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On the Riesz–Fischer theorem for vector-valued functions

by

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Dedicated to Professor A. Zygmund on the occasion of 50th anniversary of his scientific research

Abstract. Let φ: $\langle 0, \infty) \to R_+$ be a nondecreasing continuous function satisfying conditions $\varphi(u)/u \to 0$ if $u \to 0$, $\varphi(u)/u \to \infty$ if $u \to \infty$, X let denote a Banach space, Ξ its dual space. Let, further, $X^\wedge$ denote a vector space consisting of sequences $x^\wedge = \{x_i\}$, $x_i \in X$.

Assuming that φ is a convex function on $X^\wedge$ one can define a modular $\varrho_\varphi(x) = \sup\sum_1^\infty \varphi(|\xi(x_i)|)$, where supremum is taken over the ball $\Xi_0 = \{\xi: \|\xi\| \leqslant 1\}$.

Investigated are the properties of the space $l^{*\varphi}(X)$, elements of which are the sequences $x^\wedge \in X^\wedge$ such that $\varrho_\varphi(\lambda x^\wedge) < \infty$ for some $\lambda > 0$. Section 2 of the paper deals with the spaces of vector functions $x(\cdot)$: $\langle a, b\rangle \to X$, of finite Riesz φ-variation (as defined in 2.1) and with the spaces $V^{*\varphi}(X)$.

In Section 3 certain remarks are made about orthogonal series of the form $(*)$ $x_1\varphi_1 + x_2\varphi_2 + \ldots$ where $x_i \in X$, and $\{\varphi_i\}$ is an orthogonal system in $\langle a, b\rangle$.

If $x(\cdot)$: $\langle a, b\rangle \to X$ is a vector function absolutely continuous in $\langle a, b\rangle$, then its Fourier coefficients are represented by $x_n = \int_{\langle a,b\rangle} \varphi_n(t)\,dx$ where the integral in this formula is a (Dunford) integral $\langle a, b\rangle$ with respect to the vector measure $x(:)$ associated with $x(\cdot)$.

Using the spaces $l^{*\varphi}(X)$, $V^{*\varphi}(X)$, where $\varphi(u) = u^2$, authors obtain the analogue of Riesz–Fischer Theorem for series of the form $(*)$.

1. In this note X always stands for a real Banach space provided with a norm $\|\cdot\|$, Ξ for its conjugate space, $\Xi_0 = \{\xi \in \Xi;\ \|\xi\| \leqslant 1\}$. H denotes the class of all zero-one sequences $\{\eta_i\}$, $\mathscr{I}$ denotes the algebra of subsets of an interval $T = \langle a, b\rangle$ whose elements are finite unions of intervals $\langle c, d\rangle$, $a \leqslant c < d \leqslant b$, $\langle d, b\rangle$, $a \leqslant d < b$ and the empty set, $\mathscr{E}$ is the σ-algebra of Lebesgue measurable subsets of T and μ is the Lebesgue measure on $\mathscr{E}$. Measurability of sets and functions are always understood with respect to μ.

$x(\cdot)$, $y(\cdot)$, ... or $x, y, \ldots$ always denote vector-valued functions from T into X, $f(\cdot)$, $g(\cdot)$, ... or $f, g, \ldots$ real-valued function on T. A series $\sum_1^\infty x_i$ of elements belonging to a Banach space is said to be *perfectly con-*

vergent if the series $\sum_1^\infty \eta_i x_i$ is convergent for any $\{\eta_i\} \in H$, and *perfectly bounded* if the set of sums $\sum_1^n \eta_i x_i$, $\{\eta_i\} \in H$, $n = 1, 2, \ldots$ is bounded.

The purpose of the present paper is to discuss some modular spaces of sequences with vector-valued terms and some modular spaces of vector-valued set functions. In fact, we shall deal with some classes of modular spaces of σ-additive set functions $x(:)$ for which the (weak) integral $\int_T f(t)\mu(dx)$ in the sense of Dunford exists for any f belonging to a modular space of scalar functions. The spaces $l^{*\varphi}(X)$, $V^{*\varphi}(X)$ we deal with have been to some extent investigated in [8] under the assumption that $\varphi(u) = u^\alpha$, $\alpha \geqslant 1$. It seems that modular spaces of this kind may be of an interest for example, in the study of orthogonal (or base) expansions with respect to systems of scalar functions but with coefficients from a Banach space. We intend to return to this subject in another publication. At present occasion we limit ourselves to a very simple application, giving in Section 3 a formulation of Riesz–Fischer's theorem for vector-valued functions. In our opinion, it is formulated in more adequate terms than that of S. Bochner and A. Taylor [3]. Another domain of possible applications could occur at studying some classes of linear operators (e.g. generalizations of (p, q)-summing operators).

1.1. In the subsequent considerations we shall denote by $x^\wedge, y^\wedge, \ldots$ sequences $\{x_i\}, \{y_i\}, \ldots$ with terms $x_i, y_i, \ldots$ belonging to X and by $a^\wedge, b^\wedge, \ldots$ sequences $\{a_i\}, \{b_i\}, \ldots$ with real terms, $X^\wedge$ stands for the linear space of all such sequences under the usual formation of addition of sequences and their multiplication by scalars. In the sequel φ denotes a continuous, nondecreasing function from $\langle 0, \infty)$ into R_+ such that $\varphi(u) = 0$ if and only if $u = 0$ (such a function will be called a *φ-function* shortly). The following conditions are of importance for us:

(o_1) $\quad \dfrac{\varphi(u)}{u} \to 0$ as $u \to 0$;

(∞_1) $\quad \dfrac{\varphi(u)}{u} \to \infty$ as $u \to \infty$.

The φ-function φ is said *to satisfy condition* (Δ_2) *for small* u if there are positive constants k, u_0 such that $\varphi(2u) \leqslant k\varphi(u)$ for $0 \leqslant u \leqslant u_0$. In the following sections φ always is tacitly assumed to be convex and to satisfy conditions (o_1), (∞_1) except of a few points where the limiting case $\varphi(u) = u$ occurs; it is always explicitly stated there. Under the assumptions (o_1), (∞_1) φ^* will denote the function *complementary* to φ, that is to say, defined for $v \geqslant 0$ by $\varphi^*(v) = \sup_{u \geqslant 0}(uv - \varphi(u))$. It is well known that φ^* is a convex φ-function satisfying (o_1), (∞_1).

Let us define the mapping $m_\varphi \colon \Xi \times X^\wedge \to \langle 0, \infty)$ by the formula

$$m_\varphi(\xi, x^\wedge) = \sum_{i=1}^\infty \varphi(|\xi(x_i)|).$$

Denote by $l^{*\varphi}(X)$ (or by $l^{*\varphi}$, if $X = R$) the class of those sequences $x^\wedge$ for which for any ξ there is a positive number λ such that $m_\varphi(\xi, \lambda x^\wedge) < \infty$, $l^{*\varphi}(X)$ is a linear space. Evidently $a^\wedge \in l^{*\varphi}$ if and only if $\sum_1^\infty \varphi(\lambda|a_i|)$ is finite for some $\lambda > 0$. Instead of $l^{*\varphi}(X)$ we shall also write $l^\alpha(X)$, l^α (when $X = R$) if $\varphi(u) = u^\alpha$, $\alpha \geqslant 1$.

1.2. The following statements are mutually equivalent:

(a) $x^\wedge \in l^{*\varphi}(X)$;

(b) if

$$(*) \qquad \varrho_\varphi(x^\wedge) = \sup_{\xi \in \Xi_0} m_\varphi(\xi, x^\wedge)$$

then there exists a constant $\lambda > 0$ such that

$$(**) \qquad \varrho_\varphi(\lambda x^\wedge) \leqslant 1$$

holds;

(c) if

$$({*}{*}{*}) \qquad n_\varphi(x^\wedge) = \sup \Big\| \sum_{i=1}^n a_i x_i \Big\|$$

where the supremum is taken over all $\{a_i\}$ $i = 1, \ldots, n$ such that $\sum_1^n \varphi^*(|a_i|) \leqslant 1$ and for all n, then $n_\varphi(x^\wedge)$ is finite.

(a) $\Rightarrow$ (b). For a fixed $x^\wedge \in l^{*\varphi}(X)$ and any $\xi \in \Xi$ set

$$\|\xi\|_\varphi = \inf\{\varepsilon > 0 \colon m_\varphi(\xi/\varepsilon, x^\wedge) \leqslant 1\}.$$

It is well known and easy to check that $\|\xi\|_\varphi$ is homogeneous and subadditive on Ξ. If $\xi_n \to \xi$ then $\liminf_n m_\varphi(\lambda\xi_n, x^\wedge) \geqslant m_\varphi(\lambda\xi, x^\wedge)$ for any $\lambda > 0$ and therefore $\liminf_n \|\xi_n\|_\varphi \geqslant \|\xi\|_\varphi$, whence, by the Orlicz–Gelfand lemma, the inequality $\|\xi\|_\varphi \leqslant k\|\xi\|$ follows. Since $\|\xi\| \leqslant 1$ implies $\|\xi\|_\varphi \leqslant k$, $m_\varphi(\lambda\xi, x^\wedge) = m_\varphi(\xi, \lambda x^\wedge)$, we obtain (**) with $\lambda = 1/k$.

(b) $\Rightarrow$ (c). Let $\sum_1^n \varphi^*(|a_i|) \leqslant 1$. The application of the Young inequality yields

$$\text{(i)} \qquad \Big| \sum_{i=1}^n a_i \xi(x_i) \Big| \leqslant \frac{1}{\lambda}\big(m_\varphi(\xi, \lambda x^\wedge) + 1\big),$$

and consequently by (b) the inequality

$$\Big\|\sum_{i=1}^{n} a_i x_i\Big\| \leqslant k,$$

holds, where $k \leqslant 2/\lambda$, $n = 1, 2, \ldots$. Thus (i) $n_\varphi(x^\wedge) \leqslant 2/\lambda$. To prove the implication (c) $\Rightarrow$ (a) let us suppose that $n_\varphi(x^\wedge) > 0$ and remark that $(*{*}*)$ implies $\sum_{i=1}^{n} |a_i|\,|\xi(x_i)| \leqslant n_\varphi(x^\wedge) r$, where $r = \sup\big(1, \sum_{1}^{n} \varphi^*(|a_i|)\big)$, $\xi \in \Xi_0$. We now apply a familiar argument. We chose $a_i \geqslant 0$ in such a manner that

$$a_i \frac{|\xi(x_i)|}{n_\varphi(x^\wedge)} = \varphi\left(\frac{|\xi(x_i)|}{n_\varphi(x^\wedge)}\right) + \varphi^*(a_i),$$

whence

$$\sum_{i=1}^{n} a_i \frac{|\xi(x_i)|}{n_\varphi(x^\wedge)} = \sum_{i=1}^{n} \varphi\left(\frac{|\xi(x_i)|}{n_\varphi(x^\wedge)}\right) + \sum_{i=1}^{n} \varphi^*(a_i) \leqslant r,$$

which yields $r = 1$, provided there is a $\xi(x_i) \neq 0$. In any case we get

(ii) $$\sum_{i=1}^{n} \varphi\left(\frac{|\xi(x_i)|}{n_\varphi(x^\wedge)}\right) \leqslant 1, \quad \text{for } n = 1, 2, \ldots,$$

thus (a) follows.

1.3. The functional $n_\varphi(x^\wedge)$ is a homogeneous norm in $l^{*\varphi}(X)$. Another homogeneous norm $\|x^\wedge\|_\varphi^c$ in $l^{*\varphi}(X)$, equivalent to $n_\varphi(x^\wedge)$, may be defined by the formula

$$\|x^\wedge\|_\varphi^c = \inf\{\varepsilon > 0\colon\ \varrho_\varphi(x^\wedge/\varepsilon) \leqslant 1\}.$$

In virtue of the inequalities (i), (ii) we have $\|x^\wedge\|_\varphi^c \leqslant n_\varphi(x) \leqslant 2\|x^\wedge\|_\varphi^c$. If $a^\wedge \in l^{*\varphi}$ then $\|a^\wedge\|_\varphi^c = \inf\{\varepsilon > 0\colon \sum_{1}^{\infty} \varphi(|a_i|/\varepsilon) \leqslant 1\}$.

1.3.1. *If $x^\wedge\colon 0, 0, \ldots, x, 0, \ldots$, then $n_\varphi(x^\wedge) = \varphi_{-1}^*(1)\|x\|$, if $x^\wedge \in l^{*\varphi}(X)$ then $\sup_i \|x_i\| \leqslant [\varphi_{-1}^*(1)]^{-1} n_\varphi(x^\wedge)$.*

This is a trivial consequence of the definition of $n_\varphi(x^\wedge)$.

1.3.2. *The space $l^{*\varphi}(X)$ provided with the norm $n_\varphi(x^\wedge)$ or $\|x^\wedge\|_\varphi^c$, respectively, is a Banach space and the inequalities*

$$\|x^\wedge\|_\varphi^c \leqslant n_\varphi(x^\wedge) \leqslant 2\|x^\wedge\|_\varphi^c$$

hold. For any $x^\wedge \in l^{*\varphi}(X)$, $a^\wedge \in l^{*\varphi^*}$ the Hölder inequality

$$\Big\|\sum_{i=1}^{\infty} a_i x_i\Big\| \leqslant n_\varphi(x^\wedge)\|a^\wedge\|_{\varphi^*}^c,$$

is satisfied, provided the series $\sum_{1}^{\infty} a_i x_i$ converges.

In view of 1.3 it remains to prove the completeness, e.g. of $n_\varphi(x^\wedge)$, and the Hölder inequality. The completeness can easily be verified applying 1.3.1, the Hölder inequality is a consequence of the definition of $n_\varphi(x^\wedge)$ and the equivalence of conditions $\sum_{1}^{\infty} \varphi^*(|a_i|) \leqslant 1$ and $\|a^\wedge\|_{\varphi^*}^c \leqslant 1$.

1.4. *If the series*

(*) $$\sum_{i=1}^{\infty} a_i x_i$$

is convergent for every $a^\wedge \in l^{*\varphi^*}$ then $x^\wedge \in l^{*\varphi}(X)$. If φ^* satisfies condition (Δ_2) for small u and $a^\wedge \in l^{*\varphi^*}$ then the series (*) is convergent for any $x^\wedge \in l^{*\varphi}(X)$.

The first part of the statement is a simple consequence of the Banach–Steinhaus theorem when applied to the sequence of operators $a^\wedge \to \sum_{1}^{n} a_i x_i$ on $l^{*\varphi^*}$. The second one follows from the Hölder inequality and the remark that, under the assumption of the (Δ_2)-condition, $\|a^{\wedge(n)} - a^\wedge\|_{\varphi^*}^c \to 0$ as $n \to \infty$, where $a^{\wedge(n)}\colon a_1, a_2, \ldots, a_n, 0, 0, \ldots$

1.5. Let us now discuss the limiting case $\varphi(u) = u$. Under this assumption the space $l^{*\varphi}(X)$ (written also as $l^1(X)$) is the space of those sequences $\{x_i\}$ for which the series $\sum_{i=1}^{\infty} |\xi(x_i)|$ is convergent for any $\xi \in \Xi$ or, equivalently, such that the sums $\sum_{1}^{n} a_i x_i$ $(n = 1, 2, \ldots)$ are bounded for every bounded sequence $a^\wedge$. This in turn is equivalent to the assumption that the series $\sum_{1}^{\infty} x_i$ is perfectly bounded in X. The space $l^1(X)$ will be equipped with the norm $n_1(x^\wedge) = \sup\big\|\sum_{1}^{n} a_i x_i\big\|$ where the supremum is taken over all n and all sequences $\{a_i\}$ such that $|a_i| \leqslant 1$. The space $l^1(X)$ provided with this norm is a Banach space. If we restrict the definition of $n_1(x^\wedge)$ to zero-one sequences we obtain another norm in $l^1(X)$ which is equivalent to $n_1(x^\wedge)$.

1.5.1. Write $\varrho_\varphi^s(x^\wedge) = \sum_{i=1}^{\infty} \varphi(\|x_i\|)$, $\lambda^{*\varphi}(X) = \{x^\wedge\colon \varrho_\varphi^s(\lambda x^\wedge) < \infty$ for some $\lambda > 0\}$. It is well known that $\varrho_\varphi^s(x^\wedge)$ is a convex modular in $\lambda^{*\varphi}(X)$ and that $\lambda^{*\varphi}(X)$ is under this modular a modular space in which a homogeneous norm may be defined, e.g. by $|||x^\wedge|||_\varphi^c = \inf\{\varepsilon > 0\colon\ \varrho_\varphi^s(x^\wedge/\varepsilon) \leqslant 1\}$. Clearly $l^{*\varphi}(X) \supset \lambda^{*\varphi}(X)$ and $\lambda^{*\varphi}(X)$ is a Banach space. If $\varphi(u) = u$ then $\lambda^{*\varphi}(X)$ (written also as $\lambda^1(X)$) is the space of absolutely convergent series. If $\varphi(u) = u^\alpha$, $\alpha \geqslant 1$, we write $\lambda^\alpha(X)$ instead of $\lambda^{*\varphi}(X)$.

1.6. The following statements are mutually equivalent:

(a) there are positive constants a, b, u_0 such that

$$(*)\qquad \psi(u) \leqslant a\varphi(bu) \quad \text{if } 0 \leqslant u \leqslant u_0;$$

(b) $l^{*\varphi}(X) \subset l^{*\psi}(X)$;

(c) $\lambda^{*\varphi}(X) \subset \lambda^{*\psi}(X)$;

(d) if $x_n^{\hat{}} \in l^{*\varphi}(X)$ and $x_n^{\hat{}} \in l^{*\psi}(X)$, then $\|x_n^{\hat{}}\|_\varphi^c \to 0$ as $n \to \infty$ implies $\|x_n\|_\psi^c \to 0$ as $n \to \infty$. The above statements concern also the limiting case $\varphi(u) = u$, $\psi(u) = u$.

(a) $\Rightarrow$ (b). Let $x^{\hat{}} \in l^{*\varphi}(X)$ and set $\lambda_0 = \varphi_{-1}^*(1)u_0/2$. Since $\|x^{\hat{}}\|_\varphi^c/n_\varphi(x^{\hat{}}) \geqslant \frac{1}{2}$ we obtain by 1.3.1 $\lambda_0\|x_i\|/\|x^{\hat{}}\|_\varphi^c \leqslant u_0$. In view of $(*)$ we have for $0 \leqslant \lambda \leqslant 1$, $\xi \in \Xi_0$,

$$\psi\left(\frac{\lambda\lambda_0|\xi(x_i)|}{\|x^{\hat{}}\|_\varphi^c}\right) \leqslant a\varphi\left(\frac{b\lambda\lambda_0|\xi(x_i)|}{\|x^{\hat{}}\|_\varphi^c}\right),$$

$$\varrho_\psi\left(\frac{\lambda\lambda_0 x^{\hat{}}}{\|x^{\hat{}}\|_\varphi^c}\right) \leqslant a\varrho_\varphi\left(\frac{b\lambda\lambda_0 x^{\hat{}}}{\|x^{\hat{}}\|_\varphi^c}\right).$$

Setting $\lambda = \inf\left((b\lambda_0)^{-1}, (b\lambda_0)^{-1}a^{-1}, 1\right)$ we get the term on the right-hand side of the last inequality less or equal 1, therefore $x^{\hat{}} \in l^{*\psi}(X)$ and $\|x^{\hat{}}\|_\psi^c \leqslant (\lambda\lambda_0)^{-1}\|x^{\hat{}}\|_\varphi^c$.

Thus (a) $\Rightarrow$ (d), too. The proof of the implication (a) $\Rightarrow$ (c) is similar. The implications (b) $\Rightarrow$ (a), (c) $\Rightarrow$ (a), (d) $\Rightarrow$ (a) are well known in the case when X is the space of scalars. The case of an arbitrary Banach space X may be immediately reduced to the former one.

1.7. Assume that a φ-function ψ fulfils condition (Δ_2) for small u and that a φ-function φ satisfies the condition $\varphi(u)/\psi(u) \to 0$ as $u \to 0$.

If, for a sequence of elements $x_n^{\hat{}} = \{x_{in}\} \in l^{*\psi}(X)$, the inequality $\|x_n^{\hat{}}\|_\psi^c \leqslant k$ holds for $n = 1, 2, \ldots$ and $\sup_i \|x_{in}\| \to 0$ as $n \to \infty$, then $\|x_n\|_\varphi^c \to 0$ as $n \to \infty$.

By 1.6 $(*)$ $x_n^{\hat{}} \in l^{*\varphi}(X)$ and we can assume $\|x_n^{\hat{}}\|_\psi^c \geqslant c > 0$ for $n = 1, 2, \ldots$ For any $\varepsilon > 0$

$$\text{(i)}\qquad \psi(u/\varepsilon) \leqslant \bar{k}\psi(u)$$

where $0 < \bar{k} < \infty$, for sufficiently small u. Choose $\eta > 0$ such that

$$\text{(ii)}\qquad \bar{k}\eta \leqslant 1.$$

Then there is a $\bar{u} > 0$ such that the inequalities (i) and

$$\text{(iii)}\qquad \varphi(u/\varepsilon) \leqslant \eta\psi(u/\varepsilon) \quad \text{if } 0 \leqslant u \leqslant \bar{u}$$

hold simultaneously. If, similarly as in the proof of 1.6, we set $\lambda_0 = \psi_{-1}^*(1)/\bar{u}12$, we get $\lambda_0 \sup_i \|x_{in}\|/\|x_n\|_\psi^c \leqslant \bar{u}$, whence by (i)

$$m_\psi\left(\xi, \frac{\lambda_0}{\varepsilon}\frac{x_n^{\hat{}}}{\|x_n^{\hat{}}\|_\psi^c}\right) \leqslant \bar{k}m_\psi\left(\xi, \frac{x_n^{\hat{}}}{\|x_n^{\hat{}}\|_\psi^c}\right) \quad \text{if } \xi \in \Xi_0,$$

$$\varrho_\psi\left(\frac{\lambda_0}{\varepsilon}\frac{x_n^{\hat{}}}{\|x_n^{\hat{}}\|_\psi^c}\right) \leqslant \bar{k}\varrho_\psi\left(\frac{x_n^{\hat{}}}{\|x_n^{\hat{}}\|_\psi^c}\right) \leqslant \bar{k}.$$

For $n \geqslant \bar{n}$, where $\bar{n}$ is sufficiently large, we have, by (iii), (ii)

$$\varrho_\varphi\left(\frac{\lambda_0}{\varepsilon}\frac{x_n^{\hat{}}}{k}\right) \leqslant \varrho_\varphi\left(\frac{\lambda_0}{\varepsilon}\frac{x_n^{\hat{}}}{\|x_n\|_\psi^c}\right) \leqslant \eta\varphi_\psi\left(\frac{\lambda_0}{\varepsilon}\frac{x_n^{\hat{}}}{\|x_n\|_\psi^c}\right) \leqslant \eta\bar{k} \leqslant 1.$$

Thus $\|x_n^{\hat{}}\|_\varphi^c \leqslant \varepsilon k\lambda_0^{-1}$ and $\|x_n^{\hat{}}\|_\varphi^c \to 0$ as $n \to \infty$ follows.

1.8. A sequence $x^{\hat{}} \in l^{*\varphi}(X)$ is called *regular* (in $l^{*\varphi}(X)$) if $n_\varphi(x^{\hat{}(n)} - x^{\hat{}}) \to 0$ as $n \to \infty$, where $x^{\hat{}(n)}$ denotes the sequence $x_1, x_2, \ldots, x_n, 0, 0, \ldots$ Let us remark that a sequence $x^{\hat{}} = \{x_i\}$ in $l^1(X)$ is regular if and only if the series $\sum_1^\infty x_i$ is perfectly convergent. As easily seen, 1.3.1 implies $\|x_i\| \to 0$ as a necessary condition for a sequence $x^{\hat{}}$ to be regular. Note that if $x^{\hat{}}$ is a regular sequence then for any $\{\eta_i\} \in H$ the sequence $\{\eta_i x_i\}$ is regular, too.

1.8.1. *The collection of all regular sequences in $l^{*\varphi}(X)$ forms a closed linear subspace of $l^{*\varphi}(X)$.*

1.8.2. *If φ satisfies condition (Δ_2) for small u and $x^{\hat{}} \in \lambda^{*\varphi}(X)$ then $x^{\hat{}}$ is regular in $l^{*\varphi}(X)$. This is also trivially true for $\varphi(u) = u$.*

For every $x^{\hat{}} \in \lambda^{*\varphi}(X)$ the series $\sum_1^\infty \varphi(\|x_i\|/\varepsilon)$ is convergent for any $\varepsilon > 0$, whence $\varrho_\varphi\left((x^{\hat{}(n)} - x^{\hat{}})/\varepsilon\right) \leqslant \sum_{i=n+1}^\infty \varphi(\|x_i\|/\varepsilon) \leqslant 1$ for sufficiently large n and $\|x^{\hat{}(n)} - x^{\hat{}}\| < \varepsilon$ follows.

1.8.3. *Let ψ be a φ-function such that $\varphi(u)/\psi(u) \to 0$ as $u \to 0$. Each of the following conditions is sufficient for a sequence $x^{\hat{}}$ to be regular in $l^{*\varphi}(X)$:*

(a) *$x^{\hat{}} \in l^{*\psi}(X)$, $x_i \to 0$, ψ satisfies condition (Δ_2) for small u,*

(b) *$x^{\hat{}} \in l^{*\psi}(X)$, $x^{\hat{}}$ is a regular sequence in $l^{*\psi}(X)$.*

The assumptions in (a) make possible an application of 1.7, thus $x^{\hat{}}$ is regular in $l^{*\varphi}(X)$. Since $\varphi(u) \leqslant \psi(u)$ for small u and, in view of 1.6 (d), the relation $\|x^{\hat{}(n)} - x^{\hat{}}\|_\psi^c \to 0$ implies $\|x^{\hat{}(n)} - x^{\hat{}}\|_\varphi^c \to 0$ as $n \to \infty$, thus (b) follows.

1.8.4. *If X is separable and every sequence belonging to $l^{*\varphi}(X)$ is regular then $l^{*\varphi}(X)$ is separable and conversely.*

If X is separable then the set $\{x^{\wedge(n)}\colon x^\wedge\in l^{*\varphi}(X),\ n=1,2,\ldots\}$ is separable. Consequently, if for any $x^\wedge\in l^{*\varphi}(X)$ $n_\varphi(x^{\wedge(n)}-x^\wedge)\to 0$, the space $l^{*\varphi}(X)$ is separable, too. Now, suppose that $l^{*\varphi}(X)$ is separable. If $x^\wedge; 0,0,\ldots,x,0,0,\ldots$ then $n_\varphi(x^\wedge)=\varphi_{-1}^{*}(1)\|x\|$, thus X is separable. We claim that all $x^\wedge$ are regular in $l^{*\varphi}(X)$. If this is not so then there are an $\varepsilon>0$ and two increasing sequences of indices $\{p_i\}$, $\{q_i\}$, $p_i<q_i<p_{i+1}$ for $i=1,2,\ldots$, such that $n_\varphi(x^{\wedge(q_i)}-x^{\wedge(p_i)})\geqslant\varepsilon$, $i=1,2,\ldots$ Denote by $Y^\wedge\subset l^{*\varphi}(X)$ the class of sequences $y^\wedge=\{y_i\}$ defined as follows: we set $y_i=x_i\eta_j$ if $p_j<i\leqslant q_j$, $j=1,2,\ldots$ where $\{\eta_j\}\in H$ and $y_j=0$ elsewhere. Clearly, for two different sequences $y^{\wedge\prime}, y^{\wedge\prime\prime}$ in $Y^\wedge$ we have $n_\varphi(y^{\wedge\prime}-y^{\wedge\prime\prime})\geqslant\varepsilon$ and since the mapping just defined of H onto $Y^\wedge$ is bijective and H is uncountable we get the contradiction with the separability of $l^{*\varphi}(X)$.

1.9. In this section we give some remarks and examples concerning the notion of the regularity and Theorems 1.8.1–1.8.3. Under our assumption that φ satisfies condition (o_1) any sequence $x^\wedge=\{x_i\}$, where $\sum_1^\infty x_i$ is perfectly convergent in X, is regular in $l^{*\varphi}(X)$. Let X be the space c_0 and x_n the unit sequence $\{\delta_{jn}\}$, $\delta_{jn}=0$ if $j\neq n$, $\delta_{nn}=1$, in c_0. The sequence $\{x_i\}$, $x_i=\{\delta_{ji}\}$, belongs to any $l^{*\varphi}(X)$ but, since $\|x_n\|=1$, it is not regular, though X is separable. More generally: if there exist in a Banach space X perfectly bounded series which are not perfectly convergent, then there are sequences $x^\wedge\in l^{*\varphi}(X)$ which are not regular. Indeed, for an arbitrary Banach space X, if $\sum_1^\infty x_i$ is a perfectly bounded series in X, the sequence $x^\wedge=\{x_i\}$ belongs to $l^{*\varphi}(X)$. On the other hand, we always can construct a perfectly bounded series $\sum_1^\infty x_i$ with $\|x_i\|=1$ for $i=1,2,\ldots$ if we only assume that perfect boundednes not always implies perfect convergence in X.

Let $X=L^\alpha\langle a,b\rangle$, $\alpha>1$, $1/\alpha+1/\alpha'=1$. Choose disjoint measurable sets e_n of positive measure in $\langle a,b\rangle$ and define functions $x_n(t)=\chi_{e_n}(t)\mu(e_n)^{-1/\alpha}$. If $\xi\in\Xi$ then $\xi(x_n)=\int_a^b x_n(t)f(t)\,dt$, $|\xi(x_n)|^{\alpha'}\leqslant\big(\int_{e_n}|x_n(t)|^\alpha dt\big)^{\alpha'/\alpha}\int_{e_n}|f(t)|^{\alpha'}dt<\infty$, whence $\sum_1^\infty|\xi(x_n)|^{\alpha'}\leqslant\int_a^b|f(t)|^{\alpha'}dt$ and consequently $x^\wedge=\{x_i\}\in l^\gamma(X)$, $\gamma\geqslant\alpha'$. Since $\|x_n\|_\alpha=1$, $x^\wedge$ is not regular. The following example is also worth remarking. Let $\{x_n\}$ be an orthonormal system of uniformly bounded functions in $\langle a,b\rangle$. Then it follows from the Young–Hausdorff–Riesz theorem that $x^\wedge=\{x_i\}\in l^\alpha(X)$, where $X=L^\alpha\langle a,b\rangle$, $\alpha\geqslant 2$, $1/\alpha+1/\alpha'=1$ (the restriction that x_i are uniformly bounded may be dropped if $\alpha=\alpha'=2$). Since $\|x_i\|_\alpha\geqslant c>0$ for $i=1,2,\ldots$ the sequence $x^\wedge$ is not regular in $l^\alpha(X)$. Concerning relations between the spaces $l^\alpha(X)$ and $\lambda^\alpha(X)$, let us recall the well known inclusions: $l^1(X)\subset\lambda^2(X)$ if $X=L^\alpha\langle a,b\rangle$, $1\leqslant\alpha\leqslant 2$, $l^1(X)\subset\lambda^\alpha(x)$ if $X=L^\alpha\langle a,b\rangle$, $\alpha\geqslant 2$.

2. For any interval $\delta\subset T$ $x(\delta)$ denotes the difference $x(t'')-x(t')$ where t'' is the right end point and t' the left end point of δ. For a function $x(\cdot)\colon T\to X$, we mean by *variation* of $x(\cdot)$ — in symbols $\operatorname{var} x$ — the value of $\sup_\pi\|\sigma(\pi)\|$, where $\sigma(\pi)=\sum_1^n x(\delta_i)$, $\pi\colon \delta_1,\delta_2,\ldots,\delta_n$ is a finite collection of disjoint intervals $\delta_i\subset T$, and the supremum is taken over all such collections. A function $x(\cdot)\colon T\to X$ is said to be *absolutely continuous* if for every $\varepsilon>0$ there is a $\lambda>0$ such that the inequality $\sum_1^n\mu(\delta_i)<\lambda$, where $\delta_1,\delta_2,\ldots,\delta_n$ are disjoint intervals in T, implies $\big\|\sum_1^n x(\delta_i)\big\|<\varepsilon$. Any such function is of bounded variation on T. Denote by $AC(X)$ the class of all absolutely continuous functions. The set $AC(X)$ becomes a Banach space under standard definitions of the sum of elements and of multiplication by scalars, and provided with the norm $\|x\|_0=\|x(a)\|+\operatorname{var}x$. The following well known fact is basic for our futher considerations:

There is a one-to-one correspondence between σ-additive set functions $x(:)\colon \mathscr{E}\to X$ which are also absolutely continuous with respect to μ and absolutely continuous functions $x(\cdot)$ on T, which are equal to 0 at a. This correspondence is established by setting $x(\langle a,t))=x(t)-x(a)=x(t)$ for $t\in T$. $x(:)$ will be called *associated* with $x(\cdot)$. In what follows we always restrict ourselves (except 2.1–2.2) to absolutely continuous functions $x(\cdot)$ such that $x(a)=0$. For such functions and measurable scalar functions $f(\cdot)$ the notion of an integral — written $\int_a^b f(t)\,dx$ — will be used. Namely, by this integral — if it exists — we mean the integral of Dunford type (the weak integral) $\int_T f(t)x(d\mu)$ where $f(\cdot)$ always is assumed to be a measurable scalar function and $x(:)$ is the set function associated with the function $x(\cdot)$ [5], [4], [6].

Let us still recall the definition of a space $L^{*\varphi}$ (an Orlicz space). For a given φ-function φ it is the Banach space of all measurable functions g for which $\int_T\varphi(\lambda|g(t)|)\,dt<\infty$ holds for some $\lambda>0$, provided with the norm $\|g\|_\varphi^c=\inf\{\varepsilon>0\colon \int_T\varphi(|g(t)|/\varepsilon)\,dt\leqslant 1\}$ (or with another B-norm equivalent to $\|\cdot\|_\varphi^c$). If $\varphi(u)=u^\alpha$, $\alpha\geqslant 1$, we shall denote by $\|\cdot\|_\alpha$ the respective norm $\|\cdot\|_\varphi^c$ and by L^α the space $L^{*\varphi}$.

2.1. For a partition π corresponding to the points $a=t_0<t_1<\ldots<t_n=b$ write $\delta_i=\langle t_{i-1},t_i)$ if $i=1,2,\ldots,n-1$, $\delta_n=\langle t_{n-1},t_n\rangle$,

$$(*)\qquad \sigma_\pi(\xi, x) = \sum_{i=1}^{n} \varphi\left(\frac{|\xi(x(\delta_i))|}{\mu(\delta_i)}\right)\mu(\delta_i),$$

$$(**)\qquad \sigma_\pi^s(x) = \sum_{i=1}^{n} \varphi\left(\frac{\|x(\delta_i)\|}{\mu(\delta_i)}\right)\mu(\delta_i),$$

and set $m_\varphi(\xi, x) = \sup\limits_\pi \sigma_\pi(\xi, x)$, $m_\varphi(x) = \sup\limits_\pi \sigma_\pi^s(x)$. We define $V^{*\varphi}(X) = \{x(\cdot):\ m_\varphi(\xi, \lambda x) < \infty$ for any $\xi \in \Xi$ and for some $\lambda = \lambda(\xi, x) > 0\}$, $W^{*\varphi}(X) = \{x(\cdot): m_\varphi(\lambda x) < \infty$ for some $\lambda = \lambda(x) > 0\}$. Evidently $V^{*\varphi}(X) \supset W^{*\varphi}(X)$ and routine arguments show that $V^{*\varphi}(X)$, $W^{*\varphi}(X)$ are linear spaces. If $X = R$, then $V^{*\varphi}(X) = W^{*\varphi}(X)$ and we shall denote by $V^{*\varphi}$ this space of scalar functions $f(\cdot)$. It is well known that $f \in V^{*\varphi}$ if and only if $f(t) = \int_a^t g(\tau)\,d\tau + c$, $a \leqslant t \leqslant b$, where g belongs to the Orlicz space $L^{*\varphi}$, i.e. $\int_a^b \varphi(\lambda|g(t)|)\,dt < \infty$ for some $\lambda > 0$ ([1], [2], [7], [9]).

2.2. *If $x \in V^{*\varphi}(X)$ then*

(a) *there exists a positive constant λ such that*

$$\varrho_\varphi(\lambda x) \leqslant 1,$$

where $\varrho_\varphi(x) = \sup\limits_{\xi \in \Xi_0} m_\varphi(\xi, x)$;

(b) *$x(\cdot)$ is absolutely continuous.*

Ad (a). If we define $\|\xi\|_\varphi$ by the formula

$$\|\xi\|_\varphi = \inf\{\varepsilon > 0:\ m_\varphi(\xi/\varepsilon, x) \leqslant 1\}$$

then the proof parallels that of 1.2. (b).

Ad (b). We can assume $\varrho_\varphi(x) \leqslant 1$. Let Δ be the union of disjoint intervals $\delta_1, \delta_2, \ldots, \delta_n$. By the convexity of φ and by the Jensen inequality the inequality

$$\text{(i)}\qquad \varphi\left(\frac{|\xi(x(\delta_1)+x(\delta_2)+\ldots+x(\delta_n))|}{\mu(\Delta)}\right)\mu(\Delta) \leqslant \sum_{i=1}^{n} \varphi\left(\frac{\xi(x(\delta_i))}{\mu(\delta_i)}\right)\mu(\delta_i),$$

holds for $\xi \in \Xi_0$. Since the sum on the right-hand side is $\leqslant 1$, we get

$$\|x(\delta_1)+x(\delta_2)+\ldots+x(\delta_n)\| \leqslant \varphi_{-1}(r)\frac{1}{r},$$

where $r = 1/\mu(\Delta)$. If $\mu(\Delta) \to 0$, then by condition (∞_1) $\varphi_{-1}(r)/r \to 0$.

For an additive set function $x(:)$: $\mathscr{E} \to X$ we define $\sigma_\pi(\xi, x(:))$ and $\sigma_\pi^s(x(:))$ analogously to (*) and (**), respectively, by replacing there δ_i by disjoint sets $e_i \in \mathscr{E}$ (if $\mu(e_i) = 0$, $x(e_i) = 0$ then the respective term in (*), (**) is supposed to be equal 0). Set $m_\varphi(\xi, x(:)) = \sup\limits_\pi \sigma_\pi(\xi, x(:))$, $\varrho_\varphi^{\sim}(x(:)) = \sup\limits_{\xi \in \Xi_0} m_\varphi(\xi, x(:))$, $m_\varphi(x(:)) = \sup\limits_\pi \sigma_\pi^s(x(:))$; using these functionals we define $V^{*\varphi}_{\mathrm{ass}}(X)$ or $W^{*\varphi}_{\mathrm{ass}}(X)$ analogously to $V^{*\varphi}(X)$ or $W^{*\varphi}(X)$, respectively. For $V^{*\varphi}_{\mathrm{ass}}(X)$ a theorem analogous to Theorem 2.2 holds. Consequently any $x(:) \in V^{*\varphi}_{\mathrm{ass}}(X)$ is absolutely μ-continuous and obviously it is associated with a function $x(\cdot)$ belonging to $V^{*\varphi}(X)$. Moreover, $\varrho_\varphi(x) \leqslant \varrho_\varphi^{\sim}(x(:))$. Conversely, if $x \in V^{*\varphi}(X)$ then its associated set function $x(:) \in V^{*\varphi}_{\mathrm{ass}}(X)$ and $\varrho_\varphi^{\sim}(x(:)) \leqslant \varrho_\varphi(x)$. Indeed, $x(:)$ is defined on $\mathscr{I}$ by setting $x(e) = x(\delta_1) + \ldots + x(\delta_n)$ if $e \in \mathscr{I}$, $e = \bigcup\limits_1^n \delta_i$, where δ_i are disjoint intervals. Assume $\varrho_\varphi(x) < \infty$. By (i) in 2.2 we have $\sigma_\pi(\xi, x(:)) \leqslant \varrho_\varphi(x)$ if $\xi \in \Xi_0$, for any partition $e_1, e_2, \ldots, e_n$ of T, where $e_i \in \mathscr{I}$. By the absolute continuity of $x(\cdot)$, $x(:)$ is absolutely continuous on $\mathscr{I}$, uniquely extendable to an absolutely continuous $x(:)$ on $\mathscr{E}$, and the last inequality remains true for any partition of T into measurable sets e_i. Therefore $\varrho_\varphi^{\sim}(x(:)) \leqslant \varrho_\varphi(x)$ and $x(:)$ is in $V^{*\varphi}_{\mathrm{ass}}(X)$. The functional $\varrho_\varphi(x)$ is a convex modular on $V^{*\varphi}(X)$ for which $\varrho_\varphi(\lambda x) \to 0$ as $\lambda \to 0$. Thus, it generates a homogeneous norm $\|x\|_\varphi^c$ in $V^{*\varphi}(X)$ defined by

$$(*)\qquad \|x\|_\varphi^c = \inf\{\varepsilon > 0:\ \varrho_\varphi(x/\varepsilon) \leqslant 1\},\qquad x \in V^{*\varphi}(X).$$

Similarly: $\varrho_\varphi(x(:))$ is a convex modular on $V^{*\varphi}_{\mathrm{ass}}(X)$, $\varrho_\varphi(\lambda x(:)) \to 0$ as $\lambda \to 0$, and a norm $\|x(:)\|_\varphi^{\mathrm{ass}}$ may be defined analogously as above for $V^{*\varphi}(X)$.

The considerations in 2–2.2 yield the following theorem:

2.3. The space $V^{*\varphi}(X)$ is a linear subspace of $AC(X)$; $V^{*\varphi}(X)$ provided with the norm $\|\cdot\|_\varphi^c$ is isometric to the space $V^{*\varphi}_{\mathrm{ass}}(X)$ equipped with the norm $\|\cdot\|_\varphi^{\mathrm{ass}}$. Under this isometry each $x(\cdot) \in V^{*\varphi}(X)$ corresponds to the set function $x(:) \in V^{*\varphi}_{\mathrm{ass}}(X)$ associated with $x(\cdot)$.

2.4. *If $x \in V^{*\varphi}(X)$ and the associated set function has a representation $x(e) = \int_e y(t)\,d\mu$ for any measurable set e, where $y(\cdot)$: $T \to X$, is measurable, then the integral*

$$\int_T \varphi(\lambda|\xi(y(t))|)\,d\mu,\qquad \xi \in \Xi_0,$$

is finite for some $\lambda > 0$,

$$\int_T \varphi(|\xi(y(t))|)\,d\mu = m_\varphi(\xi, x),\qquad \|x\|_\varphi^c = \sup_{\xi \in \Xi_0} m_\varphi(\xi, x)\ ([7]).$$

2.4.1. Spaces $l^{*\varphi}(X)$ and $V^{*\varphi}(X)$ may be considered as special cases of spaces of vector-valued set functions. The former ones regarded as space of set functions over the ring N_f of finite sets of positive integers for which $x(\{n\}) = x_n$, $n = 1, 2, \ldots$, the latter ones as spaces of set functions over the algebra $\mathscr{E}$ of measurable sets (or over $\mathscr{I}$) on which the set functions $x(:)$ associated with $x(t)$ are defined. The sums $\sum\limits_1^n a_i x_i$, $\sum\limits_1^n a_i x(e_i)$, respectively, are nothing else but $\int\limits_T f(t)\mu(dx)$, (the integral in the sense of Dunford), where f if a simple function $\sum\limits_1^n a_i \chi_{e_i}$, $e_i \in N_f$ or $e_i \in \mathscr{E}(\mathscr{I})$, respectively. Therefore Theorem 1.2 is paralleled by the following theorem which may be proved similarly (note that (a) $\Rightarrow$ (b) is contained in 2.2).

2.5. The following statements are mutually equivalent:

(a) $x(\cdot) \in V^{*\varphi}(X)$;

(b) there exists a constant $\lambda > 0$ such that

$$\varrho_\varphi(\lambda x) \leqslant 1;$$

(c) if

$$(+) \qquad n_\varphi(x) = \sup\Big(\Big\|\sum_{i=1}^n a_i x(\delta_i)\Big\|\Big),$$

where the supremum is taken over all collections of disjoint intervals δ_i, $i = 1, 2, \ldots, n$ such that $\sum\limits_1^n \varphi^*(|a_i|)\mu(\delta_i) \leqslant 1$ and over all n, then $n_\varphi(x)$ is finite.

Remark. If $n_\varphi(x) < \infty$ then $x(\cdot) \in AC(X)$ and if we define a supremum analogously as in (c), replacing δ_i by measurable sets e_i, $x(\cdot)$ by $x(:)$, we get the same lowest upper bound $n_\varphi(x)$.

2.5.1. *The functional $n_\varphi(x)$ is a homogeneous norm on $V^{*\varphi}(X)$. The space $V^{*\varphi}(X)$ provided with this norm is a Banach space, the inequalities $\|x\|_\varphi^c \leqslant n_\varphi(x) \leqslant 2\|x\|_\varphi^c$ hold* (comp. [7], [4]).

2.5.2. *If $x \in V^{*\varphi}(X)$, $\varrho_{\varphi^*}(\lambda f) < \infty$ for every $\lambda > 0$ (if, e.g., $f \in L^{*\varphi^*}$, φ^* satisfies condition (Δ_2)) then the integral $\int\limits_T f(t)\,dx$ exists, and Hölder's inequality*

$$(*) \qquad \Big\|\int_T f(t)\,dx\Big\| \leqslant n_\varphi(x)\|f\|_{\varphi^*}^c$$

holds.

The hypotheses in 2.5.2 make possible an application of Theorem 5.2 from [7].

2.5.3. *If the integral $\int\limits_T f(t)\,dx$ exists for every $f \in L^{*\varphi^*}$ then $x \in V^{*\varphi}(X)$.*

Define $\varrho(f) = \sup\limits_{e \in \mathscr{E}} \big\|\int\limits_e f(t)\,dx\big\|$. As is easily seen $\operatorname{var} x < \infty$. Let $\|f_n - f\|_{\varphi^*}^c \to 0$ as $n \to \infty$. Set $e_n = \{t\colon |f_n(t) - f(t)| \geqslant \varepsilon\}$. Since f_n converges in measure

to f we have the relation $\mu(e_n) \to 0$ and consequently $\big\|\int\limits_{e_n} f(t)\,dx\big\| \to 0$, $\big\|\int\limits_{T-e_n} f(t)\,dx - \int\limits_T f(t)\,dx\big\| \to 0$. But $\big\|\int\limits_{T-e_n} f_n(t)\,dx - \int\limits_{T-e_n} f(t)\,dx\big\| \leqslant 2\varepsilon \operatorname{var} x$, thus

$$\varrho(f_n) \geqslant \Big\|\int_{T-e_n} f(t)\,dx\Big\| - 2\varepsilon \operatorname{var} x,$$

$\liminf\limits_{n\to\infty} \varrho(f_n) \geqslant \varrho(f)$. Since $\varrho(f)$ is a homogeneous and subadditive functional on $L^{*\varphi^*}$, we obtain $\varrho(f) \leqslant k\|f\|_{\varphi^*}$. Setting a simple function for f and applying 2.5 we infer $x \in V^{*\varphi}(X)$.

The existence of integrals $(*)$ $\int\limits_T f(t)\,dx$ for $f \in L^{*\varphi^*}$, $x \in V^{*\varphi}(X)$, depends either on conditions imposed on φ^* or on $V^{*\varphi}(X)$, that is on φ and X. A counter-example [7], p. 323–324, shows that for each φ^* which does not satisfy condition (Δ_2), and for $X = c_0$ the integral does not exist for certain $f \in L^{*\varphi^*}$. On the other hand, if $x(\cdot)$ satisfies e.g. the Lipschitz condition $\|x(t+h) - x(t)\| \leqslant k|h|$, $t, t+h \in T$, the integral $(*)$ exists, no matter whatever be φ. More generally, the same remains true if the norm $\|x\|_\varphi^{\mathrm{ass}}$ of the set function associated with $x(\cdot)$ is absolutely continuous [7].

2.6. For $\varphi(u) = u^a$, $a > 1$, denote by $\|x\hat{}\|_a$ if $x\hat{} \in l^{*\varphi}(X)$, and by $\|x\|_a$, if $x \in V^{*\varphi}(X)$, the respective norms $\|x\hat{}\|_\varphi^c$, $\|x\|_\varphi^c$. It can be easily checked that $\|x\hat{}\|_a = \sup\limits_{\xi \in \Xi_0} \big(\sum\limits_1^\infty |\xi(x_i)|^a\big)^{1/a}$, $\|x\|_a = \sup\limits_{\pi, \xi \in \Xi_0} \Big(\sum\limits_1^n \big(\big|\xi(x(\delta_i))\big|/\mu(\delta_i)\big)^a \mu(\delta_i)\Big)^{1/a}$, where intervals δ_i correspond to a partition $a = t_0 < t_1 \ldots, < t_n = b$. The complementary function to $\varphi(u) = u^a$ is $\varphi^*(v) = (a-1)a^{-\beta}v^\beta$, where $1/a + 1/\beta = 1$. By definitions 1.2 $(*{*}*)$, 1.3, we obtain $\|a\hat{}\|_{\varphi^*}^c = k^{1/\beta}\|a\hat{}\|_\beta$, $n_a(x\hat{}) = k^{-1/\beta}\|x\hat{}\|_a$ where $k = (a-1)a^{-\beta}$. Hence the Hölder inequality 2.5.2 $(*)$ assumes the familiar form

$$(*) \qquad \Big\|\sum_{i=1}^\infty a_i x_i\Big\| \leqslant \|x\hat{}\|_a \|a\hat{}\|_\beta;$$

here the series $\sum\limits_1^\infty a_i x_i$ is convergent for every $x\hat{} \in l^a(X)$, $a\hat{} \in l^\beta$, $\|a\hat{}\|_\beta = \big(\sum\limits_1^\infty |a_i|^\beta\big)^{1/\beta}$. Analogously $\|f\|_{\varphi^*}^c = k^{1/\beta}\|f\|_\beta$ if $f \in L^\beta$, $n_a(x) = k^{-1/\beta}\|x\|_a$, if $x \in V^a(X)$, and

$$(**) \qquad \Big\|\int_T f(t)\,dx\Big\| \leqslant \|x\|_a \|f\|_\beta;$$

here the integral $\int\limits_T f(t)\,dx$ exists for every $x \in V^a(X)$, $f \in L^\beta$, $\|f\|_\beta = \big(\int\limits_T |f(t)|^\beta\,dt\big)^{1/\beta}$.

3. Assume that $\{\varphi_i\}$ is an orthonormal system of scalar functions on T. In this section we shall deal with the orthogonal series of the form

$$(*) \qquad x_1\varphi_1(t) + x_2\varphi_2(t) + \ldots$$

where x_i are elements of a Banach space X. It is known that the theory of series of such kind differs in some points from the more simple, classical case $X = R$. The difficulties arise already when attempting to give an adequate formulation of the Riesz–Fisher theorem. Having this in mind, we will base our considerations on the fact that one forms the Fourier series (*) for vector-valued absolutely continuos functions $x(\cdot)$ only, defining their Fourier coefficients as

$$x_n = \int_T \varphi_n(t)\,dx \quad \text{for } n = 1, 2, \ldots$$

In any case, if $x \in V^2(X)$, then x_n makes sense, by Theorem 2.5.2. If $x(\cdot)$ is a scalar function, its Fourier coefficients reduce to the common Fourier coefficients.

3.1. *For any $x^\wedge = \{x_i\} \in l^2(X)$ the series*

$$(*) \qquad x(t) = \sum_{i=1}^{\infty} x_i \int_a^t \varphi_i(\tau)\,d\tau$$

is uniformly convergent on T.

Indeed, the series $\sum\limits_1^\infty a_i^2(t)$, where $a_i(t) = \int\limits_a^t \varphi_i(\tau)\,d\tau$, is known to be uniformly convergent on T. Since for $a^\wedge = \{a_i\} \in l^2$, $\|a^\wedge\|_2 = \big(\sum\limits_1^\infty a_i^2\big)^{1/2}$, we get, by 2.6 (*), for sufficiently large m, n,

$$\Big\| \sum_{i=m}^{n} x_i \varphi_i(t) \Big\| \leqslant \|x^\wedge\|_2 \Big(\sum_{i=m}^{n} a_i^2(t) \Big)^{1/2} < \varepsilon, \qquad t \in T.$$

3.2. *If $x^\wedge = \{x_i\} \in l^2(X)$ then the function $x(\cdot)$ defined by* 3.1 (*) *belongs to $V^2(X)$, its Fourier coefficients are x_i, and the following inequality holds*

$$\|x\|_2 \leqslant \|x^\wedge\|_2.$$

Let $\delta_1, \delta_2, \ldots, \delta_n$ be consecutive intervals corresponding to the partition $a = t_0 < t_1 < \ldots, < t_n = b$. Set $g(t) = a_i \mu(\delta_i)^{-1/2}$ if $t \in \delta_i$, $i = 1, 2, \ldots, n$. Then, for $\xi \in \Xi_0$, the inequalities

$$\sum_{j=1}^{n} a_j \frac{\xi(x(\delta_j))}{\mu(\delta_j)^{1/2}} = \sum_{i=1}^{\infty} \xi(x_i) \int_T g(t)\varphi_i(t)\,dt \leqslant \Big(\sum_{i=1}^{\infty} \xi^2(x_i) \Big)^{1/2} \Big(\int_T g^2(t)\,dt \Big)^{1/2}$$

$$= \|x^\wedge\|_2 \Big(\sum_{j=1}^{n} a_j^2 \Big)^{1/2},$$

imply

$$\Big(\sum_{j=1}^{n} \xi^2 \Big(\frac{x(\delta_j)}{\mu(\delta_j)} \Big) \mu(\delta_j) \Big)^{1/2} \leqslant \|x^\wedge\|_2.$$

Hence $x \in V^2(X)$, $\|x\|_2 \leqslant \|x^\wedge\|_2$. Similarly we have

$$\text{(i)} \qquad \|x_m\|_2 \leqslant \|x^\wedge\|_2,$$

where $x_m(t) = \sum\limits_{i=1}^{m} x_i \int\limits_a^t \varphi_i(\tau)\,d\tau$. By Hölder's inequality 2.6 (**) and by (i) the inequality

$$\text{(ii)} \qquad \Big\| \int_T f\,d(x - x_m) \Big\| \leqslant \|x - x_m\|_2 \|f\|_2 \leqslant 2\|x\|_2 \|f\|_2$$

is satisfied. Since $\int\limits_a^u dx_m = x_m(u) - x_m(a) \to x(u) - x(a)$, for any simple function $s(t) = \sum\limits_1^m a_i \chi_{\delta_i}(t)$ the relation

$$\text{(iii)} \qquad \int_T f(t)\,dx_m \to \int_T f(t)\,dx \quad \text{as } m \to \infty$$

holds. But the set of simple functions s is dense in L^2, so, by (ii), relation (iii) holds for any $f \in L^2$. Substituting $f(t) = \varphi_i(t)$ and remarking that $\int\limits_T f(t)\,dx_m = x_i$ for $m \geqslant i$ we see that $x_i = \int\limits_T \varphi_i(t)\,dx$.

3.2.1. *If $x \in V^2(x)$ then the sequence $x^\wedge = \{x_i\}$ of Fourier coefficients of $x(\cdot)$ belongs to $l^2(X)$ and the Bessel inequality*

$$\|x^\wedge\|_2 \leqslant \|x\|_2$$

holds.

In view of 2.6 (**) we have $\big\|\sum\limits_1^n a_i x_i\big\| = \big\|\int\limits_T \big(\sum\limits_1^n a_i \varphi_i(t)\big)\,dx\big\| \leqslant \|x\|_2 \|a^\wedge\|_2$ for every $a^\wedge \in l^2$, $n = 1, 2, \ldots$ whence $x^\wedge \in l^2(X)$, and $\|x^\wedge\|_2 = \sup\limits_{\xi \in \Xi_0} \big(\sum\limits_1^\infty \xi^2(x_i)\big)^{1/2} \leqslant \|x\|_2$.

If the orthonormal system $\{\varphi_i\}$ is complete in L^2, then it is complete with respect to $V^2(X)$. Indeed, if $\int\limits_T \varphi_i(t)\,dx = 0$ for $i = 1, 2, \ldots$ then $\int\limits_T f(t)\,dx = 0$ for any $f \in L^2$, by the Hölder inequality, for, orthogonal polynomials are dense in L^2. Setting $f_u(t) = \chi_{\langle a,u \rangle}(t)$, where $a \leqslant u \leqslant b$, we obtain $x(u) = 0$. Thus, as a corollary to 3.2, 3.2.1 we have the following analogue of the classical Riesz–Fischer's theorem;

3.2.2. *If the orthonormal system $\{\varphi_i\}$ is complete in L^2, then there is a bijective mapping between $l^2(X)$ and $V^2(X)$ such that the sequence $x^\wedge = \{x_i\} \in l^2(X)$ corresponds to the function $x(\cdot) \in V^2(X)$ with the Fourier coefficients x_i.*

The Parseval's identity

$$\|x^\wedge\|_2 = \|x\|_2,$$

holds for any pair of corresponding $x^\wedge$, $x(\cdot)$, so that this mapping is isometric.

3.2.3. *Denote $s_n(\cdot) = \sum_1^n x_i \int_a^t \varphi_i(\tau)d\tau$, $n = 1, 2, \ldots$ If $x^\wedge \epsilon l^2(X)$ is regular in $l^2(X)$ and $x(\cdot) = \sum_1^\infty x_i \int_a^t \varphi_i(\tau)d\tau$ then*

$$(*) \qquad \|s_n - x\|_2 \to 0 \quad \text{as } n \to \infty.$$

If $x \epsilon V^2(X)$, $x_i = \int_T \varphi_i(\tau)dx$ and the relation $(*)$ is satisfied, then the sequence $x^\wedge = \{x_i\}$ of the Fourier coefficients of $x(\cdot)$ is regular and the series $\sum_1^\infty \int_T \varphi_i(\tau)dx \int_a^t \varphi_i(\tau)d\tau$ is perfectly convergent in $V^2(X)$.

The statement follows immediately from 3.2, 3.2.1 and the definition of the regularity in $l^2(X)$ of a sequence $\{x_i\}$.

COROLLARY. *Assume that the orthonormal system $\{\varphi_i\}$ is complete in L^2. If any sequence in $l^2(X)$ is regular, in particular if $l^2(X)$ is separable, then the expansion 3.1 $(*)$ any functions in $V^2(X)$ is convergent to $x(\cdot)$ in $V^2(X)$ and the convergence is perfect.*

If for any $x \epsilon V^2(X)$ the expansion 3.1 $(*)$ converges to $x(\cdot)$, then any sequence in $l^2(X)$ is regular; in particular if X is separable then $l^2(X)$ is separable.

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On the best values of the constants in the theorems of M. Riesz, Zygmund and Kolmogorov*

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Abstract. Let f be a real 2π-periodic function and $\tilde{f}$ its conjugate. Then; (i) The least value of the constant A_p in M. Riesz's theorem ($\|\tilde{f}\|_p \leqslant A_p\|f\|_p$, $p > 1$, $f \epsilon L^p$) is $\tan(\pi/2p)$ if $1 < p \leqslant 2$ (and hence $\cot(\pi/2p)$ if $p > 2$). (ii) The only possible values of the constant A in Zygmund's theorem ($\|\tilde{f}\|_1 \leqslant A(1/2\pi)(\int_{-\pi}^{\pi}|f|\log^+|f|) + B$, $f \epsilon L\log^+ L$) are those $> 2/\pi$. (iii) For non-negative functions the least value of the constant B_p in Kolmogorov's theorem ($\|\tilde{f}\|_p \leqslant B_p\|f\|_1$, $p < 1$, $f \epsilon L^1$) is $(\cos(p\pi/2))^{-1/p}$. (iv) The constant A_p in (i) is also best possible for real non-periodic functions in R^1 (instead of the conjugate function it is now considered the Hilbert transform). The proof of these results makes use of a refinement of the inequality on which A. Calderòn's proof of the theorem of M. Riesz is based (see A. Zygmund; Trigonometric Series, Ch. VII, section 2, Cambridge Un. Press, 1968).

1. Introduction. The purpose of this paper is to examine the constants appearing in the theorems of M. Riesz, Zygmund and Kolmogorov ([4], Chapter VII, Section 2). In Section 2 we examine the case of real functions which are non-negative and 2π-periodic and we obtain sharp estimates of these constants. It turns out that the above mentioned theorems can be considered as instances of the same inequality (see Theorem 2.4 and the remarks following it). In Section 3 we consider real 2π-periodic functions of variable sign. Although the results are not as complete as in Section 2, we are able to prove that the least value of the constant A_p in M. Riesz's theorem ($\|\tilde{f}\|_p \leqslant A_p\|f\|_p$, $p > 1$, $f \epsilon L^p$) is $\tan(\pi/2p)$ if $1 < p \leqslant 2$ (and hence $\cot(\pi/2p)$ if $p \geqslant 2$). The proof of this result (theorem 3.7) is based on a refinement of the device, due to A. Calderòn, used in [4] for the proof of M. Riesz's theorem (only the Theorems 2.4 and 2.12 (c) from Section 2 are needed for the proof). T. Gokhberg and N. Krupnik have obtained the same result for special values of p ($p = 2^n$, $n = 1, 2, \ldots$). They have also proved that $A_p \geqslant \cot(\pi/2p)$ for all $p \geqslant 2$ and conjectured that this estimate is best possible. ~~(see [1]). In Section 4~~ we discuss some related results concerning non-periodic functions.

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