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Invariant systems of conjugate harmonic functions associated with compact Lie groups

by

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Abstract. Conjugate systems of harmonic functions, anologous to Riesz systems, are defined for compact Lie groups. The corresponding H^p spaces theory is sketched as well as a generalization of the F. and M. Riesz theorem.

1. Introduction. It is well known that the theory of functions of a complex variable plays an important role in Fourier analysis on the line or on the circle. There are several ways of associating a theory of functions with Fourier analysis on Euclidean space $\boldsymbol{R}^n = \{x = (x_1, x_2, \ldots, x_n)\colon x_j$ real for $j = 1, 2, \ldots, n\}$. One of these ways can be described as follows: Given a function $f \in L^p(\boldsymbol{R}^n)$ we consider the $(n+1)$-tuple

$$F(x_0, x) = \big(u_0(x_0, x), u_1(x_0, x), \ldots, u_n(x_0, x)\big)$$

defined on the upper half-space $\boldsymbol{R}^{n+1}_+ = \{(x_0, x)\colon x \in \boldsymbol{R}^n, x_0 > 0\}$, where

$$u_0(x_0, x) = c_n \int\limits_{\boldsymbol{R}^n} f(x-y)\, \frac{x_0}{(|y|^2 + x_0^2)^{(n+1)/2}}\, dy$$

is the Poisson integral of f and

$$u_j(x_0, x) = c_n \int\limits_{\boldsymbol{R}^n} f(x-y)\, \frac{y_j}{(|y|^2 + x_0^2)^{(n+1)/2}}\, dy,$$

$j = 1, 2, \ldots, n$, are the n conjugate Poisson integrals of f. The function F can then be thought of as a generalized analytic function. Indeed, it is easy to see that the components of F satisfy the equations

$$\frac{\partial u_j}{\partial x_k} = \frac{\partial u_k}{\partial x_j} \quad \text{and} \quad \sum_{i=0}^{n} \frac{\partial u_i}{\partial x_i} = 0, \tag{1.1}$$

$j, k = 0, 1, \ldots, n$, which reduce to the Cauchy–Riemann equations when

$n = 1$. Stein and Weiss [10] have developed a theory of H^p spaces for $(n+1)$-tuple valued functions F satisfying (1.1) and the condition

$$\int_{\boldsymbol{R}^n} |F(x_0, x)|^p\, dx \leqslant c < \infty \tag{1.2}$$

for all $x_0 > 0$. The chief feature of this theory is that results on the boundary behaviour of F (as x_0 approaches 0) can be obtained for certain $p < 1$. The basic tool used in order to do this is the fact that if F satisfies (1.1), then $|F|^p$ is subharmonic for $p \geqslant (n-1)/n$. This is a consequence of the following simple fact about matrices (see [10]): suppose $\mathfrak{M} = (m_{ij})$, $i, j = 0, 1, 2, \ldots, n$, is a symmetric real matrix with trace 0 then its operator norm $\|\mathfrak{M}\|$ and its Hilbert–Schmidt norm $|||\mathfrak{M}||| = (\sum_{i,j} m_{ij}^2)^{1/2}$ satisfy

$$\|\mathfrak{M}\|^2 \leqslant \frac{n}{n+1} |||\mathfrak{M}|||^2. \tag{1.3}$$

It is our intention in this paper to show how equations similar to (1.1) can be associated to a class of Lie groups in such a way that a subharmonicity result can be obtained which permits us to obtain an H^p space theory analogous to the classical one. We do this in the next section. In Section 3 we indicate how the H^p space theory can be developed. We then apply it to obtain an extension of the theorem of F. and M. Riesz. We also indicate how the operators corresponding to the M. Riesz transforms

$$(R_j f)(x) = \lim_{x_0 \to 0} u_j(x_0, x)$$

can be used to obtain results concerning Jacobi polynomials by examining more closely the special case $G = SU(2)$.

2. Invariant Riesz systems of conjugate harmonic functions. Let G be an n-dimensional Lie group and $\mathscr{G}$ its Lie algebra (the left invariant derivations). We assume that $\mathscr{G}$ is endowed with an inner product $(\ ,\ )$ satisfying

$$([X, Y], Z) = -(Y, [X, Z]). \tag{2.1}$$

This equality implies that $\mathscr{G}$ is the Lie algebra of a compact group (see Hochschield [5], p. 142). In fact, if G is a semi-simple compact group we can choose the inner product to be the negative of the Killing form. When G is abelian (and not necessarily compact) obviously any inner product can be chosen.

Let us choose an orthonormal basis $\{X_1, X_2, \ldots, X_n\}$ of $\mathscr{G}$. We can then define a *Laplacian* Δ for G by letting

$$\Delta = \sum_{j=1}^{n} X_j^2.$$

It is easy to see (using (2.1)) that Δ is bi-invariant.

Our purpose will be to study harmonic functions on $\boldsymbol{R}_+ \times G = \{(x_0, x): x \in G, x_0 > 0\}$. Following Stein [9], this means that we shall study twice-differentiable functions $u(x_0, x)$, defined on $\boldsymbol{R}_+ \times G$, satisfying

$$\varDelta u = \left(\frac{\partial^2}{\partial x_0^2} + \Delta\right) u = 0.$$

We shall write $X_0 = \partial/\partial x_0$ and, thus, $\varDelta = \sum_{j=0}^{n} X_j^2$. We also extend the inner product to the direct sum $\boldsymbol{R} \oplus \mathscr{G}$ by requiring $\{X_0, X_1, \ldots, X_n\}$ to be an orthonormal basis. It will be convenient to identify $(n+1)$-tuples $F = (f_0, f_1, \ldots, f_n)$ with the elements $f = \sum_{j=0}^{n} f_j X_j$ of $\boldsymbol{R} \oplus \mathscr{G}$.

We can now present the extension of the generalized Cauchy–Riemann equations (1.1) to the case obtained when $\boldsymbol{R}^n$ is replaced by G. We shall say that a function $F = (f_0, f_1, \ldots, f_n)$ mapping $\boldsymbol{R}_+ \times G$ into $\boldsymbol{R}^{n+1}$ is a (left) invariant Riesz system if it is differentiable and satisfies the equations

$$\begin{cases} \text{(a)} \quad X_i f_j - X_j f_i = ([X_i, X_j], F) = a_{ij}(F) = a_{ij} \\ \qquad\qquad\qquad\qquad \text{for } i, j = 0, 1, \ldots, n; \\ \text{(b)} \quad \sum_{j=0}^{n} X_j f_j = 0. \end{cases} \tag{2.2}$$

An example of such an invariant Riesz system is obtained by choosing a harmonic function u on $\boldsymbol{R}_+ \times G$ and forming its "gradient:" $F = (X_0 u, X_1 u, \ldots, X_n u)$.

PROPOSITION 2.3. *If $F = (f_0, f_1, \ldots, f_n)$ is an invariant Riesz system, then each of the components $f_0, f_1, \ldots, f_n$ is harmonic*(1).

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(1) The method used by Coifman and Weiss [2] to obtain the existence of a $p < 1$ for which theorem (2.4) is true can also be employed here once this proposition is established.

Proof. Using (2.1), the identification $\boldsymbol{R}^{n+1}$ with $\boldsymbol{R}\oplus\mathscr{G}$ we made above and (2.2), part (a), in that order, we have:

$$\sum_{k=0}^{n}[X_j, X_k]f_k = \sum_{k=0}^{n}\sum_{i=0}^{n}(X_i, [X_j, X_k])X_i f_k$$
$$= \sum_{k=0}^{n}\sum_{i=0}^{n}([X_i, X_j], X_k)X_i f_k = \sum_{i=0}^{n}X_i([X_i, X_j], F)$$
$$= \sum_{i=0}^{n}X_i(X_i f_j - X_j f_i) = \Delta f_j - \sum_{i=0}^{n}X_i X_j f_i.$$

But, by (2.2) part (b),

$$\sum_{i=0}^{n}X_i X_j f_i = X_j\sum_{i=0}^{n}X_i f_i + \sum_{i=0}^{n}[X_i, X_j]f_i = -\sum_{k=0}^{n}[X_j, X_k]f_k.$$

Thus,

$$\Delta f_j = \sum_{i=0}^{n}X_i X_j f_j + \sum_{k=0}^{n}[X_j, X_k]f_k = 0.$$

This proves the proposition.

Now consider a C^2 function s defined on a domain $\mathscr{D}\subset \boldsymbol{R}_+\times G$ such that $\Delta s\geqslant 0$. By repeating the usual argument it can be shown that such a function restricted to a compact set $K\subset\mathscr{D}$ attains its maximum on the boundary ∂K. From this it follows that if h is a harmonic function defined on a neighborhood of K such that $h\geqslant s$ on ∂K, then $h\geqslant s$ on K. We use this last property of s in order to define the notion of a subharmonic function as is done for the ordinary Laplace operator (see Rado [6]). Observe that a decreasing limit of subharmonic functions in subharmonic.

The main result of this paper is the following one:

THEOREM 2.4. *Let F be (left) invariant Riesz system; then $s = |F|^p$ is subharmonic for $p\geqslant (n-1)/n$.*

We shall reduce this result to inequality (1.3) by establishing the lemma:

LEMMA 2.5. *Suppose $\mathfrak{M}$ is an $n\times n$ real matrix with trace 0 and F a column vector satisfying $(\mathfrak{M}-\mathfrak{M}')F = AF = 0$, where $\mathfrak{M}'$ is the transpose of $\mathfrak{M}$, then*

$$|\mathfrak{M}F|^2\leqslant\frac{n}{n+1}|||\mathfrak{M}|||^2|F|^2. \tag{2.6}$$

Proof. Let $\mathfrak{N} = \mathfrak{M}-\frac{1}{2}A$. Then $\mathfrak{N}$ is symmetric and has trace 0. Thus, by (1.3), $\|\mathfrak{N}\|^2\leqslant [n/(n+1)]|||\mathfrak{N}|||^2$. Since $\mathfrak{M}F = \mathfrak{N}F$, inequality (2.6) will follow if we can show that $|||\mathfrak{N}|||^2\leqslant |||\mathfrak{M}|||^2$. But

$$\begin{aligned}|||\mathfrak{N}|||^2 &= \operatorname{tr}(\mathfrak{N}\mathfrak{N}') = \operatorname{tr}(\mathfrak{M}-\tfrac{1}{2}A)(\mathfrak{M}'-\tfrac{1}{2}A')\\ &= \operatorname{tr}(\mathfrak{M}\mathfrak{M}')+\tfrac{1}{4}\operatorname{tr}(AA')+\tfrac{1}{2}\{\operatorname{tr}(\mathfrak{M}A)-\operatorname{tr}(A\mathfrak{M}')\}\\ &= |||\mathfrak{M}|||^2+\tfrac{1}{4}|||A|||^2+\tfrac{1}{2}\{\operatorname{tr}(A\mathfrak{M})-\operatorname{tr}(A\mathfrak{M}')\}\\ &= |||\mathfrak{M}|||^2+\tfrac{1}{4}|||A|||^2-\tfrac{1}{2}\operatorname{tr}(AA') = |||\mathfrak{M}|||^2-\tfrac{1}{4}|||A|||^2\leqslant |||\mathfrak{M}|||^2.\end{aligned}$$

In order to apply this lemma to obtain Theorem 2.4 let $s_\varepsilon = (|F|^2+\varepsilon^2)^{p/2}$ for $\varepsilon > 0$. It suffices to show that $\Delta s_\varepsilon\geqslant 0$ for $p\geqslant (n-1)/n$. An elementary computation, taking into account the fact that the components of F are harmonic (by Proposition 2.3), shows

$$\Delta s_\varepsilon = p(|F|^2+\varepsilon^2)^{(p-4)/2}\Big[(p-2)\sum_{i=0}^{n}(F\cdot X_i F)^2+(|F|^2+\varepsilon^2)\sum_{i=0}^{n}|X_i F|^2\Big]$$
$$\geqslant p(|F|^2+\varepsilon^2)^{(p-4)/2}\Big[(p-2)\sum_{i=0}^{n}(F\cdot X_i F)^2+|F|^2\sum_{i=0}^{n}|X_i F|^2\Big].$$

When $p\geqslant 2$ it is clear that the last expression is non-negative and, therefore, s_ε is subharmonic. If $p<2$ the last expression is non-negative if and only if

$$\sum_{i=0}^{n}(X_i F\cdot F)^2\leqslant\frac{1}{2-p}|F|^2\sum_{i=0}^{n}|X_i F|^2. \tag{2.7}$$

But the matrix $\mathfrak{M} = (m_{ij})$, where $m_{ij} = X_i f_j$, $j = 0, 1, \ldots, n$, and the n-tuple F satisfy the conditions of Lemma 2.5. The fact that $\operatorname{tr}\mathfrak{M} = 0$ is the assumption that (2.2), part (b), is satisfied. While, using (2.2), part (a), and (2.1) we obtain

$$\sum_{j=0}^{n}a_{ij}(F)f_j = \Big(\sum_{j=0}^{n}f_j[X_i, X_j], F\Big) = ([X_i, F], F) = -(X_i, [F, F]) = 0.$$

That is, $(\mathfrak{M}-\mathfrak{M}')F = AF = 0$. Thus, by (2.6), we see that (2.7) is satisfied for $1/(2-p)\geqslant n/(n+1)$ or, equivalently, $p\geqslant (n-1)/n$. This proves Theorem 2.4.

3. Some applications. We shall first indicate how Theorem 2.4 can be used in order to obtain an H^p-space theory. Let G be a Lie group of the type we are considering and F: $\boldsymbol{R}_+\times G\to\boldsymbol{R}^{n+1}$ a left invariant Riesz system. We say that F belongs to the class $H^p = H^p(G)$ for $p>0$ if and only if

$$\int_G |F(x_0, x)|^p dx\leqslant c<\infty \tag{3.1}$$

for all $x_0 > 0$. The results concerning the boundary behaviour of functions in the class H^p, which were developed in chapter VI of the book of Stein and Weiss [11] for the case $G = \boldsymbol{R}^n$, can easily be extended to the case treated here. The method is essentially the same. The basic tools needed to carry out this extension, such as the result of Calderón [1] concerning non-tangential convergence can be found in the article of Widman [12] (by expressing Δ in terms of local coordinate systems). The same is also true for the area theorem treated in chapter VII of Stein's book [8]. For this reason we will not carry out the details here.

As can be expected from the classical situation, the theorem of F. and M. Riesz extends to the case of compact Lie groups by making use of properties of functions in $H^1(G)$. These properties are easily obtained from Theorem 2.4 without making use of the local behaviour discussed above. Let us indicate, briefly, how this can be done.

We define the Poisson kernel associated with G by letting, as is done by Stein [9],

$$P_{x_0}(x) = \sum_{a\in\mathscr{A}} d_a e^{-\lambda_a x_0}\chi_a(x),$$

where $\mathscr{A}$ is an index set for all unitary irreducible representations of G, χ_a are the corresponding characters, $d_a = \chi_a(e)$ and $\lambda_a \geqslant 0$ are defined by $\Delta\chi_a = -\lambda_a^2\chi_a$.

We can now state the extension of the Riesz brothers' theorem:

THEOREM 3.2. *Let* $\mu = (\mu_0, \mu_1, \ldots, \mu_n)$ *be a vector-valued finite measure on* G *such that* $P_{x_0} * \mu$ *is an invariant Riesz system* (²), *then* μ *is absolutely continuous.*

Proof. Since μ is a finite measure it follows that $F = P_{x_0} * \mu$ belongs to $H^1(G)$. Moreover,

$$\lim_{x_0\to\infty} F(x_0, x) = \big(\mu_0(G), \mu_1(G), \ldots, \mu_n(G)\big) = c, \tag{3.3}$$

uniformly in x. We now claim that there exists an $f = (f_0, f_1, \ldots, f_n)$ with $f_j \in L^1(G)$, $j = 0, 1, \ldots, n$, such that

$$F(x_0, x) = (P_{x_0} * f)(x). \tag{3.4}$$

Once this equality is established theorem (3.2) follows from the uniqueness of Poisson integrals.

The first step for obtaining (3.4) it to show that there exists $g \in L^p(G)$, $p = n/(n-1) > 1$, such that

$$(P_{x_0} * g) \geqslant |F(x_0, x)|^{(n-1)/n} \equiv s(x_0, x). \tag{3.5}$$

(²) If the Riesz transforms are defined as in Stein [9], then $\mu_1, \mu_2, \ldots, \mu_n$ are the n Riesz transforms of μ_0.

Since $p > 1$ and G is compact we have, for $x_0 > 0$,

$$\int_G s(x_0, x)\,dx \leqslant \Big(\int_G [s(x_0, x)]^p\,dx\Big)^{1/p} = \Big(\int_G |F(x_0, x)|\,dx\Big)^{1/p}.$$

But, since $F \in H^1(G)$, this means that $\int_G s(x_0, x)\,dx \leqslant A < \infty$ for all $x_0 > 0$. Let $s_\varepsilon(x) = s(\varepsilon, x)$ and $u_\varepsilon(x_0, x) = (P_{x_0} * s_\varepsilon)(x)$, where $\varepsilon > 0$. We now claim that

$$u_\varepsilon(x_0, x) \geqslant s(x_0+\varepsilon, x) \tag{3.6}$$

for all $(x_0, x) \in \boldsymbol{R}_+ \times G$. This is clearly true for $x_0 = 0$. Moreover, the integral $\int_G s(x_0, x)\,dx$ is convex as a function of x_0 since

$$\frac{\partial^2}{\partial x_0^2}\int_G s(x_0, x)\,dx = \int_G \Delta s(x_0, x)\,dx \geqslant 0\,(^3).$$

The fact that this integral is also a bounded function of x_0 implies that it decreases to (see (3.3))

$$\lim_{x_0\to\infty}\int_G s(x_0, x)\,dx = |c|^{(n-1)/n} = \lim_{x_0\to\infty} s(x_0, x).$$

Hence,

$$\lim_{x_0\to\infty} u_\varepsilon(x_0, x) = \int_G u_\varepsilon(0, x)\,dx = \int_G s(\varepsilon, x)\,dx$$

$$\geqslant \lim_{x_0\to\infty}\int_G s(x_0, x)\,dx = \lim_{x_0\to\infty} s(x_0, x).$$

Thus, for each $\delta > 0$ there exists $a > 0$ such that $u_\varepsilon(x_0, x)+\delta \geqslant s(x_0+\varepsilon, x)$ whenever $x_0 \geqslant a$. This and the maximum principle clearly imply (3.6).

The functions s_ε form a bounded family in $L^p(G)$. Thus, there exists a sequence $\{\varepsilon_k\}$ decreasing to 0 such that s_{ε_k} converges weakly to a function $g \in L^p(G)$. In particular, using (3.6), we have

$$(P_{x_0} * g)(x) \geqslant s(x_0, x)$$

for each $x_0 > 0$. Hence,

$$g^*(x) \equiv \sup_{x_0>0}(P_{x_0} * g)(x) \geqslant s(x_0, x).$$

But, by Lemma 1 on p. 48 of Stein [9], $g * \in L^p(G)$. Using the same lemma and standard arguments we obtain the existence of the (almost everywhere) limits

$$\lim_{x_0\to 0}(P_{x_0} * d\mu)(x) = \big(f_0(x), f_1(x), \ldots, f_n(x)\big) = f(x).$$

(³) See Stein [9], p. 50, for this equality. If $\Delta s(x_0, x)$ is not defined we can approximate s (from above) by C^2 functions as we did in the proof of Theorem 2.4.

Since $(g^*)^{1/p} \in L^1(G)$ and dominates $|F(x_0, x)|$ we can apply the dominated convergence theorem of Lebesgue to obtain the L^1-convergence of $F(x_0, \cdot)$ to the function f. From this, (3.4) and the fact that $d\mu(x) = f(x)\,dx$ follow immediately and Theorem 3.2 is established.

Many other results connected with H^p-space theory also admit similar extensions. For example, the theory involving the Lusin area function, as developed in chapter VII of Stein [8], can be carried out in this situation as well. In this connection see also the results of Fefferman and Stein [4], [7].

The Riesz transforms alluded to above are defined in Stein [9], where a proof is given of their boundedness as operators on $L^p(G)$, $1 < p < \infty$. These Riesz transforms add a novel feature to the harmonic analysis on semi-simple compact groups that does not appear in the commutative case. When $G = SU(2)$ there exist linear combinations of these transforms that are "shift" operators with respect to a "canonical" basis of $L^2(G)$ (see Coifman and Weiss [3]). The extension of the Riesz brothers' theorem obtained here implies a corresponding theorem for expansions of some Jacobi polynomials (which are connected with the elements of this basis). That this phenomenon is more general is evident from the fact that there exists a root-space decomposition of the associated Lie algebra.

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On regular temperate distributions

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Abstract. There are given some conditions which imply that a locally Lebesgue integrable function u defines a temperate distribution (u, σ) by the relation $(u, \sigma) = \int_{\mathbf{R}^n} u(x)\sigma(x)\,dx$ with the integral converging absolutely for every function rapidly decaying at infinity. It is shown that the assertion included in [1] about the necessity of one of these conditions is not true.

1. Basic notations. The variable in the n-dimensional real Euclidean space $\mathbf{R}^n$ will be denoted by $x = (x_1, \ldots, x_n)$. By α we shall denote multi-indices, that is, n-tuples $(\alpha_1, \ldots, \alpha_n)$ of non-negative integers. We set $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$ with $D_j = \dfrac{\partial}{\partial x_j}$. Similarly we write $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$.

A complex valued function φ defined in $\mathbf{R}^n$ is said to be a C^∞ function if it possesses continuous partial derivatives of all orders. By C_0^k we denote the set of all functions in C^∞ with compact support in $\mathbf{R}^n$.

By S or $S(\mathbf{R}^n)$ we denote the set of all functions $\sigma \in C^\infty$ such that

$$\sup_x |x^\beta D^\alpha \sigma(x)| < \infty \tag{1}$$

for all multi-indices α and β. The topology in S is defined by semi-norms in the left-hand side of (1).

A continuous linear functional (u, σ) on S is called a *temperate distribution*. The set of all temperate distributions is denoted by S'.

We denote by $L_1^{\mathrm{loc}}(\mathbf{R}^n)$ the space of locally Lebesgue integrable functions, i.e. Lebesgue integrable on any compact subset of $\mathbf{R}^n$. We identify every function $u \in L_1^{\mathrm{loc}}(\mathbf{R}^n)$ with the distribution u defined by:

$$(u, \varphi) = \int_{\mathbf{R}^n} u(x)\varphi(x)\,dx \quad \text{for} \quad \varphi \in C_0^\infty(\mathbf{R}^n). \tag{2}$$

A temperate distribution u is called *regular* if there exists a function $u \in L_1^{\mathrm{loc}}(\mathbf{R}^n)$ such that

$$(u, \sigma) = \int_{\mathbf{R}^n} u(x)\sigma(x)\,dx \quad \text{for} \quad \sigma \in S(\mathbf{R}^n) \tag{3}$$