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Invariant means on L^∞

by

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To Professor Antoni Zygmund on the 50th anniversary of his first mathematical publication.

Abstract. Let G be a locally compact, infinite, amenable group. It is shown that there exist many invariant means on $L^\infty(G)$. If G is compact, its Haar integral is therefore not the only invariant mean on $L^\infty(G)$. If G is not discrete, it follows that there exist bounded linear operators on $L^\infty(G)$ which commute with translations but do not commute with convolutions by continuous functions with compact support.

I. Introduction. It is part of the folklore of harmonic analysis that (roughly speaking) every linear operator Ψ that commutes with translations is convolution with something (a measure, an integrable function of some specified type, or a distribution) and that Ψ therefore also commutes with convolutions, i.e., that

$$f*\Psi g = \Psi(f*g) \tag{1}$$

for all g in the domain of Ψ and for all suitable f. The present paper exhibits two situations in which this "theorem" fails.

To be more specific, let G be a locally compact group (not necessarily abelian), and define left and right translation operators L_s and R_s, for every $s\in G$, by

$$(L_sf)(x) = f(sx), \qquad (R_sf)(x) = f(xs) \qquad (x\in G) \tag{2}$$

if f is a function with domain G, and by

$$(L_s\mu)(E) = \mu(sE), \qquad (R_s\mu)(E) = \mu(Es) \tag{3}$$

if μ is a measure on G. A space X of functions or measures on G is then said to be *invariant* if $L_sX = X = R_sX$ for every $s\in G$, and an operator Ψ on X is said to *commute with translations* if

$$L_s\Psi = \Psi L_s \quad\text{and}\quad R_s\Psi = \Psi R_s \tag{4}$$

for every $s\in G$.

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For many of the invariant spaces X that occur in harmonic analysis it has been proved (especially when G is abelian) that any bounded linear operator Ψ that satisfies (4) also satisfies (1), essentially for all f for which the convolutions make sense for every $g \in X$. See, for instance [4] or ([2], Chap. 16); also ([8], pp. 161–162, 197–198).

But when $X = L^\infty(G)$ (the space of all essentially bounded Haar-measurable functions on G, modulo those that vanish a.e.) and also when $X = M(G)$ (the space of all regular complex Borel measures on G), the implication (4) $\to$ (1) has so far been in doubt, even in the case in which G is the unit circle. These problems are discussed in ([2], § 16.2.6) and in ([4] p. 125, p. 130).

For $M(G)$ it is, however, very easy to see that (4) may hold without (1), if G is *any* non-discrete locally compact group: Every $\mu \in M(G)$ has a unique decomposition

$$\mu = \mu_d + \mu_c \tag{5}$$

where μ_d is discrete and μ_c is continuous. Define

$$\Psi\mu = \mu_d. \tag{6}$$

It is clear that Ψ is linear, that $\|\Psi\| = 1$, and that Ψ commutes with translations. If G is not discrete, there exists a continuous $\lambda \in M(G)$ such that $\lambda \geqslant 0$ and $\lambda(G) = 1$, Then $\lambda * \mu$ is continuous for every $\mu \in M(G)$ ([3] Th. (19.16)) so that $\Psi(\lambda * \mu) = 0$. But if $\mu \in M(G)$ is chosen so that $\mu_d \geqslant 0$ and $\mu_d(G) = 1$, then $\lambda * \Psi\mu \neq 0$.

For $L^\infty(G)$, the problem is not quite so easy. A negative answer is obtained below (Theorem 4.1) for non-discrete G which are abelian or, more generally, for those that are amenable (Definition 3.2) although I suspect that the same negative result holds for every infinite compact G. It is also shown that no restriction on the range of Ψ can force a positive answer, since the operators Ψ that give the counter examples have one-dimensional ranges. They are therefore essentially invariant means on $L^\infty(G)$.

I shall now introduce a class of sets which is useful in the construction of such means, and which may be of some independent interest.

II. Permanently positive sets.

DEFINITION 2.1. Let G be a locally compact group. A Borel set $E \subset G$ is said to be *permanently positive* (PP, for brevity) if all intersections of the form

$$\bigcap_{i=1} x_i E y_i \tag{7}$$

have positive Haar measure, for every natural number n and for every choice of $x_1, \ldots, x_n, y_1, \ldots, y_n$ in G.

A *PP-filter* is a nonempty collection Ω of PP sets, with the following two properties:

(i) If $A \in \Omega$ and $B \in \Omega$ then $A \cap B \in \Omega$.

(ii) If $A \in \Omega$, $x \in G$, $y \in G$, then $xAy \in \Omega$.

EXAMPLES 2.2. (a) *Dense open sets are PP. The collection of all dense open subsets of G is a PP-filter.*

This is true because the intersection of any two dense open sets is dense, and since nonempty open sets have positive Haar measure.

(b) *If E is PP and if Ω is the collection of all intersections* (7), *then Ω is a PP-filter.*

This is obvious, from the definitions. It follows that *every PP set lies in some PP-filter.*

(c) If G is compact, it is clear that every PP set is dense. This is not true in general. For instance, if $G = R$ (the real line) and if E is any union of intervals of unbounded lengths, then E is PP. One can also remove open sets of very small measure from these intervals, and thus produce a PP set in R which is closed and nowhere dense. This is in marked contrast with (d):

(d) *Suppose G is compact, $A \subset G$ is PP, $B \subset G$ is dense and open. Then $A \cap B$ is PP.*

To prove this, put $E = A \cap B$, and note that (7) is the intersection of A' and B', where

$$A' = \bigcap_{i=1}^{n} x_i A y_i, \qquad B' = \bigcap_{i=1}^{n} x_i B y_i. \tag{8}$$

Since G is compact, finitely many translates of B' cover G. If it were true that $m(A' \cap B') = 0$, the intersection of some finite collection of translates of A' would have measure 0, which is impossible if A is PP. Thus $m(A' \cap B') > 0$, which proves that E is PP.

However, even in compact groups it is not true that the intersection of any two PP sets is PP:

THEOREM 2.3. *Every infinite compact group G contains a PP set whose complement is also PP.*

Proof. Let S be the collection of all Borel sets in G, modulo sets of Haar measure 0, metrized by

$$d(A, B) = \int_G |\chi_A - \chi_B|\, dm, \tag{9}$$

where χ_A and χ_B are the characteristic functions of A and B, respectively, and m is the normalized Haar measure of G. It is then easily verified (and well known) that S is a *complete* metric space.

For $n = 1, 2, 3, \ldots$, define Q_n to be the collection of all $E \in S$ such that

$$m\Big(\bigcap_{i=1}^{n} x_i E y_i\Big) = 0 \tag{10}$$

for some choice of $x_1, \ldots, x_n, y_1, \ldots, y_n$ in G, and let Q'_n be the collection of all $E \in S$ whose complements E^c belong to Q_n.

We shall prove that each Q_n is closed in S and that Q_n has empty interior.

Once this is done, the same is true of Q'_n, since the mapping $E \to E^c$ is an isometry of S onto S. By Baire's theorem, some $E \in S$ belongs therefore to no Q_n and to no Q'_n. Any such E satisfies the requirements of the theorem.

Fix $E \in Q_n$. By the regularity of m, and Theorem 2.4 below, there exist dense open sets $V_k \supset E$ such that $d(E, V_k) < 1/k$, for $k = 1, 2, 3, \ldots$ Each V_k is PP, hence lies outside Q_n. The interior of Q_n is therefore empty.

Finally, suppose $A \in S$ lies in the closure of Q_n. Choose $\varepsilon > 0$. There exists $E \in Q_n$ with $d(A, E) < \varepsilon/n$, and there exist x_i, y_i so that (10) holds. The inequality

$$d\Big(\bigcap_{i=1}^{n} A_i, \bigcap_{i=1}^{n} B_i\Big) \leqslant \sum_{i=1}^{n} d(A_i, B_i), \tag{11}$$

valid for arbitrary $A_i, B_i \in S$, follows easily from (9). Now (10) and (11) imply that

$$m\Big(\bigcap_{i=1}^{n} x_i A y_i\Big) = d\Big(\bigcap_{i=1}^{n} x_i A y_i, \bigcap_{i=1}^{n} x_i E y_i\Big) \leqslant \sum_{i=1}^{n} d(x_i A y_i, x_i E y_i) = n\, d(A, E),$$

so that

$$m\Big(\bigcap_{i=1}^{n} x_i A y_i\Big) < \varepsilon. \tag{12}$$

The left side of (12) is a continuous function of $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ on the compact space G^{2n}, the cartesian product of $2n$ copies of G. By (12), the greatest lower bound of this function is 0. By compactness of G^{2n}, the value 0 is attained. Thus $A \in Q_n$, Q_n is closed, and the proof is complete.

THEOREM 2.4. *Suppose G is a locally compact group which is not discrete and which is generated by one of its compact subsets, m is a left Haar measure on G, and $\varepsilon > 0$. Then G contains a dense open set E with $m(E) < \varepsilon$.*

Proof. Since G is not discrete, its identity element has neighborhoods U_n, with compact closure, such that $m(U_n) < 1/n$. It follows from ([3], Th. (8.7)) that G has a compact normal subgroup $N \subset \cap U_n$, with $m(N) = 0$ such that G/N is separable. The union of some countable collection of

cosets of N is therefore dense in G. The regularity of m shows that these cosets can be covered by open sets of small measure, whose union E has $m(E) < \varepsilon$.

III. Invariant means.

DEFINITION 3.1. Let $B(G)$ be the vector space of *all* bounded complex functions on the group G. Let X be an invariant subspace of $B(G)$ which contains the constants. A linear functional M on X is then said to be an *invariant mean* on X if

(i) $ML_s f = Mf = MR_s f$ for all $f \in X$ and $s \in G$,

(ii) $M1 = 1$, and

(iii) $|Mf| \leqslant \sup_{x \in G} |f(x)|$, for all $f \in X$.

A familiar argument (see [7], pp. 109–110) shows that (ii) and (iii) imply

(iv) $Mf \geqslant 0$ if $f \geqslant 0$ and $f \in X$.

Invariant means on $L^\infty(G)$ are defined in the same way, except that the sup in (iii) is replaced by the *essential* supremum.

DEFINITION 3.2. If there exists an invariant mean on $B(G)$, then G is called *amenable.*

In [6], von Neumann showed (generalizing earlier work of Banach) that all abelian groups, as well as all solvable groups, are amenable; on the other hand, if G contains a free subgroup with two generators (for instance, if G is the group of all rotations in R^3), then G is not amenable.

Frequently, amenability is defined in such a way that only one half of condition 3.1 (i) is assumed. But then there exists a (possibly different) mean on $B(G)$ which satisfies the full condition. See ([3], Th. (17.11)). Section 17 of [3] contains a good introduction to invariant means.

Remark 3.3. If G is compact, then G need not be amenable, but nevertheless there is always an invariant mean on $L^\infty(G)$, namely the Haar integral; on $C(G)$, this is the only one. The point of Theorem 3.4 below is that there are many others on $L^\infty(G)$, at least when G is amenable.

Relatively little work seems to have been done on invariant means on $L^\infty(G)$; see [5]. For infinite amenable groups it is known [1] that there are at least 2^c invariant means on $B(G)$, where c is the cardinality of the continuum.

If $f \in L^\infty(G)$, $Z(f)$ will denote the set of all $x \in G$ at which $f(x) = 0$.

THEOREM 3.4. *Suppose that G is a locally compact amenable group, and Ω is a PP-filter in G. Then there exists an invariant mean M on $L^\infty(G)$ with the following property: if $f \in L^\infty(G)$ and $Z(f)$ contains some member of Ω, then $Mf = 0$.*

Thus M identifies any two functions in $L^\infty(G)$ which coincide on some member of Ω.

Proof. $L^\infty(G)$ is a commutative Banach algebra, with respect to pointwise multiplication. Let J be the set of all $f\in L^\infty(G)$ such that $Z(f)$ contains some member of Ω. Since Ω satisfies 2.1 (i), J is an ideal. Since $1\notin J$, J is contained in a maximal ideal, and is therefore (by the Gelfand–Mazur theorem) annihilated by some homomorphism h of $L^\infty(G)$ onto the complex field.

The amenability of G implies that of the direct product $G\times G$ ([3], Th. (17.14)]. Hence there is an invariant mean Λ on $B(G\times G)$.

Since $\|h\|=1$, the formula

$$(\Phi f)(x,y)=h(L_xR_yf) \tag{13}$$

associates with each $f\in L^\infty(G)$ a function $\Phi f\in B(G\times G)$, and one can therefore define

$$Mf=\Lambda\Phi f\qquad \big(f\in L^\infty(G)\big). \tag{14}$$

Since Ω satisfies 2.1 (ii), $L_xR_yf\in J$ for all $x,y\in G$ if $f\in J$. Hence $Mf=0$ if $f\in J$.

Since $h(1)=1$ and $\Lambda 1=1$, we have $M1=1$.

Since $\|h\|=1$, we have $|Mf|\leqslant\|f\|_\infty$.

It remains to be proved that M satisfies 3.1 (i), i.e., that

$$ML_sR_tf=Mf \tag{15}$$

or all $s\in G$, $t\in G$, and $f\in L^\infty(G)$.

The relations

$$L_uL_v=L_{vu},\qquad L_uR_v=R_vL_u,\qquad R_uR_v=R_{uv} \tag{16}$$

follow directly from (2). They imply that

$$(\Phi L_sR_tf)(x,y)=h(L_xR_yL_sR_tf)=h(L_{sx}R_{yt}f)=(\Phi f)(sx,yt)$$

for any s,t,x,y in G. In other words, using self-explanatory notation,

$$\Phi L_sR_tf=L_{(s,e)}R_{(e,t)}\Phi f, \tag{17}$$

where e is the unit element of G. Since Λ is an invariant mean on $B(G\times G)$ (17) implies that

$$ML_sR_tF=\Lambda\Phi L_sR_tf=\Lambda L_{(s,e)}R_{(e,t)}\Phi f=\Lambda\Phi f=Mf.$$

Thus (15) holds, and the proof is complete.

IV. Some consequences.

THEOREM 4.1. *Suppose G is a locally compact group which is amenable and not discrete. Then there exists a bounded linear operator Ψ on $L^\infty(G)$ such that*

(i) *Ψ commutes with translations*

(ii) *each Ψf is a constant function, and*

(iii) *for some $g\in L^\infty(G)$ and some continuous f with compact support,*

$$f*\Psi g\neq\Psi(f*g). \tag{18}$$

Proof. Let m be a left Haar measure on G. Let f be a fixed continuous function on G, with compact support, such that $f\geqslant 0$ and

$$\int_G f\,dm=2. \tag{19}$$

Then there exists $\delta>0$ such that

$$\int_B f\,dm<1\qquad\text{whenever } m(B)<\delta. \tag{20}$$

The support of f generates an open subgroup G_0 of G. By Theorem 2.4, G_0 contains a dense open set E_0 with $m(E_0)<\delta$. Pick one point u in each coset of G_0, let E be the union of the corresponding translates uE_0, and put $A=E\cap E^{-1}$.

This gives us a dense open set A. Since A is PP, Theorem 3.4 shows that there exists an invariant mean M on $L^\infty(G)$ such that

$$M\chi_A=1, \tag{21}$$

where χ_A is the characteristic function of A. Define $\Psi\colon L^\infty\to L^\infty$ by

$$(\Psi\varphi)(x)=M\varphi\qquad\big(x\in G,\ \varphi\in L^\infty(G)\big). \tag{22}$$

Then $\Psi L_s=\Psi=\Psi R_s$ because M is invariant, and $L_s\Psi=\Psi=R_s\Psi$ because every Ψf is a constant. Thus Ψ commutes with translations.

Put $g=1-\chi_A$. By (21) and (22), $\Psi g=0$, so that

$$f*\Psi g=0. \tag{23}$$

On the other hand,

$$(f*g)(x)=\int_G f(y)g(y^{-1}x)\,dm(y)=2-\int_G f(y)\chi_A(y^{-1}x)\,dm(y).$$

Since $A=A^{-1}$,

$$\chi_A(y^{-1}x)=\chi_A(x^{-1}y)=\chi_{xA}(y),$$

so that

$$(f*g)(x)=2-\int_{xA} f\,dm\qquad(x\in G). \tag{24}$$

Since $m\big(G_0\cap(xA)\big)<\delta$, it follows from (20) and (24) that

$$(f*g)(x)>1\qquad(x\in G). \tag{25}$$

Consequently,

$$\Psi(f*g) \geqslant 1, \tag{26}$$

because M is a mean. Now (18) follows from (23) and (26).

THEOREM 4.2. *Suppose G is a compact group. Let Y be the subspace of $L^\infty(G)$ that consists of all finite linear combinations of the functions*

$$\varphi - L_s\varphi \quad \text{and} \quad \varphi - R_s\varphi \quad \big(s \in G,\ \varphi \in L^\infty(G)\big), \tag{27}$$

and let C be the space of all continuous complex functions on G.

If G is amenable, and if A is a PP set in G with $m(A) < 1$, where m is the normalized Haar measure on G, then

(a) $\operatorname*{ess\,sup}_{x \in A} g(x) \geqslant 0$ *for every $g \in Y$, and*

(b) *χ_A is not in the L^∞-closure of $C + Y$.*

Proof. By Theorem 3.4 there is an invariant mean M on $L^\infty(G)$ such that $M\chi_A = 1$.

If (a) were false, there would be a $g \in Y$ for which $g(x) \leqslant -1$ a.e. on A. Hence $M_g \leqslant -1$. But the invariance of M implies that $M_g = 0$ for every $g \in Y$. This contradiction proves (a).

If $f \in C$, then $Mf = \int f dm$, by the uniqueness of the Haar integral. Hence, for any $g \in Y$,

$$2\|\chi_A - f - g\|_\infty \geqslant \Big|M(\chi_A - f - g) - \int_G (\chi_A - f - g)\,dm\Big|$$
$$= \Big|M\chi_A - \int_G \chi_A\,dm\Big| = 1 - m(A),$$

which proves (b).

Remarks 4.3. There are several ways in which one might try to prove Theorem 4.1 for compact groups that are not necessarily amenable.

First, the functions Φf that occur in the proof of Theorem 3.4 might be Haar-measurable on $G \times G$, in which case the mean Λ could be replaced by the Haar integral over $G \times G$. This would give Theorem 3.4 (hence also 4.1) without amenability.

However, this approach does not seem very promising. In fact, I am unable to decide whether

$$x \to h(L_x f) \tag{28}$$

defines a measurable function on the unit circle T, for every $f \in L^\infty(T)$, if h is a complex homomorphism of $L^\infty(T)$.

Secondly, it is conceivable that there always exists an invariant mean on the subspace $\Phi\big(L^\infty(G)\big)$ of $B(G \times G)$. This again would be enough to prove 3.4 and 4.1.

A third possibility is to try to prove Theorem 4.2 (b) directly, and without using amenability. In fact, it seems quite likely that the L^∞-distance between χ_A and Y is 1, at least when A is dense and open. In that case, the Hahn–Banach theorem would give a linear functional M on L^∞, of norm 1, which vanishes on Y (hence is invariant), and which satisfies $M\chi_A = 1$. This M would give Theorem 4.1.

Postscript (added September 24, 1971). F. P. Greenleaf's book "Invariant Means on Topological Groups" (Van Nostrand, 1969) came to my attention after completion of the present paper. In the terminology used there, my Theorem 4.1 implies the existence of means on $L^\infty(G)$ which are invariant but not *topologically* invariant. That this phenomenon does not happen on the space of all uniformly continuous bounded functions on G is proved on p. 27 and on p. 101 of Greenleaf's book.

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