By (36), we have

$$4\gamma^2 \geqslant v_n[c_{(43)}2^{-3}\theta^2 - 16\delta^2\eta^{-2} - 8\delta\theta],$$

and we see that this leads to a contradiction if both $\delta$ and $\gamma$ are chosen suitably small. Therefore, (34) must hold and the proof of Theorem 3 is complete.

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*Received October 8, 1971* (416)

STUDIA MATHEMATICA, T. XLIV. (1972)

# An inequality for the indefinite integral of a function in $L^q$

by

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*To A. Zygmund, real analyst*

**Abstract.** Let $p, q$ be positive numbers with $1 = 1/p + 1/q$. Let $F$ be continuously differentiable on the positive reals and zero at the origin. Let $g$ denote the $p$th power of $|F|$. J. Moser has shown that if $q$ is at least 2 and the $q$ norm of $F'$ is at most 1 then the integral of $\exp(g(x) - x)$ is bounded by a constant depending only on $q$. A new proof of this is given, and the result extended to all $q > 1$.

In his paper "*A sharp form of an inequality by N. Trudinger*", J. Moser ([2], Theorem 1) proves that if $D$ is a bounded domain in $R^n$, $n \geqslant 2$, and $u$ is a $C^1$ function with compact support in $D$ such that $\int_D |\operatorname{grad} u(x)|^n dx \leqslant 1$ then $\int_D \exp \alpha_n |u(x)|^{n/(n-1)} dx \leqslant c_n$ for certain constants $\alpha_n$, $c_n$ independent of $u$.

Earlier N. Trudinger [3] proved this for some $\alpha > 0$.

Moser elegantly reduces the question to the following one-dimensional inequality which he proves for $q \geqslant 2$.

The present paper contains a new proof which incidentally works for $q > 1$.

THEOREM 1.1. *Let $q$ and $p$ denote positive numbers with $1 < q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ (Hölder conjugates). Let $f$ be Lebesgue measurable on $(0, \infty)$, let $\int_0^\infty |f(x)|^q dx \leqslant 1$, and let $F(x) = \int_0^x f(t)\,dt$. Then there exists a number $C_q$ depending only on $q$ such that*

$$\int_0^\infty e^{|F(x)|^p} e^{-x} dx \leqslant C_q. \tag{1.2}$$

In what follows we consider only non-negative functions $f$. An equivalent theorem arises through use of the substitution $x = \log\frac{1}{u}$:

* Partially supported by the National Science Foundation under NSF grant GP 28271.

THEOREM 1.3. *Let $g$ be defined on $(0,1)$. Suppose $g$ is measurable, non-negative, and $\int_0^1 g(u)^q \frac{du}{u} \leqslant 1$. Write $Tg(x) = \int_x^1 g(u)\frac{du}{u}$, $0<x<1$. Then (with $p$, $q$, and $C_q$ as before),*

$$\int_0^1 \exp\big(Tg(x)\big)^p dx \leqslant C_q.$$

In Section 2, Theorem 1.1 is shown to be equivalent to an assertion about the measure of the set of points where $x-F(x)^p<y$, namely that $m\big(x-F(x)^p<y\big)\leqslant \text{const}\, y$. The derivation of the estimate occupies Sections 3, 4, 5.

The function $A_p(x) \equiv (\exp x^p)-1$, which might as well appear in (1.2) and (1.3) determines an Orlicz space $L_{A_p}$, the dual of the Orlicz space determined by $\bar{A}_p$ (which is essentially $x(\log^+ x)^{1/p}$). The linear map $g\to Tg$ of (1.3) which carries $L^q(I, du/u)$ ($I$ the unit interval) into $L_{A_p}(I, dx)$ is the transpose of the map $f\to F(x) = \int_0^x f(t)\,dt$, so we get the following endpoint case of Hardy's inequality.

THEOREM 1.4. *If $1<p<\infty$, $\int_0^1 f(t)\big(\log^+ f(t)\big)^{1/p}\,dt \leqslant 1$, and $F(x) = \int_0^x f(t)\,dt$, then for a certain number $K_p$ depending only on $p$,*

$$\int_0^1 F(x)^p \frac{dx}{x} \leqslant K_p.$$

Inequalities of this nature may be found in Zygmund's book [4], e.g., Chapter V (8.22), Chapter XII, Examples 5, 6.

**2. An equivalent variant of Theorem 1.1.** In this section we show Theorem 1.1 equivalent to the following lemma, in its turn equivalent to an assertion about the increasing distribution function of $t-F(t)^p$.

LEMMA 2.1. *Let $f$ belong to $L^q(I, dx)$, $f\geqslant 0$. There exists a number $K_q$ depending only on $q$ such that whenever*

$$\|f\|_q \equiv \Big\{\int_0^1 f(t)^q dt\Big\}^{1/q} \leqslant 1,$$

*then*

$$x\int_0^1 \exp\{-x(t-F(t)^p)\}\,dt \leqslant K_q$$

*for all $x>0$.*

Suppose this holds. An application of Fatou's lemma gives (1.1) when we know (1.2) holds for any one of our functions $f\geqslant 0$ which is also zero from some point on. Thus if $f(t) = 0$ for $t>x$,

$$I \equiv \int_0^\infty \exp\big(F(t)^p - t\big)\,dt = \int_0^x \exp\big(F(t)^p-t\big)\,dt + \exp\big(F(x)^p - x\big).$$

By Hölder's inequality $F(x)^p\leqslant x$ so the second term is bounded by 1. This and the change of variable $t\to xt$ gives the estimate

$$I \leqslant x\int_0^1 \exp\Big\{-x\Big(t-\frac{1}{x}F(xt)^p\Big)\Big\}\,dt+1 = J+1.$$

Now $F_x(t) \equiv x^{-1/p} F(xt)$ is the indefinite integral of $f_x(t) = x^{1/q}f(xt)$, and $f_x\in L^q(I, dx)$ with $\|f_x\|_q\leqslant 1$. By (2.1),

$$J \leqslant \sup_{y>0} y \int_0^1 \exp\{-y(t-F_x(t)^p)\}\,dt \leqslant K_q.$$

Next suppose Theorem 1.1 holds. Let $f$ satisfy the conditions of Lemma 2.1. Extend $f$ to be 0 for $t>1$, and apply Theorem 1.1 to $f_{1/x}$:

$$C_q \geqslant \int_0^\infty \exp\big(F_{1/x}(t)^p - t\big)\,dt = x\int_0^\infty \exp\{-x(t-F(t)^p)\}\,dt.$$

Lemma 2.1 is a consequence of the next two lemmas.

LEMMA 2.2. *Let $u$ be a Lebesgue measurable function defined on $(0,1)$ with $u(s)>0$ a.e. Let $m(u<y)$ denote the measure of the set of points $s$ for which $u(s)<y$ and set*

$$M = \sup_{y>0} m(u<y)/y.$$

*Then*

$$e^{-1}M \leqslant \sup_{x>0} x\int_0^1 e^{-xu(s)}\,ds \leqslant M.$$

LEMMA 2.3. *If $1<q<\infty$, $0\leqslant f\in L^q(I, dx)$, $\|f\|_q\leqslant 1$ and $F(t) = \int_0^t f(u)\,du$ there exists a number $K_q$ depending only on $q$ such that*

$$m\big(t-F(t)^p<y\big)\leqslant K_q y \quad \text{for all } y>0, \tag{2.4}$$

*where $p = q/(q-1)$ is the Hölder conjugate of $q$.*

Lemma 2.3 is proved in Sections 3, 4, 5.

Proof of Lemma 2.2.

$$I(x) \equiv x\int_0^1 e^{-xu(s)}\,ds = x^2\int_0^\infty m(u<y)\,e^{-xy}\,dy$$

$$= x^2\int_0^\infty \frac{m(u<y)}{y}\,ye^{-xy}\,dy \leqslant Mx^2\int_0^\infty ye^{-xy}\,dy = M.$$

Let $y>0$ be fixed. Then because $m(u<z)$ does not decreases as $z$ increases,

$$I(x) \geqslant x^2 m(u<y)\int_y^\infty e^{-xz}\,dz = \frac{m(u<y)}{y}\cdot yx^2\frac{e^{-xy}}{x}.$$

If we let $x=1/y$ we get

$$\sup_x I(x) \geqslant m(u<y)/ey,$$

and the lemma follows.

**3. A property of the indefinite integrals of functions in $L^q$.** If $f\geqslant 0$ $\|f\|_q = 1$ and $F(x) = \int_0^x f(t)\,dt$ we have by Hölder's inequality that $F(x)^p\leqslant x$, with strict inequality unless $f = c\chi_{(0,x)}$ (where $c = x^{-1/q}$). Then $F(t)^p<t$ for other values of $t$; Lemma 2.3 gives information on the distribution of values of the difference.

In what follows we use repeatedly the following result, which is a weakened paraphrase of Theorem 10 in the paper [1] of Hardy, Littlewood and Polya. For convenience a proof is included in Section 5.

LEMMA 3.1. (HLP): *Let $f$, $g$ be non-negative and monotone non-increasing functions defined on an interval $(0,a)$. Suppose that each is integrable and that*

$$\int_0^x f(t)\,dt \geqslant \int_0^x g(t)\,dt, \qquad 0<x<a.$$

*Let $C$ be any convex increasing function which is 0 at 0. Then*

$$\int_0^a C\big(f(t)\big)\,dt \geqslant \int_0^a C\big(g(t)\big)\,dt.$$

We begin the proof of Lemma 2.3 with two simplifications. First, it is enough to show that an estimate (2.4) holds for step functions. Second, the rearrangement $f^*$ of the steps of $f$ in decreasing order, beginning at 0 gives a new function $F^*(t) = \int_0^t f^*(u)\,du$ with $F^*(t)\geqslant F(t)$, all $t$. Hence $t-F^*(t)^p\leqslant t-F(t)^p$, so we may assume $f$ is non-increasing.



If $y\geqslant 1/q$ we have $m\big(t-F(t)^p<y\big)\leqslant qy$. Now let $y$ be fixed, $0<y<1/q$. Write $(t-y)_+^{1/p}$ for the function which is zero for $0\leqslant t\leqslant y$ and $(t-y)^{1/p}$ for $t>y$. Then the open set of points $t$ for which $t-F(t)^p<y$ may be written as

$$\{t\colon F(t)>(t-y)_+^{1/p}\} = (0,t_0)\cup\bigcup_{k=1}^n (s_k,t_k),$$

where the open intervals indicated are pairwise disjoint. They are finite in number because $F$ is piecewise linear. Let the notation be chosen so that $t_k\leqslant s_{k+1}$, $1\leqslant k\leqslant n$.

To illustrate the idea of the proof let us first find an estimate $t_0\leqslant K_0y$ for some $K_0$. It is clear that $t_0>y$. If $t_0>qy$ we proceed as follows. The line passing through the origin and the point $\big(qy,(qy-y)^{1/p}\big)$ on the graph of $(t-y)_+^{1/p}$ is a tangent line, with slope $m_y = (q-1)^{1/p}q^{-1}y^{-1/q}$, thus $G_y$, $\equiv m_yt(t\leqslant qy)$, else $(t-y)_+^{1/p}$, is concave, increasing and 0 at 0. Thus the derivative $g_y$ of $G_y$ is non-increasing. Since $F$ is concave, increasing and 0 at 0, $F(t)/t$ decreases. Thus if $t_0>qy$, $F(qy)>G_y(qy)$ so $F(t)\geqslant G_y(t)$, $0<t<t_0$. Now apply Lemma 3.1, which we shall refer to as HLP, to get

$$1\geqslant\int_0^{t_0} f(t)^q\,dt\geqslant\int_0^{t_0} g_y(t)^q\,dt = m_y^q qy + \int_{qy}^{t_0}\left(\frac{1}{p}(t-y)^{-1/p}\right)^q dt \tag{3.2}$$

$$= p^{-q+1}+p^{-q}\log(t_0-y)/(qy-y) = p^{-q+1}+p^{-q}\log(p-1)(K-1),$$

where $K = t_0/y\geqslant q$. Hence $K$ is dominated by the root $K_0$ of the equation $1 = p^{-q+1}+p^{-q}\log(p-1)(K-1)$. It is crucial in what follows to note that even if $t_0<qy$, we still have $s_1\geqslant qy$. This follows from the concavity of $F$; the part of its graph to the right of $t_0$ lies under the line through the origin and the point $\big(t_0,(t_0-y)^{1/p}\big)$.

**4. A procedure for estimating $m\big(t-F(t)^p<y\big)$.** Having now set the stage we turn to the main part of the argument. In case $t_0>qy$ we add to the intervals $(s_k,t_k)$ the interval $(qy,t_0)$. However, we keep the same notation.

Let a function $F_1$ be defined by requiring that

(4.1a) $F_1(0) = 0$,

(4.1b) $F_1$ be continuous,

(4.1c) $F_1(t) = (t-y)^{1/p}$ if $s_k\leqslant t\leqslant t_k$ for some $k$,

(4.1d) the graph of $F_1$ consist of line segments otherwise.

In particular, $F_1(t) = tF(s_1)/s_1 = t(s_1-y)^{1/p}s_1^{-1}$ for $0\leqslant t\leqslant 1$. The function $F_1$ is the smallest concave increasing function which is 0 at 0 and equal to $(t-y)^{1/p}$ in $\bigcup_{k=1}^n(s_k,t_k)$. Moreover, $F(t)\geqslant F_1(t)$, $0\leqslant t\leqslant t_n$.

Then by HLP,

$$\int_0^{t_n} f(t)^q dt \geqslant \int_0^{t_n} f_1(t)^q dt, \quad \text{where } f_1 = F_1'. \tag{4.2}$$

For the sake of definiteness we combine intervals with a common endpoint, but keep the same notation.

Now the idea of the proof is to replace $F_1$ by a similar, simpler function $F_2$ which satisfies the conditions (4.1), but with $(s_1, t_1)$ and $(s_2, t_2)$ replaced by $(s_2-(t_1-s_1), s_2)$. That is, we move the first curved part of the graph of $F_1$ to the right, staying in the graph of $(t-y)_+^{1/p}$, and above an interval of length $(t_1-s_1)$, until we have reached the next curved part of the graph of $F_1$. Then we combine these adjacent curved parts into one, and complete the graph of $F_2$ with the line segment from the origin to the point $(s_2-(t_1-s_1), \{s_2-(t_1-s_1)-y\}_+^{1/p})$. We continue in this way until the curved parts are combined into one (when we reach $F_n$). The graphs of $F_k$ and $F_{k+1}$ cross each other. In order to apply the idea of HLP we have to show that the passage from $F_k$ to $F_{k+1}$ only decreases the right-hand side of formula (4.2). It is enough to do this for $F_1$ and $F_2$. We need another lemma, which is proved in Section 5.

LEMMA 4.3. *Let $a, b, d$ be positive numbers with $a < b$, $a+d < b$. Let $a \leqslant x \leqslant b-d$. Consider the function $G_x(t)$, the integral over $(0, t)$ of*

$$g_x(u) \equiv \begin{cases} \dfrac{(x+d)^{1/p}-x^{1/p}}{d} & x \leqslant u < x+d, \\ \dfrac{1}{p}u^{-1/q}, & \text{otherwise, for } u > 0. \end{cases}$$

*($G_x(t) = t^{1/p}$ except in $(x, x+d)$, where $G_x$ is linear.) Then on the interval $(a, b-d)$ the function*

$$Q(x) \equiv \int_a^b g_x(t)^q dt$$

*is strictly increasing.*

This is applied as follows. The right-hand side of (4.2) can be written

$$\int_0^{s_1} f_1(t)^q dt + \int_{s_1}^{t_1} f_1(t)^q dt + \int_{t_1}^{s_2} f_1(t)^q dt + \int_{s_2}^{t_2} f_1(t)^q dt.$$

By (4.3) the sum of the second and third terms is only decreased if we move the linear part of the graph of $F_1$ above $(t_1, s_2)$ to the *left* to a new position above the interval $(s_1, s_2-(t_1-s_1))$, still as a chord of $(t-y)_+^{1/p}$. We then get new second and third terms

$$\int_{s_1}^{s_2-(t_1-s_1)} h_1(t)^q dt + \int_{s_2-(t_1-s_1)}^{s_2} h_1(t)^q dt,$$



where in the new second term $h_1$ is constant (the slope of the chord joining the point $(s_1, (s_1-y)^{1/p})$ to the point $(s_2-(t_1-s_1), \{s_2-(t_1-s_1)-y\}_+^{1/p}v)$, and in the new third term $h_1(t) = p^{-1}(t-y)^{-1/q}$. This gives a function $H_1$ which is concave, increasing and 0 at 0. Hence $H_1(t) \geqslant F_2(t)$ on $0 \leqslant t \leqslant s_2-(t_1-s_1)$, since on this interval $F_2(t)$ has the form $ct$, and $H_1(s_2-(t_1-s_1)) = F_2(s_2-(t_1-s_1))$. Again apply HLP and the argument just preceding to conclude that

$$\int_0^{t_n} f_1(t)^q dt \geqslant \int_0^{t_n} f_2(t)^q dt.$$

As mentioned before we finally get to $F_n$ in which all the curved parts have been combined into one curved part of the graph of $(t-y)_+^{1/p}$, the part lying over the interval $(t_n-\mu, t_n)$, where $\mu = \sum_{k=1}^{n}(t_k-s_k)$. Thus

$$F_n(t) = \begin{cases} t(t_n-\mu-y)^{1/p}(t_n-\mu)^{-1}, & 0 \leqslant t \leqslant t_n-\mu, \\ (t-y)^{1/p}, & t_n-\mu < t \leqslant t_n. \end{cases}$$

From (4.2) and the chain of inequalities just ended we get

$$1 \geqslant \int_0^{t_n} f_n(t)^q dt = \left(\frac{t_n-\mu-y}{t_n-\mu}\right)^{q-1} + p^{-q}\log\frac{t_n-y}{t_n-\mu-y},$$

(compare with formula (3.2)) or

$$1 > (1-r)^{q-1} + p^{-q}\log(1+Kr);$$

we have written $r = y(t_n-\mu)^{-1}$, $\mu = Ky$, and replaced $Kr(1-r)^{-1}$ by $Kr$ to get the strict inequality. Since $t_n-\mu \geqslant s_1 \geqslant qy$, $r \leqslant 1/q$, so we consider

$$u(t) \equiv (1-tq^{-1})^{q-1} + p^{-q}\log(1+Ktq^{-1}), \quad 0 \leqslant t \leqslant 1.$$

Now $u(0) = 1$, and $u(1) \geqslant 1$ if $K$ is sufficiently large. It remains to show that by (possibly) increasing $K$, $u(t) \geqslant 1$ for $0 \leqslant t \leqslant 1$.

Now

$$u'(t) = (q-1)(1-tq^{-1})^{q-2}(-q^{-1}) + p^{-q}K(q+Kt)^{-1},$$

$$u''(t) = (q-1)(q-2)(1-tq^{-1})^{q-3}q^{-2} - p^{-q}K^2(q+Kt)^{-2}.$$

If $q \leqslant 2$, $u'' < 0$ so that if $u(1) \geqslant 1$ we have $u(t) \geqslant 1$, $0 \leqslant t \leqslant 1$. Suppose $q > 2$, and $u'(t) = 0$ for some $t$, $0 < t < 1$. Then for such $t$,

$$\begin{aligned} u''(t) &= (q-2)q^{-1}(1-tq^{-1})^{-1}p^{-q}K(q+Kt)^{-1} - p^{-q}K^2(q+Kt)^{-2} \\ &= P(t)\{(q-2)(q+Kt) - q(1-tq^{-1})K\}, \end{aligned}$$

where $P(t)$ is a positive function of $t$. The expression in curly brackets is negative for $0<t<1$ if $K>q(q-2)$. Hence if $K$ is so large that $u(1)\geqslant 1$, and also $K>q(q-2)$, we have $u(t)\geqslant 1$ for $0\leqslant t\leqslant 1$, as desired.

**5. Proofs of two lemmas.**

Proof of Lemma 3.1. Our application is to $C(x)=x^q$. Write

$$C(x)=\int_0^x c(y)\,dy$$

where $0\leqslant c(y)$, and $c$ is monotone non-decreasing. Integration by parts gives

$$C(x)=\int_0^x (x-y)\,dc(y)+xc_0,$$
$$=\int_0^\infty (x-y)_+\,dc(y)+xc_0,$$

where $(x-y)_+=\max(0,x-y)$ and $c_0=\lim\limits_{y\to 0}c(y)$. Thus

$$\int_0^a C\big(f(t)\big)\,dt=\int_0^\infty\int_0^a \big(f(t)-y\big)_+\,dt\,dc(y)+c_0\int_0^a f(t)\,dt,$$

so the lemma follows once we show that

$$\int_0^a \big(f(t)-y\big)_+\,dt\geqslant\int_0^a \big(g(t)-y\big)_+\,dt$$

for all $y>0$. Let $m(f>y)$ denote the measure of the set of points $t$ in which $f(t)>y$, with a similar expression for $g$. In each case, the set measured is an interval with endpoints 0 and $m(f>y)$ (or $m(g>y)$).

We have to show that

$$D\equiv\int_0^{m(f>y)} f(t)\,dt-ym(f>y)-\int_0^{m(g>y)} g(t)\,dt+ym(g>y)$$

is non-negative.

If $A\equiv m(f>y)\geqslant m(g>y)\equiv B$ write

$$D=\int_B^A f(t)\,dt-y(A-B)+\Big(\int_0^B f(t)\,dt-\int_0^B g(t)\,dt\Big),$$



which is non-negative (i) since $f(t)>y$ for $t<A$ and (ii) by the hypothesis. If $A<B$ write

$$D=y(B-A)-\int_A^B f(t)\,dt+\Big(\int_0^B f(t)\,dt-\int_0^B g(t)\,dt\Big),$$

which is non-negative since $t>A$ implies $f(t)\leqslant y$.

Proof of Lemma 4.3. Write $C(y)$ for $y^q$, set $G(t)=t^{1/p}$, $g(t)=G'(t)=\dfrac{1}{p}t^{-1/q}$. Then

$$Q(x)=\int_a^x C\big(g(t)\big)\,dt+dC\Big(\frac{G(x+d)-G(x)}{d}\Big)+\int_{x+d}^b C\big(g(t)\big)\,dt,$$

$$Q'(x)=C\big(g(x)\big)+C'\Big(\frac{G(x+d)-G(x)}{d}\Big)\big(g(x+d)-g(x)\big)-C\big(g(x+d)\big)$$
$$=C\big(g(x)\big)-C\big(g(x+d)\big)-C'\Big(\frac{G(x+d)-G(x)}{d}\Big)\big(g(x)-g(x+d)\big)$$
$$=p^{-q}\Big(\frac{1}{x}-\frac{1}{x+d}\Big)-q\Big(\frac{(x+d)^{1/p}-x^{1/p}}{d}\Big)^{q-1}\Big(\frac{1}{p}x^{-1/q}-\frac{1}{p}(x+d)^{-1/q}\Big).$$

Consider the expressions

$$A=p^{-q}\Big(\frac{1}{x}-\frac{1}{x+d}\Big)\Big[\frac{q}{p}\Big(\frac{1}{x^{1/q}}-\frac{1}{(x+d)^{1/q}}\Big)\Big]^{-1},$$

and

$$B=\Big(\frac{(x+d)^{1/p}-x^{1/p}}{d}\Big)^{q-1}.$$

To show $Q'(x)>0$ we show that for each $x>0$, the functions of $d$ defined by $A$ and $B$, which have the same limit as $d$ diminishes to 0, have ratio $R=B/A$ less than 1 for positive values of $d$. If we put $d=xt$, $0<t<\infty$,

$$R=p^{q-1}q\Big(\frac{(1+t)^{1/p}-1}{t}\Big)^{q-1}\frac{\big((1+t)^{1/q}-1\big)(1+t)^{1/p}}{t}.$$

Put $s^p=1+t$, $s>1$, to get

$$R=p^{q-1}q\Big(\frac{s-1}{s^p-1}\Big)^q\Big(\frac{s^p-1+1-s}{s^p-1}\Big)=p^{q-1}q(rp^{-1})^{q-1}(1-rp^{-1}),$$

where $r/p = (s-1)(s^p-1)^{-1}$. As $s$ increases from 1 to $+\infty$, $r$ decreases from 1 to 0 ($p/r$ is the slope of a certain chord of a convex function). Since the function $qr^{q-1}(1-rp^{-1})$ attains its maximum value of 1 at $r=1$, the lemma is proved.

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*Received November 11, 1971* (435)

STUDIA MATHEMATICA, XLIV. (1972)

# Sur les coefficients des séries de Fourier dont les sommes partielles sont positives sur un ensemble

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**Sommaire.** Si $E$ est un intervalle, les ocoefficients $c_n$ sont bornés. Si $E$ a certaines proprietés arithmetiques, $c_n = O(e^{\varepsilon n})$ pour tout $\varepsilon > 0$. On discute en particulier le cas où $E$ est un dénombrable fermé.

Ce travail a son origine dans le problème, toujours ouvert, de l'existence d'une série trigonométrique dont les sommes partielles tendent vers $+\infty$ sur un ensemble de mesure $> 0$. On sait que la réponse est négative pour les séries de Haar et les séries de Walsh ([7]), et que la réponse est positive si on remplace les sommes partielles par les sommes de Poisson ([6], [1]) même si on impose aux coefficients d'être "presque" de carré sommable ([5]).

Il est facile de voir (Proposition 1) que le problème mentionné est équivalent au suivant: existe-t-il une série trigonométrique dont les sommes partielles sont $\geqslant 0$ sur un ensemble de mesure $> 0$, et dont les coefficients ne soient pas bornés? Cela nous amène á étudier l'ordre de grandeur des coefficients $r_n$ d'une série trigonométrique

$$\sum_0^\infty r_n \cos(nt+\varphi_n) \tag{1}$$

dont les sommes partielles $S_n(t)$ sont $\geqslant 0$ sur un ensemble fermé $E$. Pour certains ensembles $E$, assez minces, les coefficients peuvent être arbitrairement grands (Proposition 5). Mais dès que $E$ admet un "point de densité arithmétique" (par exemple si $E$ est de mesure $> 0$) on a $r_n = O(e^{\varepsilon n})$ pour tout $\varepsilon > 0$ (Proposition 4). Si $E$ est dénombrable, on n'a pas nécessairement $r_n = O(1)$; mais on peut choisir $E$ dénombrable de façon que $r_n = O(\omega_n)$, où $\omega_n$ est une suite tendant vers l'infini arbitrairement donnée (Proposition 7). Si $E = [0, 2\pi]$, on sait par un théorème de Helson que $r_n = o(1)$ [2]. Il n'en est plus ainsi si $E$ est un intervalle contenu dans $]0, 2\pi[$ (alors $r_n = 1$ et $\varphi_n$ constant pour $n \geqslant 1$, et $r_0$ assez grand convient); dans ce cas cependant, on a $r_n = O(1)$ (Proposition 2).