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The method of Grothendieck–Ramirez and weak topologies in $C(\boldsymbol{T})$

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Abstract. In [11] Ramirez used the Completion Theorem of Grothendieck (quoted below as (CT)) to characterize Fourier transforms (exactly Fourier–Stieltjes transforms) of measures on a locally compact abelian group and their uniform limits. Our aim is to develop this method by applying it to restrictions of Fourier transforms and their limits, as well as to problems which arise if some conditions like continuity (i.e. vanishing on points) are imposed on the measures under consideration.

It may be added that characterizing $B(G^\wedge)$ or $B(G^\wedge)^-$, $B(E)$ or $B(E)^-$ (see notations below) is to a large extent an open problem, as not even the idempotents of $B(\boldsymbol{Z})^-$ are known. A considerable progress is due to Drury [3] who proved that the characteristic function of a Sidon set is in $B(\boldsymbol{Z})^-$ (compare 4.).

Further on, we are going to investigate and to compare several topologies and several notions of weak convergence in the space $C(\boldsymbol{T})$ of complex continuous functions on the circle or in some of its subspaces defined by special assumptions about the spectra. It turns out that the method mentioned in the title or some results derived from it can be effectively used for this kind of research.

0. Notations. We denote

by $f^\wedge$, $\mu^\wedge$ the Fourier (Fourier–Stieltjes) transform of a function, a measure,

by $G^\wedge$ the dual (i.e. character) group for an abelian locally compact (LCA-) group G,

by $C(X)$ the algebra of (complex) continuous functions on compact X with sup-norm,

by $C_0(G)$ the algebra of continuous functions on LCA-group G, vanishing at infinity; $C_0(G) = C(G)$ for G compact,

by $C_b(X)$ the algebra of bounded continuous functions on X, with sup-norm,

by $C^E(G)$ the space of continuous functions on (abelian) compact group G with spectrum in $E \subset G^\wedge$ (with sup-norm),

by $A(G^\wedge)$ the algebra of Fourier transforms of integrable functions on G with norm induced by $L_1(G)$ $\big(A(G^\wedge) \simeq L_1(G)\big)$,

by $A(E)$ the algebra of restrictions of elements of $A(G^\wedge)$ to the set $E \subset G^\wedge$, with quotient norm, by $M(K)$ the space of finite complex regular Borel measures μ with carrier in $K \subset G$ and with norm $\|\mu\| = \sup_{\|\varphi\|_\infty \leq 1} \left|\int_G \varphi(t)\,d\mu(t)\right|$ where φ runs over $C_0(G)$,

by $M_c(K)\,(M_d(K))$ the subspace of $M(K)$ consisting of continuous (discrete) measures,

by $B(G\hat{\ })$ the algebra of Fourier—Stieltjes transforms of elements of $M(G)$ with norm induced by $M(G)\,(B(G\hat{\ }) \simeq M(G))$,

by $B_c(G\hat{\ })\,(B_d(G\hat{\ }))$ the subalgebra of $B(G\hat{\ })$ consisting of transforms of continuous (discrete) measures,

by $B(E)\,(B_c(E),\ B_d(E))$ the algebra of restrictions of elements of $B(G\hat{\ }) \times \times (B_c(G\hat{\ }),\ B_d(G\hat{\ }))$ to the set $E \subset G\hat{\ }$, with quotient norm,

by $B(E, K)$ $(B_c(E, K),\ B_d(E, K))$ the subspace of $B(E)$ $(B_c(E),\ B_d(E))$ consisting of restrictions to E of transforms of measures (continuous measures, discrete measures) with carrier in K,

by $B(E)^-$, $B_c(E)^-$ etc. the uniform (i.e. sup-norm) closure of $B(E)$, $B_c(E)$ etc.

by $\varphi|_K$, $\mu|_K$ the restrictions of functions or measures to the set K.

1. Characterization of $B_c(G\hat{\ })$, $B_d(G\hat{\ })$ and their closures. Avoiding a generality beyond our need we reproduce the Completion Theorem in a specialized form as follows:

(CT) (Grothendieck, see [9], p. 271). *Let (X, Y) be a dual system of two locally convex linear spaces, i.e. $X \subset Y'$, $Y \subset X'$, $\langle x, y\rangle = 0$ for each y implies $x = 0$ and $\langle x, y\rangle = 0$ for each x implies $y = 0$. Weak topology in X (in Y) will mean the topology induced by Y (by X). Let K_m $(1 \leq m < \infty)$ be convex centered sets in X, weakly bounded, weakly closed and such that $\bigcup_m K_m = X$. Let further $\overline{Y}$ denote the completion of Y in the topology of uniform convergence on the K_m's. Then $\overline{Y}$ consists precisely of those elements of Y' which are weakly continuous on K_m's.*

Putting $\Gamma = G\hat{\ }$, for a non-discrete LCA-group G, $X = M(\Gamma)$, $Y = M(G)$, or equivalently $B(\Gamma)$, $\langle\lambda, \mu\rangle = \int_\Gamma \mu\hat{\ }\,d\lambda = \int_G \lambda\hat{\ }\,d\mu\,(\lambda \in M(\Gamma), \mu \in M(G))$ and $K_m = B_m = \{\lambda\colon \|\lambda\| \leq m\}$, it is easily proved (see Lemma 1 below) that, for a sequence $\mu_n \in M(G)$, the uniform convergence on B_1 (and then on every B_m) is equivalent to the uniform convergence of the Fourier transforms $\mu_n\hat{\ }$ on Γ. So we infer that a necessary and sufficient condition for a function $f \in C_b(\Gamma)$ to belong to $B(\Gamma)^-$ is that for a net λ_α in $M(\Gamma)$ with $\|\lambda_\alpha\| \leq 1$ the following implication holds

(i) if $\int_\Gamma \mu\hat{\ }(x)\,d\lambda_\alpha(x) \to 0$ for every $\mu \in M(G)$ then $\int f(x)\,d\lambda_\alpha(x) \to 0$.

If instead of B_m the balls $B_m\hat{\ } = \{\lambda \in M(\Gamma)\colon \|\lambda\hat{\ }\|_\infty \leq m\}$ are used, (CT) can still be applied. It turns out (Lemma 1 below) that the uniform convergence of a sequence $\{\mu_n\}$ on $B_m\hat{\ }$ is equivalent to the norm convergence in $M(G)$. Since this space is complete we see that a necessary and sufficient condition for a function $f \in C_b(\Gamma)$ to belong to $B(\Gamma)$ is that (i) holds for nets λ_α such that $\|\lambda_\alpha\hat{\ }\|_\infty \leq 1$.

A more detailed analysis leads Ramirez in [11] to other characterizations of functions in $B(\Gamma)^-$ or in $B(\Gamma)$. In particular, he proves that $f \in B(\Gamma)^-$ is equivalent to the implication

if $\lambda_n\hat{\ }(t) \to 0$ $(1 \leq n < \infty)$ for every $t \in G$, $\|\lambda_n\| \leq 1$, then $\int f\,d\lambda_n \to 0$

and $f \in B(\Gamma)$ to the implication

if $\lambda_n\hat{\ }(t) \to 0$ $(1 \leq n < \infty)$ for every $t \in G$, $\|\lambda_n\hat{\ }\|_\infty \leq 1$, then $\int f\,d\lambda_n \to 0$.

For compact G in both implications the point-wise convergence $\lambda_n\hat{\ }(t) \to 0$ can be replaced by uniform convergence (compare Theorem 6 below). Thus owing to his dealing with convex sets in some adjoint Banach spaces he can dispose of nets and express the required characterizations in terms of simple (i.e. countable) sequences.

For the sake of completeness we prove the announced Lemma 1.

LEMMA 1. *The uniform convergence of a sequence $\{\mu_n\}$ $(\mu_n \in M(G))$ on the balls B_m is equivalent to the uniform convergence of $\hat{\mu}_n$. The uniform convergence of $\{\mu_n\}$ on the balls $B_m\hat{\ }$ is equivalent to the norm convergence.*

The first part of the Lemma is proved at once by taking for λ one point measures. To show the second let

$$\lim_{n,k\to\infty} \sup_{\|\hat{\lambda}\|_\infty < 1} |\langle\lambda, \mu_n\rangle - \langle\lambda, \mu_k\rangle| = 0.$$

Since those elements of $C_0(G)$ which are of the form $\lambda\hat{\ }$ $(\lambda \in M(\Gamma))$ make a dense set we have $\sup_{\|\lambda\hat{\ }\|_\infty \leq 1} \left|\int \lambda\hat{\ }(t)\,d\mu_n(t)\right| = \|\mu_n\|$, so the above Cauchy condition is equivalent to norm fundamentality, hence to norm convergence.

It can be easily seen that the system $(M(\Gamma), M_c)$ (M_c stands for $M_c(G)$) is still a dual system. In fact, since $A(\Gamma) \cap C_0(\Gamma)$ is dense in $C_0(\Gamma)$ it would be even enough that $\langle\lambda, \mu\rangle = 0$ hold for all absolutely continuous measures μ to have $\lambda = 0$. The balls B_m and $B_m\hat{\ }$ are obviously bounded in the weak topology induced by M_c since they are bounded in that induced by the whole of $M(G)$.

LEMMA 2. *The balls B_m and $B_m\hat{\ }$ are M_c-weakly closed.*

Proof. Let $\|\lambda_\alpha\| \leq m$ and $\langle\lambda_\alpha, \mu\rangle \xrightarrow{\alpha} \langle\lambda, \mu\rangle$ for every $\mu \in M_c$. One has to show that $\|\lambda\| \leq m$. Suppose this is not the case. Then we have $\|\lambda\| > m + \delta$ for a suitable $\delta > 0$. It exists a function $f \in C_0(\Gamma)$ such that $\|\varphi\|_\infty \leq 1$ and $\left|\int f\,d\lambda\right| > m + \delta$. Since the transforms $\mu\hat{\ }$ of absolutely continuous measures are uniformly dense in $C_0(\Gamma)$ there is a $\mu \in M_c$ with $\left|\int \mu\hat{\ }\,d\lambda\right| > m + \delta$ and $|\mu\hat{\ }| \leq 1$ everywhere on Γ — a contradiction since $|\langle\lambda_\alpha, \mu\rangle| \leq m$ for every α.

Let now $\|\lambda_a^{\hat{}}\|_\infty \leqslant m$ and $\langle\lambda_a, \mu\rangle \xrightarrow{a} \langle\lambda, \mu\rangle$ $(\mu \in M_c)$. One has to show that $\|\lambda^{\hat{}}\|_\infty \leqslant m$. If not, we have $|\lambda^{\hat{}}(t)| > m$ on an open set $U \subset G$. If μ is a positive measure with carrier in U and total variation $\|\mu\| = 1$ then $\int |\lambda^{\hat{}}(t)|\,d\mu(t) > m$. Hence, for $d\nu = e^{-i\arg\lambda^{\hat{}}(t)}d\mu$ it follows $\int \lambda^{\hat{}}(t)\,d\nu(t) = \int |\lambda^{\hat{}}(t)|\,d\mu > m$, but since $\|\nu\| \leqslant 1$ we have $\left|\int \lambda_a^{\hat{}}\,d\nu(t)\right| \leqslant m$ which is impossible if μ and so ν are continuous.

THEOREM 1. *A function $f \in C_b(\Gamma)$ is in $B_c(\Gamma)^-$ if and only if, for $\|\lambda_a\| \leqslant 1$,*

(ii) *the net convergence $\int_\Gamma \mu^{\hat{}}(x)\,d\lambda_a(x) \to 0$ for every $\mu \in M_c$ implies $\int f d\lambda_a \to 0$.*

THEOREM 2. *A function $f \in C_b(\Gamma)$ is in $B_c(\Gamma)$ if and only if, for $\|\lambda_a^{\hat{}}\| \leqslant 1$, the implication* (ii) *holds.*

In view of Lemma 2, both theorems are a direct consequence of (CT), applied to B_m's or $B_m^{\hat{}}$'s respectively.

Following Ramirez [12] we now turn to discrete measures, i.e. elements of $M_d = M_d(G)$. We consider the system $(M(\Gamma), M_d)$ and state again that it is a dual system: If $\lambda \neq 0$ then we can choose a compact F so that $|\lambda|(\Gamma \setminus F) < \frac{1}{2}\|\lambda\|$. The Fourier transforms of discrete measures are uniformly dense in the space of almost periodic functions on Γ. Hence, for every $f \in C(F)$ and $\varepsilon > 0$ there exists a $\mu \in M_d(G)$ such that $\|\mu^{\hat{}}|_F - f\|_\infty < \varepsilon$. So there is a discrete μ such that $\|\mu^{\hat{}}\|_\infty \leqslant 1$ and $\left|\int_F \mu^{\hat{}}\,d\lambda\right| > \frac{1}{2}\|\lambda\|$. Hence we have $\int_\Gamma \mu^{\hat{}}\,d\lambda \neq 0$.

LEMMA 3. *The balls B_m and $B_m^{\hat{}}$ are M_d-weakly closed.*

Proof. Let $\|\lambda_a\| \leqslant m$ and $\langle\lambda_a, \mu\rangle \xrightarrow{a} \langle\lambda, \mu\rangle$ for every $\mu \in M_d$. If we suppose $\|\lambda\| > m + \delta > m$ we can choose a function f as in the proof of Lemma 2 and a compact $F \subset \Gamma$ such that $|\lambda|(\Gamma \setminus F) < \frac{1}{2}\delta$. Then we have $\left|\int_F f d\lambda\right| > m + \frac{1}{2}\delta$. We can find a $\mu \in M_d$ with $\left|\int_F \mu^{\hat{}}\,d\lambda\right| > m + \frac{1}{2}\delta$ and $\|\mu^{\hat{}}\|_\infty \leqslant 1$, so $\left|\int_\Gamma \mu^{\hat{}}\,d\lambda\right| > m$. This leads to a contradiction like in the proof just quoted.

To see that the balls $B_m^{\hat{}}$ are M_d-weakly closed we rewrite the second part of the proof of Lemma 2 choosing μ discrete which is obviously possible. Then ν is also discrete and the same contradiction appears.

Hence, in view of (CT) and of the fact that $B_d(\Gamma)^-$ consists precisely of almost periodic functions we have the known theorem

THEOREM 3 (Ramirez, [12]). *A function $f \in C_b(\Gamma)$ is almost periodic if and only if, for $\|\lambda_a\| \leqslant 1$*

(iii) *the net convergence $\int_G \lambda_a^{\hat{}}(t)\,d\mu(t) \to 0$ for every $\mu \in M_d$ implies $\int_\Gamma f d\lambda_a \to 0$.*

It is nearly obvious that the net convergence $\int \lambda_a^{\hat{}}(t)\,d\mu \to 0$ for every $\mu \in M_d$ is equivalent (in view of $|\lambda_a^{\hat{}}(t)| \leqslant 1$) to the net convergence $\lambda_a^{\hat{}}(t) \to 0$ everywhere. So Theorem 3 can be formulated as follows:

THEOREM 3′ (see also [12]). *A function $f \in C_b(\Gamma)$ is almost periodic if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ and a finite system of points $t_1, \ldots, t_N \in G$ such that $\|\lambda\| \leqslant 1$, $|\lambda^{\hat{}}(t_j)| < \delta$ $(1 \leqslant j \leqslant N)$ implies $\left|\int f d\lambda\right| < \varepsilon$.*

Remark. The second part of this theorem („only if”) is nearly trivial. In fact, if f is a character $t \in G$ we have $\int_\Gamma f d\lambda = \lambda^{\hat{}}(-t)$ and the implication in question becomes tautologic. It holds then obviously for trigonometric polynomials and so one has just to apply the approximation theorem for almost periodic functions.

THEOREM 4. *A function $f \in C_b(\Gamma)$ is in $B_d(\Gamma)$ if and only if, for $\|\lambda_a^{\hat{}}\|_\infty \leqslant 1$ the implication* (iii) *holds; equivalently: if and only if, for $\|\lambda_a^{\hat{}}\|_\infty \leqslant 1$, $\lambda_a^{\hat{}}(t) \to 0$ everywhere on G implies $\int f d\lambda_a \to 0$.*

2. Spectra with appropriate compact. We now admit that for a discrete set $E \subset \Gamma$ there exists an appropriate compact K in G. This means that there is a constant $c > 0$ such that for every trigonometric polynomial $P(t) = \sum_{n=1}^{N} a_n x_n(t)$ with spectrum in E (i.e. $x_n \in E$; they are called E-polynomials) one has $\sup_{t \in K} |P(t)| \geqslant c \sup_{t \in G} |P(t)|$. There are classes of sets known to have appropriate compacts, e.g. Sidon sets (see further in this para) or the so-called harmonious sets on the line (see [10], p. 9). It is obvious that an arithmetic progression has an appropriate compact. The canonic isomorphism $P \to P|_K$ between the normed space $PT(E)$ of E-polynomials in $C_b(G)$ and the normed space of their restictions to K (a linear subspace of $C(K)$) induces an isomorphism between their dual spaces. Thus, to every bounded linear functional F on the first space corresponds a measure μ on K such that, for every $P \in PT(E)$, $F(P) = \int_K P(t)\,d\mu(t) = \sum_{n=1}^{N} a_n \mu^{\hat{}}(x_n)$. So, every bounded linear functional on $PT(E)$ is represented by a class of measures the Fourier transforms of which coincide on E and this class contains a measure with support in K.

We are going to consider the dual system $(M(E), B(E, K))$. $B(E, K)$ is a quotient space $M(K)/J$ with $J = \{\mu \in M(K)\colon \mu^{\hat{}}(t) = 0 \text{ for } t \in E\}$. To prove that it is really a dual system we assume first that for a $\mu \in M(K)$ we have $\langle\lambda, \mu\rangle = \int_E \mu^{\hat{}}\,d\lambda = 0$ for all $\lambda \in M(E)$. Taking for λ one point measures we see that $\mu \in J$. Suppose now that $\langle\lambda, \mu\rangle = \int_K \lambda^{\hat{}}(t)\,d\mu = 0$ for all $\mu \in M(K)$. Since K is appropriate to E, the restriction $f \to f|_K$ is an isomorphism of the space of almost periodic functions on G with spectrum in E. So we have $\lambda^{\hat{}}(t) = 0$ everywhere in G and finally $\lambda = 0$.

Let B_m^E and $B^{\hat{}E}_m$ denote the balls $\{\lambda \in M(E)\colon \|\lambda\| \leqslant m\}$ and $\{\lambda \in M(E)\colon \|\lambda^{\hat{}}|_K\|_\infty \leqslant m\}$. They are bounded in the weak topology induced by $B(E, K)$ since this is the same topology as that induced by $M(G)$.

LEMMA 4. *The balls B_m^E and $B\hat{\ }_m^E$ are weakly closed.*

In fact, if we had $\lambda_a \in B_m^E$, $\langle\lambda_a, \mu\rangle \to \langle\lambda, \mu\rangle$ for every $\mu \in M(K)$ and $\|\lambda\| > m+\delta$ then we would construct as in the proof of Lemma 2 a measure $\mu \in M(G)$ such that $\|\mu\hat{\ }\|_\infty \leqslant 1$ and $|\langle\lambda, \mu\rangle| = \left|\int_E \mu\hat{\ }\, d\lambda\right| > m+\delta$. But there is a $\mu_1 \in M(K)$ with $\mu\hat{\ } = \hat{\mu_1}$ on E. So $|\langle\lambda, \mu_1\rangle| > m+\delta$; this leads to a contradiction since $|\langle\lambda_a, \mu_1\rangle| \leqslant m$.

To see that also $B\hat{\ }_m^E$ are weakly closed we must proceed in an obvious way: If $\|\hat{\lambda_a}|_K\|_\infty = \left(\sup_{t\in K} |\hat{\lambda_a}(t)|\right) \leqslant m$, $\langle\lambda_a, \mu\rangle \to \langle\lambda, \mu\rangle$ $(\mu \in M(K))$ and $|\lambda\hat{\ }(t)| > m$ at a point $t \in K$ then choosing for μ a suitable one point measure at t with $\|\mu\| = 1$ we have $\langle\lambda, \mu\rangle > m$ — a contradiction.

LEMMA 5. *The uniform convergence of a net (μ_a) $(\hat{\mu_a}|_E \in B(E, K))$ on the balls B_m^E signifies the uniform convergence of $\hat{\mu_a}$ on E. The uniform convergence of a net (μ_a) on the balls $B\hat{\ }_m^E$ signifies the convergence of μ_a with respect to the quotient norm in $B(E, K)$.*

In fact, if $\sup_{\|\lambda\|\leqslant 1} |\langle\lambda, \mu_a\rangle| \xrightarrow{a} 0$ then $\sup_{x\in E} |\hat{\mu_a}(x)| \xrightarrow{a} 0$, by taking λ one point measures, whereas the converse is obvious. Further, since the functions $\lambda\hat{\ }|_K$ with $\lambda \in B\hat{\ }_1^E$ make a dense set in the unit ball of the space $C^E(K)$ consisting of restrictions of almost periodic functions on G with spectrum in E the expression

$$\sup_{\lambda\in B_1^E} |\langle\lambda, \mu_a\rangle| = \sup_{\lambda\in B_1^E} \left|\int_K \lambda\hat{\ }(t)\, d\mu_a(t)\right| = \sup_{\lambda\in B_1^E} \left|\int_E \mu\hat{\ }(t)\, d\lambda(t)\right|$$

is the norm of μ_a considered as a functional on $C^E(K)$ and thus equal to $\inf\|\mu\|$ with respect to all $\mu \in M(K)$ such that $\mu\hat{\ } = \hat{\mu_a}$ on E. Hence the second part of the lemma follows.

If we apply (CT) to the dual system $(M(E), B(E, K))$ and to the balls B_m^E and $B\hat{\ }_m^E$ and if we use the completeness of $B(E, K)$ for the quotient norm we get the following

THEOREM 5. *A function $f \in C_b(E)$ (i.e. a bounded function on E) is in $B(E)^-$ if and only if, for $\lambda_a \in M(E)$, $\|\lambda_a\| \leqslant 1$*

(i) *the net convergence $\int_E \mu\hat{\ }(x)\, d\lambda_a(x) \to 0$ $(\mu \in M(G))$ implies $\int_E f d\lambda_a \to 0$.*

The function f is in $B(E)$ if and only if, for $\|\hat{\lambda_a}\|_\infty \leqslant 1$ (i) holds.

In the second part of this Theorem we have replaced the ball $B\hat{\ }_1^E$ by $\{\lambda \in M(E);\ \|\hat{\lambda_a}\|_\infty \leqslant 1\}$. This is allowed since the compact K is appropriate to E and so the norms $\|\lambda\hat{\ }\|_\infty$ and $\|\lambda\hat{\ }|_K\|_\infty$ are equivalent for $\lambda \in M(E)$. It is also without importance whether we use $B(E)$ or $B(E, K)$ for the same reason, since $\int_\Gamma \mu\hat{\ }(x)\, d\lambda_a(x)$ depends only on the values $\mu\hat{\ }$ assumes on E. Thus we could eliminate the compact K from the above statement. Essential is only the fact that the set E does have an appropriate compact. Actually, this is essential only for the second part of Theorem 5, as we shall see in the sequel.

Evidently, in the case G compact we may take for E an arbitrary set in Γ. This case will be treated separately in 3.

Remark. For E having an appropriate compact we can follow more closely Ramirez's method which enables us to simplify the characterizations furnished by Theorem 5. Namely, we can consider each $\lambda \in M(E)$ as an operator on $C(K)$ defined by $\lambda\hat{\ }(t)\varphi(t)$ $(\varphi \in C(K))$ and thus introduce in $M(E)$ the strong and the weak operator topology. Repeating Ramirez's reasoning we find firstly that the weak continuity of a $f \in C_b(E)$ on the balls B_m^E or $B\hat{\ }_m^E$, referred to in (CT), is equivalent to the weak operator continuity, and secondly that that the strong and the weak continuity on these balls coincide. The first statement is nearly obvious since the weak operator continuity of f on B_m^E or $B\hat{\ }_m^E$ means the following implication: If $\|\lambda_a\| \leqslant 1$ resp. $\|\hat{\lambda_a}\| \leqslant 1$ and $\int_K \hat{\lambda_a}(t)\varphi(t)\, d\mu(t) \to 0$ for every $\mu \in M(K)$ and $\varphi \in C(K)$ then $\int f d\lambda_a \to 0$, whereas in the definition of the weak continuity determined by the dual system $(M(E), B(E, K))$ the function φ is just replaced by 1. The second statement follows from the fact that for convex sets (here B_m^E and $B\hat{\ }_m^E$) the weak and the strong operator closure coincide. Now, using the strong operator topology we can confine ourselves to sequences and so we get

THEOREM 6. *A bounded function f on E is in $B(E)^-$ $(B(E))$ if and only if for $\lambda_n \in M(E)$, $\|\lambda_n\| \leqslant 1$ $(\lambda_n \in M(E))$*

(ii) $\lim_{n\to\infty} \|\hat{\lambda_n}\|_\infty = 0$ *implies* $\lim_{n\to\infty} \int f d\lambda_n = 0$.

$E \subset \Gamma$ is a Sidon set ($E \in$Sid) if every bounded function on E can be extended to the transform $\mu\hat{\ }$ of a $\mu \in M(G)$. Every Sidon set in a metrizable LCA-group Γ has an appropriate compact and even every compact in G with a non-void interior is appropriate to every Sidon in Γ [2]. On the other hand, it is known that if the compact K is appropriate to $E \subset \boldsymbol{R}$ then E is a Sidon set provided the restrictions $\mu\hat{\ }|_E$ $(\mu \in M(K)$ or $M(\boldsymbol{R}))$ are uniformly dense in $C_b(E)$ [6]. Thus, on account of Theorem 6, we get

THEOREM 7. *If a set $E \subset \boldsymbol{R}$ has an appropriate compact and if for any $f \in C_b(E)$, $\lambda_n \in M(E)$, $\|\lambda_n\| \leqslant 1$ $(1 \leqslant n < \infty)$* (ii) *holds then E is Sidon.*

Conversely, if E is Sidon then (ii) holds for every $f \in C_b(E)$ as can be seen by putting $f = \mu\hat{\ }$ since then $\int_\Gamma f d\lambda_n = \int_G \hat{\lambda_n}\, d\mu$.

COROLLARY. As to Sidon sets in a discrete group Γ, it is trivial that they have an appropriate compact, G. Hence, on account of Theorem 7,

(*) *A set E in a discrete abelian group is Sidon if and only if, for $\lambda_n \in M(E)$, $\|\lambda_n\| \leqslant 1$ and any bounded f on E* (ii) *holds, and if and only if* (ii) *holds for $\lambda_n \in M(E)$ and any bounded f on E.*

That the restriction $\|\lambda_n\| \leqslant 1$ is unessential, follows not only from Theorem 7 but also directly from the equivalence of the norms $\|\lambda\|$ and $\|\hat{\lambda}\|_\infty$ for $\lambda \in M(E)$, which is valid for Sidon sets in any LCA-group and can be taken as a definition of a Sidon set in any discrete abelian group [13].

The reason for which we could pass from the characterizations by nets to those by sequences rests upon using the system $(M(E), B(E, K))$ where all measures on K, i.e. all functionals on $C(K)$, are involved. This enabled us to replace the weak topology by the weak operator topology and then to go over to the strong one. But this would have been impossible if only discrete or purely continuous measures had been used; in such a case there would be no hope for a characterization by means of sequences. But even if we tried to use nets like in paragraph 1, there would be a serious difficulty in treating the spaces $B_c(E, K)$ or $B_d(E, K)$ as members of the dual system instead of $B(E, K)$. Namely, we do not know whether for any continuous measure μ on G there is a *continuous* measure μ_1 on K or, may be, on some bigger but fixed compact such that $\hat{\mu}_1(x) = \hat{\mu}(x)$ for $x \in E$, nor we are able to answer the analogous question for discrete measures. It seems to be an open problem. Without this knowledge we could not prove that the balls B_m^E or $\hat{B}_m^E$ are weakly closed.

However, if we resign from using $\hat{B}_m^E$ (and thus from characterizing elements of $B(E)$ or $B_c(E)$ or $B_d(E)$) we can dispose of the existence of an appropriate compact for E and obtain nevertheless a net characterization of $B(E)^-$, $B_c(E)^-$ or $B_d(E)^-$ and in case G σ-compact even a sequential characterization for $B(E)^-$, as will be seen in the next paragraph.

3. Characterizing of $B(E)^-$, $B_c(E)^-$ and $B_d(E)^-$. We consider the systems $(M(E), B(E))$, $(M(E), B_c(E))$ and $(M(E), B_d(E))$. They are dual systems because $\langle\lambda, \mu\rangle = 0$ for every element μ of $B(E)$ or $B_c(E)$ or $B_d(E)$ is equivalent to $\langle\lambda, \mu\rangle = 0$ for every $\mu \in M(G)$ or $M_c(G)$ or $M_d(G)$ respectively and this is sufficient to have $\lambda = 0$ even without the restriction "carrier $\lambda \subset E$", as stated in 1. To see that the balls B_m^E are weakly closed in each of these systems we may apply Lemma 4 with an obvious change (actually a simplification) resulting from using the whole of G instead of K. Hence, (CT) leads to the following theorem:

THEOREM 8. *A necessary and sufficient condition for a function $f \in C_b(E)$ to be in $B(E)^-$, $B_c(E)^-$ or $B_d(E)^-$ is that for every $\varepsilon > 0$ there be a $\delta > 0$ and a finite system of measures $\mu_1, \ldots, \mu_n \in M(G)$, $M_c(G)$ or M_d respectively such that $\lambda \in M(E)$, $\|\lambda\| \leqslant 1$ and $\left|\int \hat{\lambda}(t)\, d\mu_i\right| < \delta$ $(1 \leqslant i \leqslant N)$ implies $\left|\int_E f d\lambda\right| < \varepsilon$.*

The first part of this theorem (for $B(E)^-$) in the case G compact can be proved also by means of a theorem about the so-called γ-linear functionals in two-norm spaces [1]. Putting $X = C^E(G) \cap A(G)$ we introduce there two norms: $\| \ \|_A$ and $\| \ \|_\infty$. Then the γ-linear functionals on X are those distributive functionals ξ for which the convergence $\|\hat{\lambda}_n - \hat{\lambda}\|_\infty \to 0$ together with $\lambda_n \in M(E)$, $\|\hat{\lambda}_n\|_A = \|\lambda_n\| \leqslant 1$ implies $\langle\xi, \hat{\lambda}_n\rangle \to \langle\xi, \hat{\lambda}\rangle$, and by [1] they can be identified with uniform limits of $\mu|_E$ $(\mu \in M(G))$.

The third part of Theorem 8 (i.e. for discrete measures) can be restated as follows:

THEOREM 8′. *A necessary and sufficient condition for a function $f \in C_b(E)$ to be the restriction of an almost periodic function on Γ is that for every $\varepsilon > 0$ there be a $\delta > 0$ and a finite system of points $t_1, \ldots, t_n \in G$ such that $\lambda \in M(E)$, $\|\lambda\| \leqslant 1$ and $|\hat{\lambda}(t_i)| < \delta$ $(1 \leqslant i \leqslant N)$ implies $\left|\int_E f d\lambda\right| < \varepsilon$.*

Theorems 8 and 8′ are straightforward generalizations of Theorems 1,3 and 3′, which appear after putting $E = \Gamma$. Like in this case we may notice that the necessity of the condition in Theorem 8′ is nearly obvious.

Remark. Had we used the simple sequence convergence instead of the net convergence, Theorem 8 and 8′, as well as 3 and 3′ would become false. In fact, in the third part of Theorem 8 it would then be claimed that f is in $B_d(E)^-$ provided that for any sequence $\{\lambda_i\}$ with $\lambda_i \in M(E)$, $\|\lambda_i\| \leqslant 1$ $(1 \leqslant i < \infty)$ the convergence $\hat{\lambda}_i(t) \to 0$ everywhere on G implies $\int_E f d\lambda_i \to 0$. However, if we take $E = \Gamma$ (like in Theorems 3 and 3′) we see from Ramirez's result (compare 1.) that this implication holds for every f belonging to the space $B(\Gamma)^-$ which is obviously larger than $B_d(\Gamma)^-$. We do not know whether the second part of Theorem 8 (for continuous measures) or at least Theorem 1 (it means the case $E = \Gamma$) remain true with sequences instead of nets. For the first part of Theorem 8 (arbitrary measures) the sequential characterization, announced at the end of 2., is possible. We have but to use once more Ramirez's argument, sketched in the remark following Theorem 5. This time λ must be considered as an operator on $C_0(G)$. Once more, the weak operator continuity on the balls B_m^E is shown to be equivalent to that induced by the dual system $(M(E), B(E))$ and, on the other hand, the weak and the strong operatory continuity coincide on B_m^E. But the strong operator convergence on these balls means here the uniform convergence on compacta. As G is σ-compact, this convergence is metric. So we get

THEOREM 9. *If G is σ-compact then a necessary and sufficient condition for a function $f \in C_b(E)$ to be in $B(E)^-$ is that $\lambda_n \in M(E)$, $\|\lambda_n\| \leqslant 1$ and $\hat{\lambda}_n(t) \to 0$ uniformly on every compact in G imply $\int_E f d\lambda_n \to 0$.*

We add few words about the characterization of $\boldsymbol{I}_0$-sets. A set E in Γ is called $\boldsymbol{I}_0$-set or Ryll-Nardzewski set (see [5]) if every bounded function on E is extendable to an almost periodic function on Γ. Theorem 8′ yields the following Corollary:

Corollary. *An isolated set $E \subset \Gamma$ is an I_0-set if and only if, for every bounded function f on E and every $\varepsilon > 0$, there is a $\delta > 0$ and a finite system of points $t_1, \ldots, t_N \in G$ such that $\lambda \in M(E)$, $\|\lambda\| \leqslant 1$ and $|\lambda^\wedge(t_i)| < \delta$ $(1 \leqslant i \leqslant N)$ implies $\left|\int_E f d\lambda\right| < \varepsilon$.*

The condition formulated in the Corollary means that for a net $\{\lambda_a\}$ with $\|\lambda_a\| \leqslant 1$ the point-wise convergence $\lambda_a^\wedge(t) \to 0$ on G implies $\int f d\lambda_a \to 0$. It is not possible to weaken this condition as a sufficient one by writing the above implication for sequences only. This follows from two facts; 1) there are Sidon sets (in $\boldsymbol{R}$ or in $\boldsymbol{Z}$, e.g.) which are not $\boldsymbol{I}_0$ [7], 2) for any Sidon set E the conditions $\lambda_n \in M(E)$, $\|\lambda_n\| \leqslant 1$ $(1 \leqslant n < \infty)$ $\lim_n \lambda_n^\wedge(t) = 0$ $(t \in G)$ imply $\int_E f d\lambda_n \to 0$ for every bounded function f on E as is seen by putting $f = \mu^\wedge$:

$$\int_E f d\lambda_n = \int_G \lambda_n^\wedge(t)\, d\mu(t) \to 0$$

on account of Lebesgue bounded convergence theorem.

In view of Theorem 7 the last implication is also sufficient for a set to be Sidon provided it has an appropriate compact. On the other hand, a set $E \subset \boldsymbol{R}$ is Sidon if and only if (α) there is a positive constant B such that $\sup_{t \in \boldsymbol{R}} |\lambda^\wedge(t)| \geqslant B\|\lambda\|$ for every $\lambda \in M(E)$ with finite support and $\beta)$ E has an appropriate compact. In fact, (α) means that E is Sidon as a subset of the discrete real line, i.e. of the group which is dual to $\boldsymbol{R}^\sim$ — the Bohr compactification of $\boldsymbol{R}$ (compare [13]); in other words, for every bounded function f on E there is a measure μ on $\boldsymbol{R}^\sim$ such that $\mu^\wedge = f$ on E; now, (β) allows to find a measure ν on $\boldsymbol{R}$ (actually on the approppriate compact) such that $\nu^\wedge = \mu^\wedge = f$ on E, so E is Sidon a s subset of $\boldsymbol{R}$ with natura topology („un Sidon topologique"). Thus, firstly, the implication

$$\lambda_n \in M(E),\ \|\lambda_n\| \leqslant 1\ (1 \leqslant n < \infty),\ \lim_n \lambda_n^\wedge(t) = 0\ (t \in \boldsymbol{R}) \Rightarrow \lim_n \int f d\lambda_n = 0$$

for every bounded f on E together with the existence of an appropriate compact fully characterizes the Sidon sets in $\boldsymbol{R}$; secondly, if instead of $\boldsymbol{R}$ we admit a discrete Γ, this implication is characteristic for Sidon sets without any further restriction (we notice that this stands explicitely in [11]) and it can be considered as a modification of the condition given in (*) (see 2.)

Now we confine ourselves to the case G compact. We want to discuss some topologies in $C(G)$ and $C^E(G)$. What was said about E having an appropriate compact applies here clearly for every set $E \subset \Gamma$. Moreover, we are released from the difficulty mentioned after the Corollary to Theorem 7 and so we can use (CT) for the dual systems $(M(E), B_c(E))$ and $(M(E), B_d(E))$ thus obtaining the following characterizations:

Theorem 10. *A function $f \in C_b(E)$ is in $B_c(E)^-$ $(B_d(E)^-)$ if and only if for $\lambda_a \in M(E)$, $\|\lambda_a\| \leqslant 1$ the convergence $\int \lambda_a^\wedge d\mu \to 0$ with respect to the dual system $(M(E), B_c(E))$ $(M(E), B_d(E))$ implies $\int f d\lambda_a \to 0$. A function $f \in C_b(E)$ is in the closure $B_c(E)^=$ of $B_c(E)$ (in the closure $B_d(E)^=$ of $B_d(E)$) with respect to the quotient norm in $B(E)$ if and only if the above implication is valid for $\|\lambda_a^\wedge\|_\infty \leqslant 1$ instead of $\|\lambda_a\| \leqslant 1$.*

The first of these propositions is just a partial reproduction of Theorem 8.

4. Kaufman sets. We call a set $E \subset \boldsymbol{Z}$ a Kaufman set or a $\boldsymbol{Ka}$-set $(E \in \boldsymbol{Ka})$ if ther eis a measure $\mu \in M_c$ $(= M_c(\boldsymbol{T}))$ and a $\delta > 0$ such that $|\mu^\wedge(n)| > \delta$ on E. It is obvious that every $\boldsymbol{Ka}$-set has density 0. Every Sidon set is a Kaufman set. Even more: every bounded function on a Sidon set E in $\boldsymbol{Z}$ can be extended to the Fourier transform of a *continuous* measure. This easily follows [14] from Drury [3] where it is proved that, for every $\varepsilon > 0$, there is a measure $\mu \in M(\boldsymbol{Z})$ such that $\mu^\wedge = 1$ on E and $|\mu^\wedge| < \varepsilon$ elsewhere. On the other hand, a $\boldsymbol{Ka}$-set needs not satisfy the known lacunarity conditions which must be fulfilled by every Sidon set [8].

A $\boldsymbol{Ka}$-set can be characterized by the condition that the function on E identically equal to 1 (or $\mathbf{1}_E$) belongs to $B_c(E)^-$. The sufficiency is obvious. Conversely, if $|\mu^\wedge| > \delta > 0$ $(\delta < 1)$ on E for a $\mu \in M_c(\boldsymbol{Z})$ then we construct a sequence of polynomials w_n with $w_n(0) = 0$, uniformly convergent to a (continous) function with values in $[0, 1]$, equal 1 for $x \geqslant \delta^2$. Then $[w_n(\mu * \mu^*)]^\wedge(n) \to 1$ uniformly on E and the measures $w_n(\mu * \mu^*)$ are obviously continuous. (We put $\mu^*(A) = \overline{\mu(-A)}$).

The above characterization implies at once that a set $E \subset \boldsymbol{Z}$ is $\boldsymbol{Ka}$ if and only if $B_c(E)^- = B(E)^-$. Further we have

Theorem 11. *$E \in \boldsymbol{Ka}$ if and only if the conditions $\lambda_a \in M(E)$, $\|\lambda_a\| \leqslant 1$, $\int \lambda_a^\wedge(t)\, d\mu(t) \to 0$ $(\mu \in M_c)$ imply $\lambda_a^\wedge(t) \to 0$ for every $t \in \boldsymbol{T}$.*

Proof. 1° Sufficiency. Supposing the above conditions satisfied we have in particular $\lambda_a^\wedge(0) = \int d\lambda_a \to 0$ and this implies in virtue of Theorem 10 that the function $\mathbf{1}_E$ is in $B_c(E)^-$. So $E \in \boldsymbol{Ka}$.

2° Necessity. If $E \in \boldsymbol{Ka}$ then not only the function $\mathbf{1}_E$ is in $B_c(E)^-$ but also all restrictions $e^{int}|_E$ of exponentials (i.e. for arbitrary t) because multiplying $\mu^\wedge$ by an exponential corresponds to a translation of μ. So, by Theorem 10, the conditions in Theorem 11 imply $\sum_{n \in E} e^{-int} \lambda_a(n) = \lambda_a^\wedge(t) \to 0$ $(t \in \Gamma)$.

We do not know whether in this characterization simple sequences can be used instead of nets. Further, we are not able to decide whether the condition $\|\lambda_a\| \leqslant 1$ in Theorem 11 can be replaced by $\|\lambda_a^\wedge\|_\infty \leqslant 1$. Doing this we get a characterization of those spectra E for which $\mathbf{1}_E$ belongs to $B_c(E)$ because $B_c(E)$ is a (may be improper) ideal in $B(E)$, so the unit element of $B(E)$ cannot be contained in its closure unless $B_c(E) = B(E)$.

Hence, the actual problem is to know whether for every $\boldsymbol{Ka}$-set this identity holds true.

In [6] it is proved among other results that if $E = \{n_k\}$ $(n_k > 0)$ is a Sidon set then there is a uncountable set of real numbers t such that the sequence $\{n_k t\}$ is not equidistributed mod 1. Modifying and actually simplifying the method of the authors of [6] we prove the following theorem which can be considered as stronger from the standpoint of our present knowledge (after [3] was published):

THEOREM 12. *If $E = \{n_k\}$ $(n_k > 0)$ is a Kaufman set then the set of t such that $\{n_k t\}$ is not equidistributed* mod 1 *is uncountable.*

Proof. If the Theorem were false we had $\lim\limits_N \frac{1}{N}\sum\limits_{k=1}^{N} e^{-in_k t} = 0$ for every t except acoun tabl set and thus $\lim\limits_n \frac{1}{N}\sum\limits_{k=1}^{N}\int\limits_0^{2\pi} e^{-in_k t}\,d\mu(t) = 0$ for every $\mu \in M_c$. In other words we had $\lim\limits_N \frac{1}{N}\sum\limits_{k=1}^{N}\hat{\mu}(n_k) = 0$ for every $\mu \in M_c$. But as $E \in \boldsymbol{Ka}$ there is a continuous measure μ_0 with $|\hat{\mu}_0(n_k)| > \delta > 0$ $(1 \leqslant k < \infty)$. Taking $\mu = \mu_0 * \mu_c^*$ we have $\hat{\mu}(n_k) > \delta^2 > 0$ and we arrive at a contradiction.

5. Weak topologies in $C(\boldsymbol{T})$. Since the space $M(\boldsymbol{T})$ is dual to $C(\boldsymbol{T})$ the question arises what happens when the weak topology of $C(\boldsymbol{T})$ (i.e. the topology induced by $M(\boldsymbol{T})$) is made still weaker by using M_c or M_d instead of M in its definition. What can be said about sequences which converge with respect to these topologies? What are the relations between such convergence of sequences or nets and the convergence everywhere on $\boldsymbol{T}$ or the uniform convergence or that of L-type (i.e., for example, $\int|\varphi_n|\,d\mu \to 0$ for some μ's)? The same problems may be posed with respect to the subspaces $C^E(\boldsymbol{T})$.

In the sequel we will write $\varphi_a(t)$ to indicate a net of functions and φ_n to indicate a simple sequence. If we write φ_ι, it means that either a or n has to be put as index according to the need. We introduce the following conditions for bounded sequences or nets in $C(\boldsymbol{T})$ or in $C^E(\boldsymbol{T})$:

(C) $\int \varphi_\iota(t)\,d\mu(t) \to 0$ for every $\mu \in M_c$,

(D) $\int \varphi_\iota(t)\,d\mu(t) \to 0$ for every $\mu \in M$.

We shall investigate the following implications:

(I) If (C) then (D).

(II) If (C) then $\int|\varphi_\iota(t)|\,d\mu(t) \to 0$ $(\mu \in M_c)$.

(II_0) If (C) then $\int|\varphi_\iota(t)|\,dt \to 0$ (dt = the differential of the Lebesgue measure).

(II_1) If (C) then $\int|\varphi_\iota(t)|\,d\mu(t) \to 0$ $(\mu \in M)$.

(III) If (C) then $\|\varphi_\iota\|_\infty \to 0$.

(IV) If (D) then $\int|\varphi_\iota(t)|\,d\mu(t) \to 0$ $(\mu \in M)$.

(IV_0) If (D) then $\int|\varphi_\iota(t)|\,dt \to 0$.

(V) If $\varphi_\iota(t) \to 0$ $(\forall t \in \boldsymbol{T})$ then (D).

(VI) If $\varphi_\iota(t) \to 0$ $(\forall t \in \boldsymbol{T})$ then $\int|\varphi_\iota(t)|\,d\mu(t) \to 0$ $(\mu \in M)$.

(VI_0) If $\varphi_\iota(t) \to 0$ $(\forall t \in \boldsymbol{T})$ then $\int|\varphi_\iota(t)|\,dt \to 0$.

(VII) If $\varphi_\iota(t) \to 0$ $(\forall t \in \boldsymbol{T})$ then $\|\varphi_\iota\|_\infty \to 0$.

Some of these implications hold true for $\iota = a$, some only for $\iota = n$. Some hold without any restriction as to the spectra of φ_ι's, some other require such restrictions. We have the following implications (of second order):

$$(\mathrm{III}) \Rightarrow (\mathrm{II}_1) \Rightarrow (\mathrm{II}) \Rightarrow (\mathrm{II}_0) \Rightarrow (\mathrm{IV}_0), \qquad (\mathrm{I}) \cap (\mathrm{IV}) \Rightarrow (\mathrm{II}_1),$$
$$(\mathrm{I}) \cap (\mathrm{IV}_0) \Rightarrow (\mathrm{II}_0),$$
$$(\mathrm{VI}) \Rightarrow (\mathrm{VI}_0) \Rightarrow (\mathrm{IV}_0), \qquad (\mathrm{VI}) \Rightarrow (\mathrm{V}), \qquad (\mathrm{VI}) \Rightarrow (\mathrm{IV}) \Rightarrow (\mathrm{IV}_0),$$
$$(\mathrm{VII}) \Rightarrow (\mathrm{VI}).$$

To deduce some of them one has to observe that $\varphi_\iota(t) \to 0$ means the same as $\int\varphi_\iota(t)\,d\mu(t) \to 0$ $(\forall \mu \in M_d)$. The other are obviuous.

In the table below we indicate which of the conditions (I)–(VII) are fulfilled and which are not for φ_ι in $C^E(\boldsymbol{T})$ in dependence of E. We ommit statements resulting from the second order implications just mentioned and we point out open questions by interrogation mark. We write (I′) and so on if φ_ι in (I) etc. are supposed members of $A(\boldsymbol{T})$ uniformly bounded in A-norm. Finally, we put in frame those statements which are characteristic for the class of spectra marked at the top of the column.

THEOREM 13. *All assertions in the following table are true. Condition* (V) *characterizes $\boldsymbol{I}_0$-sets among Sidon sets.*

	E arbitrary	$E \in \boldsymbol{Ka}$	$E \in \boldsymbol{Sid}$	$E \in \boldsymbol{I}_0$	E finite
$\iota = n$	(I′) no (II) yes (VI) yes (VII′) no	(I) ? (I′) yes (III) ? (VI) yes (VII) ?	(I)–(VII) yes	(I)–(VII) yes	(I)–(VII) yes
$\iota = a$	(I′)–(VII′) no	(I) ? $\boxed{\text{(I′) yes}}$ (IV_0') no (V′) no	(I) yes (IV_0') no (V′) no	(I) yes (IV_0') no (V) yes	$\boxed{\text{(I)–(VII) yes}}$

Proof. Observe that 1° $I_0 \subset \boldsymbol{Sid} \subset \boldsymbol{Ka}$, 2° the statement in frame in the second column is another version of Theorem 11, 3° for spectra lying in a Sidon set there is no difference between (I) and (I′) and so on 4° it is obvious that all conditions (I)–(VII) hold for E finite. So it remains to prove (α) for sequences with arbitrary E (I′) does not hold, (II) holds and (VI) holds; (β) for sequences with spectra in a Sidon set (III) and (VII) hold; (γ) (V) for nets characterizes I_0-sets among Sidon sets; (δ) if (IV_0') holds for nets, then E is finite.

The first statement in (α) is elementary. It can be proved by taking the sequence

$$\varphi_n(t) = \frac{1}{N+1}\sum_{n=0}^{N} e^{-int}.$$

These functions have A-norm equal 1.

For every $\mu \in M_c$ one has by Wiener's theorem $\int\limits_0^{2\pi} \varphi_n(t)\,d\mu(t) = \dfrac{1}{N+1}\sum\limits_{n=0}^{N} \hat{\mu}(n) \to 0$. However $\varphi_n(0) = 1$ $(1 \leqslant n < \infty)$.

To prove the second statement in (α) it is enough to consider only positive measures and real functions. If only (II_0) had to be proved it would be sufficient to make use of the following statement included in [4]; if $\{F_n\}$ is a sequence of sets in $[0,1]$ of Lebesgue measure exceeding a $\delta > 0$ then there is a subsequence $\{F_{n_k}\}$ such that card $\big(\bigcap\limits_{k=1}^{\infty} F_{n_k}\big) = \mathfrak{c}$. In fact, if $\int\limits_0^1 |\varphi_n(t)|\,dt$ does not tend to 0, then there are numbers $\eta > 0$ and $\delta > 0$, a sign $(-1)^{\varepsilon}$ ($\varepsilon = 0$ or 1) and a sequence $\{n_k\}$ such that $(-1)^{\varepsilon}\varphi_{n_k}(t) > \eta$ on a set F_k of Lebesgue measure $> \delta$. Suppose $\varepsilon = 1$. We choose a subsequence $\{F_{k_l}\}$ such that the set $\bigcap\limits_{l=1}^{\infty} F_{k_l}$ is uncountable. It then carries a positive measure $\nu \in M_c$ of norm 1. Evidently, $\int\limits_0^1 \varphi_{n_{k_l}}(t)\,d\nu(t) \geqslant \eta$ for every l which contradicts the assumption. To obtain (II) we need a generalization of the statement of Erdös, Kestelman and Rogers in [4], namely we must replace the Lebesgue measure by an arbitrary positive continuous finite Borel measure μ_0. Such a generalization has been furnished by Ryll-Nardzewski. We sketch its proof with the author's permission:

Let χ_n be the characteristic function of F_n. We may admit that the sequence $\{\chi_n\}$ is *-weakly convergent in $L_\infty(\mu_0)$ to a function φ, say. Then $\int\limits_0^1 \varphi(t)\,d\mu_0 \geqslant \delta$. We may assume that there is a constant $a > 0$ such that $\varphi(t) > a$ everywhere; in fact, if this were not the case, we could replace in the sequel the interval $[0,1]$ by a closed set F of positive μ_0-measure on which such inequality does hold; then the sets F_n would be replaced by $F_n \cap F$. So we have

$$\lim_n \mu_0(F_n \cap Q) \geqslant a\mu_0(Q)$$

for every Borel set Q because $\mu_0(F_n \cap Q) \to \int\limits_Q \varphi(t)\,d\mu_0$. Now, we construct by induction a sequence $n_1 < n_2 < \ldots$ and closed sets $A_{(\varepsilon_1,\ldots,\varepsilon_k)}$ ($\varepsilon_i = 0$ or 1) of positive measure μ_0 such that

(i) $$F_{n_i} \cap \ldots \cap F_{n_k} \supset \dot{\bigcup_{(\varepsilon_1,\ldots,\varepsilon_k)}} A_{\varepsilon_i\ldots\varepsilon_k},$$

(ii) $A_{\varepsilon_1\ldots\varepsilon_k 0}$ and $A_{\varepsilon_1\ldots\varepsilon_k 1}$ are disjoint and both included in $A_{\varepsilon_i\ldots\varepsilon_k}$.

As $\mu_0(F_n \cap A_{\varepsilon_1\ldots\varepsilon_k}) > 0$ for every n sufficiently large and for all systems $(\varepsilon_1, \ldots, \varepsilon_k)$ and since the measure μ_0 is continuous we can find a $n_{k+1} > n_k$ and, for every system $(\varepsilon_1, \ldots, \varepsilon_k)$, two disjoint closed sets $A_{\varepsilon_1\ldots\varepsilon_k 0}$ and $A_{\varepsilon_1\ldots\varepsilon_k 1}$ of positive μ_0-measure such that (i) and (ii) be fulfilled for $k+1$. Since $\bigcap\limits_{k=1}^{\infty} F_{n_k} \supset \bigcap\limits_{k=1}^{\infty} \bigcup\limits_{(\varepsilon_1,\ldots,\varepsilon_k)} A_{\varepsilon_i\ldots\varepsilon_k}$, the set $\bigcap\limits_{k=1}^{\infty} F_{n_k}$ has power $\mathfrak{c}$.

The third statement in (α) follows from the Lebesgue bounded convergence theorem.

To obtain the second statement in (β) observe that the space $C^E(\boldsymbol{T})$ with E Sidon is isomorphic to $l_1(E)$; this follows from the equivalence of the norms $\|\ \|_\infty$ and $\|\ \|_A$ in $C^E(\boldsymbol{T})$. Since the assumption of (VII) for a bounded sequence means the weak convergence and since in l_1 the weak and the strong convergence of a sequence coincide, we have $\|\varphi_n\|_\infty \to 0$, indeed.

As (I′) holds for $\boldsymbol{Ka}$-sets and so (I) holds for Sidon sets, we obtain the first statement in (β) owing to the fact that (III) is an obvious consequence of (I) and (VII).

Let now $E \in I_0$, $\varphi_\alpha \in C^E(\boldsymbol{T})$, $\|\varphi_\alpha\|_\infty \leqslant K$ (so $\|\varphi_\alpha\|_A \leqslant L$) and $\varphi_\alpha(t) \to 0$ $(t \in \boldsymbol{T})$. If $\mu \in M(\boldsymbol{T})$ then, according to Corollary to Theorem 8′ (with $f(n) = \hat{\mu}(-n)$ on E, $\lambda_\alpha(n) = \hat{\varphi}_\alpha(n)$, whence $\|\lambda_\alpha\| \leqslant L$), we have $\sum\limits_{n \in E} \hat{\varphi}_\alpha(n)\hat{\mu}(-n) \to 0$ but this can be written as $\int \varphi_\alpha(t)\,d\mu(t) \to 0$, so (V) holds for nets if $E \in I_0$.

Further, suppose that E is Sidon and (V) holds for nets in $C^E(\boldsymbol{T})$. For any bounded function f on E there is a measure μ on $\boldsymbol{T}$ such that, for $n \in E$, $f(n) = \hat{\mu}(n)$ or else $f(n) = \hat{\nu}(-n)$ with $\nu = \overline{\mu^*}$. If $\varphi_\alpha \in C^E(\boldsymbol{T})$, $\|\varphi_\alpha\|_\infty \leqslant K$ (so again $\|\varphi_\alpha\|_A \leqslant L$) and $\varphi_\alpha(t) \to 0$ $(t \in \boldsymbol{T})$, then $\sum\limits_{n \in E} \hat{\varphi}_\alpha(n)\hat{\nu}(-n) = \int \varphi_\alpha(t)\,d\nu(t) \to 0$. Thus $\sum\limits_{n \in E} f(n)\hat{\varphi}_\alpha(n) \to 0$ for every $f \in C_b(E)$, and applying Corollary to Theorem 8′ in opposite direction we see that $E \in I_0$. So (γ) is proved.

Finally, we must prove (δ). Before doing this we insert a lemma.

LEMMA (Ryll-Nardzewski). *Given any finite set $\varphi_1, \ldots, \varphi_k$ of elements of l_∞ and any $\varepsilon > 0$ there is an element $x \in l_1$ with $\|x\|_1 = 1$ such that $|\varphi_i(x)| < \varepsilon$ $(1 \leqslant i \leqslant k)$ but $\|x\|_2 = \dfrac{1}{\sqrt{2}}$.*

Proof. If $\varphi_i = \{a_n^{(i)}\}(1 \leqslant n < \infty)$, we choose a sequence of integers $0 < n_1 < n_2 \ldots$ such that the sequences $\{a_{n_j}^{(i)}\}$ converge. Taking j sufficiently large we have $|a_{n_{j+1}}^{(i)} - a_{n_j}^{(i)}| < \varepsilon$ $(1 \leqslant i \leqslant k)$. We put $x = \{\beta_n\}$ where $\beta_{n_j} = \frac{1}{2}$, $\beta_{n_{j+1}} = -\frac{1}{2}$, $\beta_n = 0$ for $n \neq n_j, n_{j+1}$. Then x satifies the assertion.

Let now E be infinite. We may suppose E to be Sidon. If (IV_0') held for nets then, for every bounded net, (D) would imply $\int |\varphi_a(t)|^2 dt \to 0$ since, for a Sidon spectrum E, the L_1-norm and the L_2-norm are equivalent in $C^E(\boldsymbol{T})$. Instead of $\int |\varphi_a(t)|^2 dt$ we may write $\sum_{n \in E} |\hat{\varphi_a}(n)|^2$, using Parseval equation. Putting $\varphi_a(t) = \sum_{n \in E} \lambda_a(n) e^{int}$ we have $\int \varphi_a(t) d\mu(t) = \sum_{n \in E} \lambda_a(n) \hat{\mu}(-n)$ and $\sum_{n \in E} |\lambda_a(n)| < L$. Since $\hat{\mu}(-n)$ can be any bounded sequence, our assumption means that in the unit ball of $l_1(E)$ the norm topology induced by $l_2(E)$ is continuous with respect to the weak topology, i.e. that induced by the dual system (l_1, l_∞). This however contradicts the Lemma.

Remarks. 1. Theorem 13 holds also for bounded sequences or nets consisting of B-measurable functions. Evidently, only positive propositions need to be verified. If all spectra are in a Sidon set there is nothing to prove because every bounded function with Sidon spectrum is continuous. So it remains to look at (II) and (VI) for sequences with arbitrary spectra. But then the proofs (see the second and third statement in (α)) remain valid without any change.

2. The boundedness of the sequence of continuous functions (not that of a net!) needs not be required in condition (C), and *a fortiori* in (D), since it is implied by the convergence assumed in (C). Here is the argument: φ_n's may be treated as linear functionals on $M_c(\boldsymbol{T})$, $\langle \varphi_n, \mu \rangle = \int \varphi_n(t) d\mu(t)$, with norm $|\varphi_n| = \sup_t |\varphi_n(t)| = \|\varphi_n\|_\infty$. In fact, inequality $|\varphi_n| \leqslant \|\varphi_n\|_\infty$ is obvious and the converse inequality holds too because the unit mass at a point t_0 where $|\varphi_n(t_0)| = \|\varphi_n\|_\infty$ can be weakly approximated by continuous measures. So the functionals are uniformly bounded in their norms, in virtue of the Banach–Steinhaus theorem. Besides, it can be seen that boundedness was not used in the proof of the second statement in (α).

If φ_n are B-measurable and possibly not continuous then (C) does not imply boundedness of the sequence $\{\varphi_n\}$. Example: $\varphi_n(t) = 0$ for

t irrational and $\varphi_n(t) = n$ otherwise. Obviously $\int_0^1 \varphi_n d\mu = 0$ for every $\mu \in M_c$. However, (C) then implies that the sequence $\{\varphi_n\}$ is bounded besides a countable set; this follows again from the Banach–Steinhaus theorem since φ_n as functional on M_c has norm $\inf_D \sup_{\boldsymbol{T} \setminus D} |\varphi_n(t)|$ where D runs over all countable sets.

3. It would be interesting to know whether there are other spectra that Sidon for which the point-wise convergence of a bounded sequence in $C^E(\boldsymbol{T})$ implies the uniform convergence(1). Especially, have ***Ka***-sets this property? (see the table in Theorem 13).

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(1) Added in proof: Such spectra do ezist as was proved by Y. Meyer (oral communication).