(y_n) is a boundedly complete ω^*-basic sequence;*

(ii) *(y_n) is ω^*-basic and $[y_n] = \widetilde{[y_n]}$;*

(iii) *$X/(y_n)^{\top}$ has a shrinking basis (x_n) with associated biorthogonal functionals (x_n^*) such that $T^*(x_n^*) = y_n$ for $n = 1, 2, \ldots$*

(c) *(y_n) is a shrinking ω^*-basic sequence if and only if $X/(y_n)^{\top}$ has a boundedly complete basis (x_n) with associated biorthogonal functionals (x_n^*) such that $T^*(x_n^*) = y_n$ for $n = 1, 2, \ldots$*

Proof. (a) follows from the comment preceding the statement of Proposition 1 and Singer's work on ω^* Schauder bases (see [18], p. 155). (c) follows from (a) and the duality between shrinking and boundedly complete bases proven by James. The equivalence of (i) and (iii) in (b) also follows from Singer's and James' work. It remains to be seen that (i) and (ii) in (b) are equivalent.

Suppose that $(x_n) \subset X$ is such that (x_n, y_n) is biorthogonal. Assume that (ii) holds and let (a_n) be a sequence of scalars for which $\left(\sum_{i=1}^{n} a_i y_i\right)$ is bounded. Then $\left(\sum_{i=1}^{n} a_i y_i\right)$ has a weak* cluster point, say y, and of course $y \in \widetilde{[y_n]}$. Thus by (ii), $y \in [y_n]$. Since by (a) (y_n) is basic, $\sum_{i=1}^{n} y(x_i) y_i \to y$. Now for arbitrary k and $n \geqslant k$, $\left(\sum_{i=1}^{n} a_i y_i\right) x_k = a_k$, so $y(x_k) = a_k$ because y is a weak* cluster point of $\left(\sum_{i=1}^{n} a_i y_i\right)$. Hence $\sum_{i=1}^{n} a_i y_i \to y$ and (i) follows. Conversely, if (i) is satisfied and $y \in \widetilde{[y_n]}$, then $\sum_{i=1}^{n} y(x_i) y_i \xrightarrow{\omega^*} y$ and hence $\left(\sum_{i=1}^{n} y(x_i) y_i\right)$ is bounded. Since (y_n) is boundedly complete, $\sum_{i=1}^{n} y(x_i) y_i$ is norm convergent, necessarily to y. Hence $y \in [y_n]$. ∎

The next result is an immediate consequence of the known results about bases mentioned above and Proposition II.1.

PROPOSITION II.2. *Let (b_n) be a basis in the Banach space Y with biorthogonal functionals (b_n^*), let $(y_n) \subset X^*$ with (y_n) ω^*-basic and suppose that (y_n) is equivalent to (b_n^*). Then $X/(y_n)^{\top}$ is isomorphic to Y and (y_n) is ω^*-equivalent to (b_n^*); i.e. for any sequence of scalars (a_n), $\Sigma a_n y_n$ converges ω^* if and only if $\Sigma a_n b_n^*$ does.*

Remark. II.1. It is easily seen that every basic sequence is equivalent to some other basic sequence which is also ω^*-basic. For let (x_n) be a basic sequence in X with biorthogonal functionals (x_n^*) and let (f_n) be the functionals in $[x_n^*]^*$ biorthogonal to (x_n^*). Then (f_n) is a ω^*-basic sequence and (x_n) and (f_n) are equivalent.

PROPOSITION II.3. *If X^* is separable, $(x_n) \subset X$, $x_n \xrightarrow{\omega} 0$, and $\limsup \|x_n\| > 0$, then (x_n) has a shrinking basic subsequence.*

Proof. By normalizing a suitable subsequence, we may assume $\|x_n\| = 1$ for $n = 1, 2, \ldots$ By passing to another subsequence, we may assume by a result of Bessaga and Pełczyński [1] that (x_n) is basic (see also our proof of III.1). Now by a well-known stability theorem (cf., e.g. [18], p. 93) there is a $\lambda > 0$ such that if $(y_n) \subset X$ and $\sum_{n=1}^{\infty} \|x_n - y_n\| < \lambda$ then (y_n) is a basic sequence equivalent to (x_n).

Let (d_n) be a countable dense subset of X^* with $d_1 = 0$. By induction we may choose a sequence $n_1 < n_2 < \ldots$ of positive integers and elements (z_n) in X to satisfy, for $i = 1, 2, \ldots$

(i) $z_i \in (d_j)_{j=1}^{i\top}$,

(ii) $\|z_i - x_{n_i}\| < \lambda 2^{-i}$.

Indeed, let $z_1 = x_1$, $n_1 = 1$. Suppose $z_1, \ldots, z_{i-1}$ and $n_1 < \ldots < n_{i-1}$ have been defined. Since $x_n \xrightarrow{\omega} 0$ we may choose $n_i > n_{i-1}$ large enough so that

$$d\left(x_{n_i}, (d_j)_{j=1}^{i\top}\right) = \sup\{|d(x_{n_i})|\colon d \in [(d_j)_{j=1}^{i}], \|d\| \leqslant 1\} < \lambda 2^{-i}.$$

Then choose z_i to satisfy (i) and (ii).

Since $\sum \|z_i - x_{n_i}\| < \lambda$, (z_i) is a basic sequence equivalent to (x_{n_i}). Suppose $f \in [z_i]^*$ and let $\tilde{f}$ be a Hahn–Banach extension of f to an element in X^*. Then there is a subsequence (d_{m_i}) of (d_i) such that $\|d_{m_i} - \tilde{f}\| \to 0$.

Now $d_{m_i}|_{[(z_j)_{j=m_i}^{\infty}]} = 0$ hence $\lim_{i\to\infty}\|\tilde{f}|_{[(z_j)_{j=m_i}^{\infty}]}\| = 0$ whence also $\|f|_{[(z_j)_{j=i}^{\infty}]}\| \to 0$. Thus (z_i), and consequently (x_{n_i}), is a shrinking basic sequence. ■

Remark. II.2. Of course it follows immediately that if X^* is separable, X contains a shrinking basic sequence. This latter result is due to Dean, Singer, and Sternbach [3].

III. Extracting ω^*-basic sequences. In this section we prove a ω^*-basic analogue to the following theorem of Bessaga and Pełczyński [1] on the existence of basic sequences: *if* $(x_n) \subset X$, $x_n \xrightarrow{\omega} 0$, *and* $\limsup\|x_n\| > 0$, *then* (x_n) *has a basic subsequence.* We show that for X separable, if $(y_n) \subset X^*$, $y_n \xrightarrow{\omega^*} 0$, and $\limsup\|y_n\| > 0$, then (y_n) contains a ω^*-basic subsequence (Theorem III.1), and if also X^* is separable, then (y_n) contains a boundedly complete ω^*-basic subsequence (Theorem III.2). Another variation of the Bessaga–Pełczyński theorem is that if $(y_n) \subset X^*$, $[y_n]^*$ is separable, $y_n \xrightarrow{\omega} 0$, and $\limsup\|y_n\| > 0$, then (y_n) contains a shrinking ω^*-basic subsequence (Theorem III.3).

Most of the results of this paper follow easily from the argument for Theorem III.1 and previously known facts. This argument is a variation of the "product" technique for producing basic sequences, due to Mazur (c.f. the proof of the Proposition of [14]).

THEOREM III.1. *Suppose that X is separable, $(y_n) \subset X^*$, $y_n \xrightarrow{\omega^*} 0$, and $\limsup\|y_n\| > 0$. Then (y_n) has a ω^*-basic subsequence, (y_{n_i}). Furthermore (y_{n_i}) may be chosen such that if $(z_{n_i}) \subset X$ is selected with (z_{n_i}, y_{n_i}) biorthogonal and S_m: $[\tilde{y}_{n_i}] \to [\tilde{y}_{n_i}]$ is defined (for $m = 1, 2, \ldots$) by $S_m y = \sum_{i=1}^{m} y(z_{n_i})y_{n_i}$ for all $y \in [\tilde{y}_{n_i}]$, then $\|S_m\| \to 1$.*

Proof. By passing to a subsequence and normalizing, we may assume that $\|y_n\| = 1$ for all n. Let (ε_n) be a sequence of positive numbers less than one such that $\sum_{n=1}^{\infty}\varepsilon_n < \infty$; consequently $\prod_{n=1}^{\infty}\frac{1}{1-\varepsilon_n} < \infty$.

Now using Helly's theorem and the compactness of the unit ball of a finite-dimensional space together with the separability of X, we may choose an increasing sequence $k_1 < k_2 < \ldots$ of positive integers and finite subsets $F_1 \subset F_2 \subset \ldots$ of the set of elements of X of norm one with the linear span of $\bigcup_{i=1}^{\infty} F_i$ dense in X, such that for each $n = 1, 2, \ldots$,

(i) for each $f \in [(y_{k_i})_{i=1}^{n}]^*$ with $\|f\| = 1$, there is an $x \in F_n$ such that $|y(x) - f(y)| \leqslant (\varepsilon_n/3)\|y\|$ for all $y \in [(y_{k_i})_{i=1}^{n}]$.

(ii) $|y_{k_{n+1}}(x)| < \varepsilon_n/3$ for all $x \in F_n$.

We claim that (y_{k_n}) is the desired ω^* basic sequence.

Fix n, let scalars $a_1, \ldots, a_n$ be given such that $\|\sum_{i=1}^{n} a_i y_{k_i}\| = 1$, choose $f \in [(y_{k_i})_{i=n}^{n}]^*$ such that $f(\sum_{i=1}^{n} a_i y_{k_i}) = 1 = \|f\|$, and choose $x \in F_n$ satisfying (i) for this f. It follows that $|(\sum_{i=1}^{n} a_i y_{k_i})(x)| \geqslant 1 - (\varepsilon_n/3)$. Then for any scalar λ,

$$\Big\|\sum_{i=1}^{n} a_i y_{k_i} + \lambda y_{k_{n+1}}\Big\| \geqslant \Big|\sum_{i=1}^{n} a_i y_{k_i}(x) + \lambda y_{k_{n+1}}(x)\Big|$$
$$\geqslant 1 - (\varepsilon_n/3) - 2(\varepsilon_n/3) \quad \text{if } |\lambda| \leqslant 2,$$
$$\geqslant 1 \quad \text{otherwise}.$$

Thus $\|\sum_{i=1}^{n} a_i y_{k_i}\| \leqslant \frac{1}{1-\varepsilon_n}\|\sum_{i=1}^{n+1} a_i y_{k_i}\|$ holds for any scalars $a_1, \ldots, a_{n+1}$. But then for any k and scalars $a_1, \ldots, a_{n+k}$,

$$\Big\|\sum_{i=1}^{n} a_i y_{k_i}\Big\| \leqslant \Big(\prod_{j=n}^{n+k-1}\frac{1}{1-\varepsilon_j}\Big)\Big\|\sum_{i=1}^{n+k} a_i y_{k_i}\Big\|,$$

from which it follows easily that (y_{k_i}) is a basic sequence. (This is the Mazur argument for producing basic sequences.) Moreover letting (f_i) denote the functionals in $[y_{k_i}]^*$ biorthogonal to (y_{k_i}) and defining P_m: $[f_i] \to [f_i]$ by $P_m f = \sum_{i=1}^{m} f(y_{k_i})f_i$, then $\|P_m\| \leqslant \prod_{n=m}^{\infty}\left(\frac{1}{1-\varepsilon_n}\right)$ and hence $\|P_m\| \to 1$ as $m \to \infty$.

Now define T: $X \to [y_{k_i}]^*$ by $(Tx)(y) = y(x)$ for all $y \in [y_{k_i}]$. To complete the proof it suffices to show that $T(X) = [f_i]$ (which yields that (y_{k_i}) is a ω^* basic sequence) and that T^* is an isometry (which shows that $\|P_m^*\| = \|S_m\|$ for all m and hence $\|S_m\| \to 1$ as $m \to \infty$).

Now if $x \in F_n$ for some n, then $\sum|y_{k_i}(x)| < \infty$, whence $Tx = \sum y_{k_j}(x)f_j \in [f_i]$ and thus $T(X) \subset [f_i]$.

We shall now show that

(*) *for all g in the linear span of (f_i) with $\|g\| = 1$ and $\varepsilon > 0$, there exists an $x \in X$ with $\|x\| = 1$ and $\|Tx - g\| < 4\varepsilon$.*

Let $0 < \varepsilon < 1$, choose N such that $\sum_{j=n}^{\infty}\varepsilon_j < \varepsilon$ and $\|P_n\| \leqslant 1+\varepsilon$ for all $n > N$ and fix $n > N$. For the sake of convenience, define $\|f\|_1 = \|f|_{[(y_{k_j})_{j=1}^{n}]}\|$ for $f \in [(f_j)_{j=1}^{n}]$; it follows that $\|f\|_1 \leqslant \|f\| \leqslant \|P_n\|\|f\|_1 \leqslant 2\|f\|_1$ for all such f. Now fix $g \in [(f_j)_{j=1}^{n}]$ with $\|g\| = 1$ and put $f = g/\|g\|_1$. Then choosing $x \in F_n$ satisfying (i) for f, we have that $\|\sum_{j=1}^{n} y_{k_j}(x)f_j - f\|_1 \leqslant \varepsilon_n/3$. Hence $\|\sum_{j=1}^{n} y_{k_j}(x)f_j - f\| \leqslant 2(\varepsilon_n/3) < (2/3)\varepsilon$; moreover $\|f_j\| = \|P_j - P_{j-1}\| \leqslant 4$ for all $j > n$, and hence by (ii), $\|\sum_{j=n+1}^{\infty} y_{k_j}(x)f_j\| < 4\sum_{j=n}^{\infty}\varepsilon_j/3 < (4/3)\varepsilon$. Thus

$\|Tx-f\| < 2\varepsilon$. But finally $\|f-g\| \leqslant \varepsilon(1+\varepsilon)$ since $1 = \|g\| \leqslant \|P_n\|\|g\|_1 \leqslant (1+\varepsilon)\|g\|_1$, whence $\|Tx-g\| < 4\varepsilon$.

Thus $(*)$ is proved, from which it follows easily that T^* is an isometry and that T has dense range in $[f_j]$, whence $T(X) = [f_j]$. ■

Remark III.1. The above argument yields that given any X, possibly nonseparable, given $(y_n) \subset X^*$ with zero a weak* cluster point of (y_n) and $0 < \limsup\|y_n\| < \infty$, then (y_n) has a basic subsequence (y_{n_i}) such that if (f_i) is the sequence biorthogonal to (y_{n_i}) in $[y_{n_i}]^*$ and $T\colon X \to [y_{n_i}]^*$ is defined by $(Tx)(y) = y(x)$ for all $y \in [y_{n_i}]$ and $x \in X$, then $TX \supset [f_i]$. (Indeed the separability of X was used only to pick (F_i) so that $[\bigcup_{i=1}^{\infty} F_i] = X$, from which it followed that $T(X) \subset [f_i]$. When X is non-separable and (y_n) is as above, we can still choose (F_i) and (k_i) to satisfy (i) and (ii). Proceeding exactly as in the proof of III.1, one obtains that $(*)$ holds; a standard iteration argument based on $(*)$ alone, yields that $T(X) \supset [f_i]$.) Thus the result of Bessaga and Pełczyński mentioned above also follows from the proof of III.1.

Remark. III.2. Theorem III.1 gives, in some sense, the best possible result (for X separable), because if $(y_n) \subset X^*$, (y_n) is ω^*-basic, and $(\|y_n\|)$ is bounded, then $y_n \xrightarrow{\omega^*} 0$. Indeed, otherwise (y_n) has a weak* cluster point, y, with $y \neq 0$. If $(x_n) \subset X$ with (x_n, y_n) biorthogonal, then evidently $y(x_n) = 0$ for $n = 1, 2, \ldots$. But $y \in \widetilde{[y_n]}$, hence $0 = \sum_{i=1}^{n} y(x_i)y_i \xrightarrow{\omega^*} y$, which contradicts the fact that y is not 0.

Before proceeding to the next theorem, we recall a renorming theorem due independently to Kadec [6] and Klee [7]: *if X^* is separable then there is a norm $|||\cdot|||$ on X equivalent to the original norm on X such that for any sequence $(x_n^*) \subset X^*$ and $x^* \in X^*$, if $x_n^* \xrightarrow{\omega^*} x^*$ and $|||x_n^*||| \to |||x^*|||$ then $x_n^* \to x^*$.* (For a simple proof of the theorem of Kadec–Klee see [18], p. 486).

THEOREM III.2. *Suppose that X^* is separable, $(y_n) \subset X^*$, $y_n \xrightarrow{\omega^*} 0$, and $\limsup\|y_n\| > 0$. Then (y_n) has a boundedly complete ω^*-basic subsequence (y_{n_i}).*

Proof. By the aformentioned renorming theorem, we may assume that for any sequence $(x_n^*) \subset X^*$ and $x^* \in X^*$, if $x_n^* \xrightarrow{\omega^*} x^*$ and $\|x_n^*\| \to \|x^*\|$, then $x_n^* \to x^*$. By Theorem III.1 we may extract a ω^*-basic subsequence (y_{n_i}) of (y_n) and find $(x_i) \subset X$ so that (x_i, y_{n_i}) is biorthogonal and if $S_m\colon \widetilde{[y_{n_i}]} \to \widetilde{[y_{n_i}]}$ is defined (for $m = 1, 2, \ldots$) by $S_m y = \sum_{i=1}^{m} y(x_i)y_{n_i}$, then

$$\text{(a)} \qquad \|S_m\| \to 1.$$

Suppose that $y \in \widetilde{[y_{n_i}]}$. Then $S_m y \xrightarrow{\omega^*} y$ because (y_{n_i}) is ω^*-basic and thus $\liminf\|S_m y\| \geqslant \|y\|$. But from (a) it follows that $\limsup\|S_m y\| \leqslant \|y\|$, hence $S_m y \to y$ and $y \in [y_{n_i}]$. Thus $\widetilde{[y_{n_i}]} = [y_{n_i}]$ and (y_{n_i}) is boundedly complete by Proposition II.1 (b). ■

We complete this section with a result that gives information even in the case where X is non-separable.

THEOREM III.3. *Suppose that $(y_n) \subset Y \subset X^*$, Y^* is separable, $y_n \xrightarrow{\omega} 0$, and $\limsup\|y_n\| > 0$. Then (y_n) contains a shrinking ω^*-basic subsequence.*

Proof. By passing to a subsequence, we may assume by Proposition II.3 that (y_n) is a shrinking basic sequence. By the proof of Theorem III.1 (see the remark immediately following the proof of III.1), we may choose a basic subsequence (y_{k_i}) of (y_n) such that if (f_i) is the sequence in $(y_{k_i})^*$ biorthogonal to (y_{k_i}) and $T\colon X \to [y_{k_i}]^*$ is defined by $Tx(y) = y(x)$ for all $x \in X$ and $y \in [y_{k_i}]$, then $T(X) \supset [f_i]$. But since (y_n) is shrinking so is (y_{k_i}), whence $[f_i] = [y_{k_i}]^*$. It follows by Proposition II.1 (c) that (y_{k_i}) is a shrinking ω^*-basic sequence. ■

Remark III.3. It is an open question if every X has a separable quotient space. Theorem III.3 yields immediately that X has this property if X^* has a subspace with a separable dual. A slight modification of the proof of Theorem III.1 yields that X has a separable quotient space under the following hypothesis: there exists a sequence $X_1, X_2, \ldots$ of subspaces of X with $X_n \subsetneqq X_{n+1}$ for all n and $X = \bigcup_{n=1}^{\infty} X_n$. Indeed, if for all n one chooses $y_n \in X^*$ of norm one with $y_n \in X_n^{\perp}$, then (y_n) has a ω^* basic subsequence. (It is trivial that if X has a separable quotient space, then there exists such a sequence (X_n) of subspaces of X).

IV. Applications of ω^*-basic sequences. The results of this section, with the exception of Theorem IV.4, are either easy consequences of the theorems of the preceding section (e.g. IV.1 and IV.2), or of the proofs of these theorems and previously known arguments (e.g. IV.3).

Proof of Theorem IV.1. (i) is an immediate consequence of Theorem III.1 and Proposition II.1 (a), because for any separable X, X^* contains a sequence (y_n) with $y_n \xrightarrow{\omega^*} 0$ and $\|y_n\| = 1$ for $n = 1, 2, \ldots$

Letting X, Y be as in (ii), by Theorem III.2 there exists a boundedly complete normalized basic sequence (y_n) in Y. Since X is separable, there exists a weak* convergent subsequence (y_{n_i}) of (y_n). Putting $z_j = y_{n_{2j+1}} - y_{n_{2j}}$ for $j = 1, 2, \ldots$, (z_j) is a boundedly complete basic sequence with $\inf\|z_j\| > 0$, and of course $z_j \xrightarrow{\omega^*} 0$. Hence by Theorem III.1, (z_j) contains a ω^* basic subsequence $(z_{k_j})_j$ of course (z_{k_j}) is also boundedly complete, and the assertions of (ii) now follow from Proposition II.1.

Finally, the first conclusion in (iii) follows from Theorem III.3 and Proposition II.1 (c), because if Y^* is separable then Y contains a sequence (y_n) with $y_n \xrightarrow{\omega} 0$ and $\|y_n\| = 1$ for $n = 1, 2, \ldots$ Now if Z is a quotient space of X then Z^* is isomorphic to a subspace of X^*. If also Z has a boundedly complete basis (z_n) with biorthogonal functionals (z_n^*), then Z is isomorphic to $[z_n^*]^*$, whence $[z_n^*]^{**}$ is isomorphic to a subspace of X^*. ■

Remark IV.1. We do not know if every separable space X has a quotient space with a shrinking basis; by IV.1 (ii), this is equivalent to whether X^* has a boundedly complete basic sequence. However the conclusion of (ii) concerning Y requires critically the assumptions of (ii). For example suppose that Y has no reflexive subspaces and $X = Y^*$. Then regarding Y as canonically imbedded in $X^* = Y^{**}$, Y has no weak* closed subspaces. If $Y = l^1$, Y is a separable conjugate space yet X is non-separable; if $Y = c_0$, X is separable but Y is not isomorphic to a subspace of a separable conjugate space; it is well known that neither c_0 nor l^1 have reflexive subspaces.

Our next result shows that if Y has no reflexive subspaces, then either Y^* is non-separable or Y cannot be imbedded in a separable conjugate space.

THEOREM IV.2. *Suppose that X^* is separable; $Y \subset X^*$, and Y^* is separable. Then Y is somewhat reflexive.*

Proof. Let Z be a subspace of Y. Then also Z^* is separable, hence contains a shrinking basic sequence (y_n) with $\|y_n\| = 1$ for $n = 1, 2, \ldots$, by Proposition II.3. Of necessity $y_n \xrightarrow{\omega} 0$ hence $y_n \xrightarrow{\omega^*} 0$, whence by Theorem III.2 (y_n) has a subsequence (y_{n_i}) which is boundedly complete. Of course (y_{n_i}) is also shrinking, hence $[y_{n_i}]$ is a reflexive subspace of Z. ■

COROLLARY IV.1. *If X^{**} is separable then both X and X^* are somewhat reflexive.*

Proof. That X^* is somewhat reflexive follows immediately from Theorem IV.2. Now suppose $Y \subset X$. Y can be considered as a subspace of the separable conjugate space X^{**} and Y^* is separable, hence Y contains a reflexive subspace by Theorem IV.3. ■

Remark IV.2. We do not know if the converse to IV.2 is true. That is, if Y is separable and somewhat reflexive, is Y^* separable and Y isomorphic to a subspace of a separable conjugate space? (The first part of this question is also raised in [2]). It follows from a recent result of Lindenstrauss [9] and the above corollary that there exists a separable somewhat reflexive space Y which is not isomorphic to a complemented subspace of any conjugate space. (A subspace Z of X is said to be complemented in X if there exists an idempotent operator on X whose range is Z.) Indeed, Lindenstrauss showed that there exists a Y with Y^{**} separable and Y^{**}/Y isomorphic to c_0 (the space of sequences which converge to zero). Our corollary yields that Y (and also Y^* and Y^{**}) are somewhat reflexive. Nevertheless if Y were isomorphic to a complemented subspace of a conjugate space, Y would be complemented in Y^{**}, which would imply that c_0 is isomorphic to a subspace of the separable conjugate space Y^{**}, an impossibility (see the remark following the proof of Theorem IV.1).

Our next result follows easily from our proof of Theorem III.1 and an argument of Lindenstrauss and Pełczyński [10]. Before proceeding to it, we need some preliminaries: L_1 denotes the space of Lebesgue integrable functions on $[0,1]$ with $\|f\| = \int_0^1 |f|\,dt$. Given a measurable set $A \subset [0,1]$, χ_A denotes its characteristic function and $|A|$ its Lebesgue measure. The Haar basis $(h_n)_{n=0}^{\infty}$ for L_1 (c.f. [18], p. 13] is defined as follows:

$$h_0 \equiv 1,$$

$$h_{2^k+i-1} = \chi_{[(2i-2)2^{-k-1},\,(2i-1)2^{-k-1}]} - \chi_{[(2i-1)2^{-k-1},\,(2i)2^{-k-1}]},$$

$(1 \leqslant i \leqslant 2^k,\ k = 0, 1, 2, \ldots)$.

Let $(h_n^*)_{n=0}^{\infty}$ denote the functionals biorthogonal to (h_n), regarded as elements of $L^\infty[0,1]$. It is known and easily seen that $[h_n^*]$ is isometric to the space of continuous functions on the Cantor discontinuum, under the supremum norm. Indeed, the linear span of (h_n^*) consists of those functions on $[0,1]$ which are equal almost everywhere to a step function which has breaks at dyadic rationals. Then letting $\{0,1\}^\infty$ denote the compact space of all sequences of zeros and ones, there is a unique surjective linear isometry $T\colon [h_n^*] \to C(\{0,1\}^\infty)$ such that $T1 = 1$ and for all n and j with $0 \leqslant j < 2^n$, $n = 1, 2, \ldots$, $T\chi_{\left[\frac{j}{2^n}, \frac{j+1}{2^n}\right]} = \chi_{A_j^n}$, where $A_j^n = \{(x_i) \in \{0,1\}^\infty\colon x_i = \varepsilon_i$ for all $1 \leqslant i \leqslant n$ and $j = \sum_{i=1}^{n} \varepsilon_i 2^{i-1}$ with $\varepsilon_i = 0$ or 1 for all $1 \leqslant i \leqslant n\}$.

We recall finally the following consequence of the Liapounoff convexity theorem [8]: If F is a finite subset of $L^\infty[0,1]$, and B is a measureable subset of $[0,1]$, then there exists a measurable subset A of B with $|A| = (1/2)|B|$ such that $\int_A f\,dt = \int_{B\sim A} f\,dt$ for all $f \in F$.

THEOREM IV.3. *Let X be a separable Banach space and Y a subspace of X^*.*

(a) *If Y is isomorphic to l^1, there exists a ω^*-basic sequence (y_n) in Y such that (y_n) is equivalent to the usual basis for l^1. Consequently c_0 is isomorphic to a quotient space of X.*

(b) *If Y is isomorphic to L_1, there exists a ω^*-basic sequence $(y_n)_{n=0}^{\infty}$ in Y such that $(y_n)_{n=0}^{\infty}$ is equivalent to the Haar basis for L_1. Consequently $C(\{0,1\}^\infty)$ is isomorphic to a quotient space of X.*

Proof. (a) follows easily from Theorem III.1 and Propositions II.1 and II.2. Let $(y_n) \subset X^*$ be a basic sequence equivalent to the usual unit vector basis for l^1. Since X is separable, (y_n) has a weak* convergent subsequence (y_{n_i}). Then $(y_{n_{2i}} - y_{n_{2i+1}}) \xrightarrow{\omega^*} 0$ and $(y_{n_{2i}} - y_{n_{2i+1}})$ is equivalent to the usual basis for l^1. Thus by Theorem IV.1, $(y_{n_{2i}} - y_{n_{2i+1}})$ has a ω^*-basic subsequence (z_i); (z_i) is equivalent to the usual basis for l^1 and hence X has a quotient space isomorphic to c_0 by Propositions II.1 and II.2.

To prove (b), we combine the proof of Theorem III.1 and the proof of Theorem 4.1 of [10] as follows:

Let Y be as in (b) and let $T\colon Y \to L_1$ be a surjective isomorphism. Let $(\varepsilon_j)_{j=0}^{\infty}$ be an increasing sequence of positive numbers with $\sum_{j=0}^{\infty} \varepsilon_j < 1$. We may choose a sequence $(F_n)_{n=0}^{\infty}$ of finite subsets of $\{x \in X\colon \|x\| = 1\}$ with the linear span of $\bigcup_{j=0}^{\infty} F_j$ dense in X, a sequence $(A_n)_{n=0}^{\infty}$ of measureable subsets of $[0,1]$ (with $A_0 = [0,1]$) and a sequence $(y_n)_{n=0}^{\infty}$ of elements of Y (with $Ty_0 \equiv 1$) satisfying the following properties for $n = 0, 1, 2, \ldots$; and for $0 \leqslant j < 2^r$, $r = 0, 1, 2, \ldots$:

(i) for all $f \in [(y_i)_{i=0}^n]^*$ with $\|f\| = 1$, there is an $x \in F_n$ such that $|y(x) - f(y)| \leqslant \varepsilon_n/3\, \|y\|$ for all $y \in [(y_i)_{i=0}^n]$;

(ii) $y_{n+1} \in F_n^{\perp}$;

(iii) $Ty_{2^r+j} = \chi_{A_{2^{r+1}+2j-1}} - \chi_{A_{2^{r+1}+2j}}$;

(iv) $|A_{2^r+j}| = 2^{-r}$;

(v) $A_{2^{r+1}+2j-1} \cup A_{2^{r+1}+2j} = A_{2^r+j-1}$;

(vi) $A_{2^{r+1}+2j-1} \cap A_{2^{r+1}+2j} = \emptyset$.

To see that this is possible, for each $f \in X$, let $\tilde{f}$ denote the unique element of $L^\infty[0,1]$ such that $\int_0^1 \tilde{f}(t)\,(Ty)(t)\,dt = y(f)$ for all $y \in Y$ and let $\{d_0, d_1, \ldots\}$ be a dense subset of $\{x \in X\colon \|x\| = 1\}$. Put $y_0 = T^{-1}(1)$, $A_0 = [0,1]$, and choose F_0 a finite set of elements of X of norm one satisfying (i) for $n = 0$, such that $d_0 \in F_0$.

Suppose that y_q and F_q have been defined for all $q < n = 2^r + j$ and that A_q has been defined for $q < 2^{r+1} + 2_{j-1}$ satisfying (i)–(vi). By the Liapounoff convexity theorem, there exist measureable sets A and B such that $A \cup B = A_{2^r+j-1}$, $A \cap B = \emptyset$, $|A| = |B|$, and

$$\int_A \tilde{f}\,dx = \int_B \tilde{f}\,dx \quad \text{for all } \tilde{f} \in F_{n-1}.$$

Define $A_{2^{r+1}+2j-1} = A$, $A_{2^{r+1}+2j} = B$, and y_n by (iii). Now using the compactness of the unit ball of a finite dimensional space, choose F_n satisfying (i) with $d_n \in F_n$. This completes the inductive definition of these objects.

As observed in [10], it follows that $(Ty_n)_{n=0}^{\infty}$ is isometrically equivalent to the Haar basis of L_1, hence (y_n) is equivalent to (h_n). Our proof of III.1 yields immediately that $(y_n)_{n=0}^{\infty}$ is a ω^* basic sequence, and thus by Propositions II.1 and II.2, $X/(y_n)^{\top}$ is isomorphic to $[h_n^*]$, which is in turn isometric to $C(\{0,1\}^\infty)$ by our preliminary remarks. ∎

Remark IV.3. Our proof of the above result and III.1 yields that if l^1 (resp. L_1) is isometric to a subspace of X^* and X is separable, then c_0 (resp. $C(\{0,1\}^\infty)$) is isometric to a quotient space of X. The proofs of Theorem IV.1 and Theorem III.1 also yield that if E is a symmetric function space (as defined in [10]) such that the Haar system (h_n) is a basis for E, then if Y is a subspace of X^* isomorphic to E and X is separable, Y contains a ω^*-basic sequence (y_n) equivalent to (h_n) (and consequently $[h_n^*]$ (in E^*) is isomorphic to a quotient space of X).

Remark IV.4. The result mentioned in the Introduction is an immediate consequence of a theorem of Milutin which asserts that $C[0,1]$ is isomorphic to $C(\{0,1\}^\infty)$ (c.f. [16]).

Remark IV.5. It is a theorem of Pełczyński [15] that if X^* contains a semi-norming subspace isomorphic to $L_1(\mu)$ for some non-purely atomic measure μ, then X contains a subspace isomorphic to l^1. It has recently been observed by James Hagler that the proof in [15] may be modified so as to yield this result without the "semi-norming" hypothesis. Actually, this result also follows from our Theorem IV.3. For if X^* contains a subspace isomorphic to $L_1(\mu)$ for some non-purely atomic μ, it also contains a subspace Y isomorphic to $L_1[0,1]$. But then there is a separable subspace Z of X such that the (separable) space Y is isomorphic to a subspace of Z^*, and hence by Theorem IV.3, $C(\{0,1\}^\infty)$ is isomorphic to a quotient space of Y. It then follows easily (c.f. [15]) that l^1 is isomorphic to a subspace of Y, and hence of X. (On the other hand, the fact that $C[0,1]$ is isomorphic to a quotient space of X provided X is separable and $L_1[0,1]$ is isomorphic to a subspace of X^*, follows immediately from the observation of Hagler and the results of [15].)

For the final result of this paper, we need the following definitions and notation:

A sequence (E_n) of finite dimensional subspaces of X is called a *finite dimensional decomposition* (f.d.d., in short) for X provided that each $x \in X$ can be written uniquely as $\sum_{n=1}^{\infty} P_n x$ with $P_n x \in E_n$. The operators $Q_n\colon X \to X$ defined by $Q_n x = \sum_{i=1}^{n} P_i x$ are uniformly bounded, satisfy $Q_n Q_m = Q_{\min(n,m)}$, and $[Q_n X] = X$. Conversely, if (Q_n) is a uniformly bounded sequence of finite rank operators on X for which $Q_n Q_m = Q_{\min(n,m)}$ and $[Q_n X] = X$ then $([Q_n - Q_{n-1}]X)$ (where $Q_0 \equiv 0$) is a f.d.d. for X, and we call $([Q_n - Q_{n-1}]X)$ the f.d.d. determined by (Q_n). If (Q_n^*) determines a f.d.d. for X^*-equivalently, if $[Q_n^* X^*] = X^*$ – then the f.d.d. determined by (Q_n) is called *shrinking*.

A subset Λ of X^* is called *norm determining over* X provided that for each $x \in X$, $\|x\| = \sup\{|a(x)|: a \in \Lambda, \|a\| \leqslant 1\}$.

THEOREM IV.4. *If X is separable then there is a $Y \subset X$ such that Y and X/Y both have finite dimensional decompositions. If also X^* is separable, then Y may be chosen so that both Y and X/Y have shrinking finite dimensional decompositions.*

Proof. By a result of Gaposkin and Kadec [4] (see also Lemma 2 of Mackey's paper [11]) X has a biorthogonal sequence (x_n, x_n^*) with $[x_n] = X$ and $[x_n^*]$ norm determining over X. It follows easily that we can choose finite sets $\sigma_1 \subset \sigma_2 \subset \ldots$, $\Delta_1 \subset \Delta_2 \subset \ldots$ so that $\sigma = \bigcup \sigma_n$ and $\Delta = \bigcup \Delta_n$ are complementary infinite subsets of the positive integers and, for $n = 1, 2, \ldots$,

(i) if $x^* \in [(x_i^*)_{i \in \Delta_n}]$ there is $x \in [(x_i)_{i \in \Delta_n \cup \sigma_{n+1}}]$ so that $\|x\| = 1$ and $|x^*(x)| > \left(1 - \frac{1}{n+1}\right)\|x^*\|$;

(ii) if $x \in [(x_i)_{i \in \sigma_n}]$ there is $x^* \in [(x_i^*)_{i \in \sigma_n \cup \Delta_n}]$ so that $\|x^*\| = 1$ and $|x^*(x)| > \left(1 - \frac{1}{n+1}\right)\|x\|$.

For $n = 1, 2, \ldots$, define $S_n: X \to X$ and $T_n: X \to X$ by $S_n x = \sum_{i \in \sigma_n} x_i^*(x) x_i$, $T_n x = \sum_{i \in \Delta_n} x_i^*(x) x_i$. We claim that, for $n = 1, 2, \ldots$,

(iii) $\|T^*_{n|(x_i)^{\perp}_{i \in \sigma_{n+1}}}\| \leqslant 1 + \frac{1}{n}$;

(iv) $\|S_{n|(x_i^*)^{\top}_{i \in \Delta_n}}\| \leqslant 1 + \frac{1}{n}$.

To see that (iii) holds, suppose that $y \in (x_i)^{\perp}_{i \in \sigma_{n+1}}$ and using (i) pick $x \in [(x_i)_{i \in \Delta_n \cup \sigma_{n+1}}]$ so that $\|x\| = 1$ and $|T_n^* y(x)| > \left(1 - \frac{1}{n+1}\right)\|T_n^* y\|$. Since $y \in (x_i)^{\perp}_{i \in \sigma_{n+1}}$, $|y(x)| = |T_n^* y(x)|$ and hence $\|y\| \geqslant \left(1 - \frac{1}{n+1}\right)\|T_n^* y\|$ so that $\|T_n^* y\| \leqslant \left(1 + \frac{1}{n}\right)\|y\|$. Thus (iii) is true and (iv) follows from (ii) in a similar manner.

We show now that if $y \in (x_i)^{\perp}_{i \in \sigma}$ then $T_n^* y \xrightarrow{\omega^*} y$. By (iii) $(T_n^* y)$ is bounded; hence has a weak* cluster point, say, x^*. Evidently $T_n^* x^* = T_n^* y$ for $n = 1, 2, \ldots$, hence $x^* - y \in (x_i)^{\perp}_{i \in \Delta_n}$ for each $n = 1, 2, \ldots$, whence $x^* - y \in (x_i)^{\perp}_{i \in \Delta}$. Since also $x^* - y \in (x_i)^{\perp}_{i \in \sigma}$, $x^* - y = 0$ and $x^* = y$. Thus $T_n^* y \xrightarrow{\omega^*} y$ and $(x_i)^{\perp}_{i \in \sigma} = [\overline{(x_i^*)_{i \in \Delta}}]$. Setting $Y = (x_i^*)^{\top}_{i \in \Delta}$ and using the obvious analogue of Proposition II.1 (a) for f.d.d.'s, we have that X/Y has a f.d.d.

Now $Y = (x_i^*)^{\top}_{i \in \Delta} = [\overline{(x_i^*)_{i \in \Delta}}]^{\top} = (x_i)^{\perp\top}_{i \in \sigma} = [(x_i)_{i \in \sigma}]$, so from (iv) it follows that $(S_{n|Y})$ determines a f.d.d. for Y. This completes the proof of the first statement.

Suppose now that X^* is separable. By the aforementioned result of Mackey and Gaposkin–Kadec, the biorthogonal sequence (x_n, x_n^*) may be chosen so that $[x_n] = X$ and $[x_n^*] = X^*$. Also, in view of the renorming theorem of Kadec–Klee mentioned in Section III, we may assume that for any sequence $(y_n) \subset X^*$ and $y \in X^*$, if $y_n \xrightarrow{\omega^*} y$ and $\|y_n\| \to \|y\|$ then $y_n \to y$. But it follows from (ii) (cf. the proof of Theorem III.2) that if $x^* \in (x_i)^{\perp}_{i \in \sigma}$ then $\|T_n^* x^*\| \to \|x^*\|$, hence $T_n^* x^* \to x^*$, whence $[(x_i^*)_{i \in \Delta}] = [\overline{(x_i^*)_{i \in \Delta}}]$. Using the f.d.d. variant of Proposition II.1 (b), we have that X/Y has a shrinking f.d.d. Of course, since $[x_n^*] = X^*$, $(S_{n|Y})$ automatically determines a shrinking f.d.d. for Y. ■

Remark IV.6. Kadec and Pełczyński have proved the following theorem related to Theorem IV.4 (see [18], p. 489): If (x_n, x_n^*) is a biorthogonal sequence with $[x_n] = X$ and $[x_n^*]$ norming over X, then the positive integers can be partitioned into disjoint infinite subsets σ and Δ with $\sigma = \bigcup \sigma_n$, $\Delta = \bigcup \Delta_n$, where (σ_n) and (Δ_n) are disjoint and finite, and such that $[(x_i)_{i \in \sigma_n}]_{n=1}^{\infty}$ (resp. $[(x_i)_{i \in \Delta_n}]_{n=1}^{\infty}$) is a finite dimensional decomposition for $(x_i^*)^{\top}_{i \in \Delta}$ (resp. for $(x_i^*)^{\top}_{i \in \sigma}$).

References

[1] C. Bessaga and A. Pełczyński, *On bases and unconditional convergence of series in Banach spaces*, Bull. Acad. Polon. Sci. 4 (1959), pp. 379–383.

[2] W. J. Davis and I. Singer, *Boundedly complete M-bases and complemented subspaces in Banach spaces* (preprint).

[3] D. W. Dean, I. Singer and L. Sternbach, *On shrinking basic sequences in Banach spaces*, Studia Math. 40 (1971), pp. 23–33.

[4] V. F. Gaposkin and M. I. Kadec, *Operator bases in Banach spaces*, Matem. Sbornik 61 (103) (1963), pp. 3–12 (Russian).

[5] R. Herman and R. Whitley, *An example concerning reflexivity*, Studia Math. 28 (1967), pp. 289–294.

[6] M. I. Kadec, *On the connection between weak and strong convergence*, Dopovidi Akad. Nauk Ukraïn RSR (1959), pp. 949–952 (Ukrainian, Russian and English summaries).

[7] V. L. Klee, *Mappings into normed linear spaces*, Fund. Math. 49 (1960), pp. 25–34.

[8] J. Lindenstrauss, *A short proof of Liapounff's convexity theorem*, J. Math. and Mech. 15 (1966), pp. 971–972.

[9] — *On James' paper 'Separable Conjugate Spaces'*, Israel J. Math. 9 (1971); pp. 279–284.

[10] — and A. Pełczyński, *Contributions to the theory of the classical Banach spaces*, J. Func. Anal. 8 (1971), pp. 225–249.

[11] G. Mackey, *Note on a theorem of Murray*, Bull. Am. Math. Soc. 52 (1946), pp. 322–325.

[12] V. D. Milman, *The geometric theory of Banach spaces*, Uspehi Matem. Nauk 25 (153) 1970, pp. 132–138 (Russian).

[13] A. Pełczyński, *Some problems in Banach and Fréchet spaces*, Israel J. Math. 2 (1964), pp. 132–138.

[14] — *A note on the paper of I. Singer 'Basic sequences and reflexivity of Banach spaces'*, Studia Math. 21 (1962), pp. 371–374.

[15] — *On Banach spaces containing $L_1(\mu)$*, Studia Math. 30 (1968), pp. 231–246.

[16] — *Linear extensions, linear averagings, and their applications to linear topological classification of spaces of continuous functions*, Dissertationes Math. (Rozprawy Mat.) 58 (1968).

[17] I. Singer, *Weak* bases in conjugate Banach spaces*, Studia Math. 21 (1961), pp. 75–81.

[18] — *Bases in Banach spaces I*, 1970.

UNIVERSITY OF HOUSTON
HOUSTON, TEXAS

UNIVERSITY OF CALIFORNIA, BERKELEY
BERKELEY, CALIFORNIA

Received June 16, 1971 (350)