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Estimates for singular integral operators in terms of maximal functions

by

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Dedicated to my teacher, Professor Antoni Zygmund.

Abstract. Various maximal functions $M(f,x)$ are associated with certain distributions in $\bigcup_{k>0} L^p_{-k}$, $1<p<\infty$. They are slight modifications and generalizations of the following function associated with a function $f(x)$ of a single variable

$$\sup_{t>0}\frac{1}{t}\left|\int_0^t f(x+s)\,ds\right|.$$

It is shown that, if K is a singular integral operator, it is possible to estimate $M(Kf,x)$ in terms of $M(f,x)$ provided the latter is finite and integrable to some power p, $p<\infty$ outside a set of finite measure.

1. Introduction. The purpose of this paper is to obtain estimates for the distributions of values of singular integrals in such a way that cancellation due to variability of sign of the functions involved is taken into account. We accomplish this by using various maximal functions associated with a given function. These are defined in terms of primitives or potentials of the latter. For example

$$M(f,x) = \sup_{t>0}\left|\frac{1}{t}\int_0^t f(x+s)\,ds\right|,$$

where f is a function on the real line, or more generally, if $\frac{d^k}{dx^k}F_k = f$ and if $R_k(x,t)$ denotes the remainder of the k term Taylor expansion of F_k at x,

$$M(f,x) = \sup_t |t^{-k}R_k(x,t)|$$

illustrate the type of maximal functions we have in mind. Actually we will discuss some variants and generalizations of these and define them also for certain kinds of distributions. Their interest lies in the fact that if

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K is a singular integral operator it is possible to estimate $M(Kf, x)$ in terms $M(f, x)$, even if this function is inordinately large or even infinite on a set of finite measure.

2. Notation and statement of results. By $x, y, \ldots, x = (x_1, \ldots, x_n)$ we denote points in n-dimensional Euclidean space R^n. If $f(x)$ is a function on R^n, $\|f\|_p$ will denote its norm in L^p. If r is an exponent, $1 \leqslant r < \infty$, r' will stand for its conjugate, i.e., $r' = r/(r-1)$. The symbols $x+y$, $|x|$ and λx, where λ is a real number, have their usual meaning. If $a = (a_1, a_2, \ldots, a_n)$ where the a_j are non-negative integers, then

$$|a| = a_1 + a_2 + \ldots + a_n, \qquad x^a = x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$$

$$a! = a_1! a_2! \ldots a_n!, \qquad \left(\frac{\partial}{\partial x}\right)^a f = f_a = \left(\frac{\partial}{\partial x_1}\right)^{a_1} \left(\frac{\partial}{\partial x_2}\right)^{a_2} \ldots \left(\frac{\partial}{\partial x_n}\right)^{a_n} f.$$

The letter c will stand for a constant, not necessarily the same in each occurrence. Finally p, q, $1 < q \leqslant p < \infty$, will denote two exponents which will remain fixed throughout this paper.

Let $F(x)$ be a complex-valued function on R^n which belongs to L^q on bounded sets. Given a positive integer m we will associate with F a function $N(F, x)$,

$$N(F, x) = \sup_{\varrho} \varrho^{-m} \Big[\varrho^{-n} \int_{|x-y|<\varrho} |F(y) - P(x, y)|^q dy\Big]^{1/q} \tag{1}$$

if there exists a polynomial in y, $P(x, y)$ of degree at most $m-1$, such that the expression on the right above is finite. If such a polynomial exists, it is unique, and thus $N(F, x)$ is well defined. If no such $P(x, y)$ exists we set $N(F, x) = \infty$.

Consider now the class $\mathcal{N}_m^{p,q}$, $1 < q \leqslant p < \infty$ of functions F with the following properties: F belongs to L^q on bounded sets, $N(F, x)$ is finite and belongs to L^p in the complement of a set of finite measure. If m is even we shall say a distribution f belongs to $\mathcal{M}_m^{p,q}$ if there exists a function F in $\mathcal{N}_m^{p,q}$ such that $\Delta^{m/2} F = f$, where Δ stands for the Laplacian. If m is odd we shall say that f is in $\mathcal{M}_m^{p,q}$ if there exists a function F which is locally integrable and such that $\Delta^{(m+1)/2} F = f$ and $\partial F/\partial x_i \in \mathcal{N}_m^{p,q}$. If m is even and $f \in \mathcal{M}_m^{p,q}$, we define the maximal function $M(f, x)$ by $M(f, x) = N(F, x)$ where $F \in \mathcal{N}_m^{p,q}$, and $\Delta^{m/2} F = f$. As we shall see, $M(f, x)$ is well defined, i.e., independent of the choice of F. Similarly, if m is odd and $f \in \mathcal{M}_m^{p,q}$ we define

$$M(f, x) = \sum_{i=1}^{n} N\left(\frac{\partial F}{\partial x_i}, x\right),$$

where $\partial F/\partial x_i \in \mathcal{N}_m^{p,q}$ and $\Delta^{(m+1)/2} F = f$. The space $\mathcal{M}_m^{p,q}$ is contained in L^p_{-m-n} (see [2] for the definition and properties of these spaces). Thus, operators in L^p_{-m-n} are well defined on $\mathcal{M}_m^{p,q}$, and we have the following result.

THEOREM 1. *Let K be an operator C_0^∞ defined by $(Kf)\hat{} = k\hat{}\, f\hat{}$, where $f\hat{}$ and $(Kf)\hat{}$ denote the Fourier transforms of f and Kf respectively, and $k\hat{}$ is a bounded function with $|k\hat{}(x)| \leqslant 1$. Suppose that the inverse Fourier transform of $k\hat{}$ coincides with a function $k(x)$ in $x \neq 0$, and that k is m times differentiable with $|k_a(x)| \leqslant |x|^{-n-m}$ for all a, $|a| = m$. Then K maps C_0^∞ into L^p_{-n-m} and has a unique continuous extension $\bar{K}$ to L^p_{-n-m}. Let f be a distribution in $\mathcal{M}_m^{p,q}$ with m even, then $g = \bar{K}f$ belongs to $\mathcal{M}_m^{p,q}$. Furthermore, if $\mu(f, t)$ and $\mu(g, t)$ denote the measures of the sets of points where $M(f, x) > t$ and $M(g, x) > t$ respectively, then*

$$\mu(g, t) \leqslant ct^{-p} \int_0^t s^{p-1} \mu(f, s)\, ds,$$

where c depends on p, q and m.

There is a second analogue of the Hardy–Littlewood maximal function associated with a distribution in $\mathcal{M}_m^{p,q}$. Its properties are described in the following

THEOREM 2. *Let f be a distribution in $\mathcal{M}_m^{p,q}$ with m even. Then there exists a function $M^*(f, x) \leqslant \infty$ with the following properties:*

(i) $M(f, x) \leqslant M^*(f, x)$,

(ii) *if $\mu(f, t)$ and $\mu^*(f, t)$ are the measures of the sets of points where $M(f, x) > t$ and $M^*(f, x) > t$ respectively then*

$$\mu^*(f, t) \leqslant ct^{-p} \int_0^t s^{p-1} \mu(f, s)\, ds$$

where c depends only on p and m,

(iii) *if the operator K in the preceding theorem is of the form*

$$Kf = \int k(x-y) f(y)\, dy$$

where $k(x)$ is integrable, m times differentiable, $|k_a(x)| \leqslant |x|^{-n-m}$ for all a, $|a| = m$, and

$$\int \sup_{|y| \geqslant |x|} |k(y)|\, dx \leqslant 1.$$

Then $M(\bar{K}f, x) \leqslant M^*(f, x)$.

As is to be expected, there are inclusion relations among the spaces $\mathcal{M}_m^{p,q}$. Some of these are given by the following result which we state without proof.

THEOREM 3. *If $r \geqslant p$, $s \leqslant q$, then $\mathscr{M}_m^{r,s} \supset \mathscr{M}_m^{p,q}$. If $f \in \mathscr{M}_m^{p,q}$ and M_1 and M_2 are the maximal functions associated with f as an element of $\mathscr{M}_m^{p,q}$ and $\mathscr{M}_m^{r,s}$ respectively, then $M_1 \geqslant M_2$, and $M_1 = M_2$ if $s = q$. If $s \leqslant p$, and $s \leqslant \left(\frac{1}{q} - \frac{1}{n}\right)^{-1}$, then $\mathscr{M}_{m+1}^{p,s} \supset \mathscr{M}_m^{p,q}$. If $f \in \mathscr{M}_m^{p,q}$ and M_1 and M_2 are the maximal functions associated with f as an element of $\mathscr{M}_m^{p,q}$ and $\mathscr{M}_{m+1}^{p,s}$ and $\mu_1(t)$ and $\mu_2(t)$ are the measures of the sets where $M_1 > t$ and $M_2 > t$ respectively, then*

$$\mu_2(t) \leqslant ct^{-p} \int_0^t s^{p-1} \mu_1(s)\, ds$$

where c depends only on p and m.

The first part of the preceding statement follows almost immediately from the definitions. The second requires an argument which is very similar to the proof of Theorem 1.

The foregoing results are also valid under more general conditions. For example, the restriction $q \leqslant p$ can be relaxed considerably. However, this greatly complicates some of the proofs and the interest of these improvements does not seem to justify the additional effort needed to obtain them. Also, our results hold for m odd if the conditions on $k(x)$ are slightly strengthened. In the one-dimensional case, however, no additonal hypotheses on $k(x)$ are necessary and our proofs apply without change.

As an illustration of the consequences of our results let us show that if the function $M(f, x)$ in Theorem 1 belongs to L^r, $r > 0$, the same holds for $M(g, x)$ and

$$\int M(g, x)^r dx \leqslant c \int M(f, x)^r dx.$$

In fact, this follows at once by multiplying by t^{r-1} the inequality relating $\mu(g, t)$ with $\mu(f, t)$ and integrating after having selected p larger than r (according to Theorem 3, we can select p arbitrarily large).

3. In this section we shall discuss the spaces $\mathscr{N}_m^{p,q}$ and the function $N(F, x)$.

LEMMA 1. *There exists an infinitely differentiable function Φ with support in $|x| \leqslant 1$ such that*

$$P(x) = \int P(y) \lambda^n \Phi[\lambda(x-y)]\, dy$$

for all polynomials $P(x)$ of degree les than or equal to m.

This is well known. See [1], Lemma 26, for example.

LEMMA 2. *If F and P are as in (1), then $F(x) = P(x, x)$ for almost all x for which the right hand side of (1) is finite.*

Proof. Evidently

$$F(x) - P(x, x) = \lim_{\varrho\to 0} \frac{1}{\omega \varrho^n} \int_{|y-x| \leqslant \varrho} [F(y) - P(x, y)]\, dy$$

almost everywhere, ω being the volume of the unit sphere, and, from Holder's inequality it follows that

$$|F(x) - P(x, x)| \leqslant \lim_{\varrho\to 0} \left[\frac{1}{\omega \varrho^{n+mq}} \int |F(y) - P(x, y)|^q dy\right]^{\frac{1}{q}} \varrho^m \leqslant \lim_{\varrho\to 0} N(F, x)\, \varrho^m = 0.$$

LEMMA 3. *Let $N(F, x_1) \leqslant t$, $N(F, x_2) \leqslant t$, and let $P(x_1, y)$ and $P(x_2, y)$ be the corresponding polynomials. Then*

$$\left|\left(\frac{\partial}{\partial y}\right)^{\alpha} P(x_1, y) - \left(\frac{\partial}{\partial y}\right)^{\alpha} P(x_2, y)\right| \leqslant ct\,[|y - x_1| + |y - x_2|]^{m-|\alpha|}$$

where c depends only on α.

Proof. Let $\lambda^{-1} = |y - x_1| + |y - x_2|$ and let Φ be the function of Lemma 1. Then

$$\left(\frac{\partial}{\partial y}\right)^{\alpha} P(x_1, y) - \left(\frac{\partial}{\partial y}\right)^{\alpha} P(x_2, y) = \int \lambda^{n+|\alpha|} \Phi_\alpha[\lambda(y-z)][P(x_1, z) - P(x_2, z)]\, dz,$$

which follows from Lemma 1 by differentiation under the integral sign. Thus if $\varrho = 2|y - x_1| + 2|y - x_2| = 2\lambda^{-1}$ we have

$$\begin{aligned}
&\left|\left(\frac{\partial}{\partial y}\right)^{\alpha} P(x_1, y) - \left(\frac{\partial}{\partial y}\right)^{\alpha} P(x_2, y)\right| \\
&\qquad \leqslant \int_{|z-x_1| \leqslant \varrho} |F(z) - P(x_1, z)|\, \lambda^{n+|\alpha|} |\Phi_\alpha[\lambda(y-z)]|\, dz + \\
&\qquad + \int_{|z-x_2| \leqslant \varrho} |F(z) - P(x_2, z)|\, \lambda^{n+|\alpha|} |\Phi_\alpha[\lambda(y-z)]|\, dz.
\end{aligned}$$

Thus, applying Hölder's inequality to these integrals we find that their sum is dominated by $ct\lambda^{n+|\alpha|} \varrho^{n+m}$, which is the desired result.

LEMMA 4. *There is at most one polynomial $P(x, y)$ of degree $m-1$ which makes the right hand side of (1) finite.*

Proof. This follows from the preceding lemma by setting $x_1 = x_2 = x$, $y = x$ and letting α range over all multindices with $0 \leqslant |\alpha| \leqslant m-1$.

LEMMA 5. *Let $a_\alpha(x) = \left(\frac{\partial}{\partial y}\right)^{\alpha} P(x, y)]_{y=x}$. Then, if $N(F, x) \leqslant t$ on a set E,*

$$\left|a_\alpha(x) - \sum \frac{1}{\beta!} (x - \bar{x})^{\beta} a_{\alpha+\beta}(\bar{x})\right| \leqslant ct\, |x - \bar{x}|^{m-|\alpha|} \tag{2}$$

for x and $\bar{x}$ in E. Furthermore, if $|\alpha| = m-1$, $a_\alpha(x)$ satisfies a Lipschitz condition on E with constant ct. If $|\alpha| < m-1$, $a_\alpha(x)$ satisfies a uniform Lipschitz condition on every bounded subset of E. The constant c depends only on m.

Proof. Inequality (2) is obtained from lemma 3 by setting $x_1 = \bar{x}$, $x_2 = y = x$. If $|\alpha| = m-1$, then $a_{\alpha+\beta}(x) = 0$ for $|\beta| > 0$, and the sum in (2) reduces to $a_\alpha(x)$, and we see that $a_\alpha(x)$ satisfies a Lipschitz condition with constant ct. Arguing by induction on j, and using (2) it follows that $a_\beta(x)$, $|\beta| = m-j$ satisfies a uniform Lipschitz condition on every bounded subset of E.

LEMMA 6. *The function $N(F, x)$ is lower semicontinuous.*

Proof. Let x_0 be a limit point of the set of points where $N(F, x) \leqslant t$. Suppose that $N(F, x_j) \leqslant t$ and $x_j \to x_0$. Then $a_\alpha(x_j)$ converges as $j \to \infty$, as is shown by Lemma 5. Let $P(x_0, y) = \lim P(x_j, y)$. Then

$$\int\limits_{|y-x_0|\leqslant\varrho} |F(y) - P(x_0, y)|^p = \lim \int\limits_{|y-x_j|\leqslant\varrho} |F(y) - P(x_j, y)|^p\, dy \leqslant t^p \varrho^{n+mp}.$$

and $N(F, x_0) \leqslant t$. Thus the set where $N(F, x) \leqslant t$ is closed for all t, $t < \infty$, i.e., $N(F, x)$ is lower semicontinuous.

THEOREM 4. *Suppose that $N(F, x)$ is locally integrable in an open set $\mathcal{O}$. Then the derivatives of order m of F coincide with locally integrable functions in $\mathcal{O}$ and*

$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha F\right| \leqslant cN(F, x)$$

for all α, $|\alpha| = m$, where c depends only on m.

Proof. Let $\eta(x) \in C_0^\infty$ and have support in $\mathcal{O}$. Let Φ be the function of Lemma 1. Then

$$\int \lambda^{n+|\alpha|} \Phi_\alpha[\lambda(y-z)]\, \eta(z)\, dz$$

converges uniformly to $\eta_\alpha(y)$ and has support in $\mathcal{O}$ for λ sufficiently large. Consequently

$$\int F(y) \left(\frac{\partial}{\partial y}\right)^\alpha \eta(y)\, dy = \lim_{\lambda\to\infty} \int F(y) \int \lambda^{n+|\alpha|} \Phi_\alpha[\lambda(y-z)]\, \eta(z)\, dz\, dy.$$

Interchanging the order of integration in the last integral, it becomes

$$\lambda^{n+|\alpha|} \int \eta(z) \int F(y) \Phi_\alpha[\lambda(y-z)]\, dy\, dz.$$

Now, if $|\alpha| = m$, Φ_α is orthogonal to all polynomials of degree $m-1$. Therefore, we may subtract $P(z, y)$ from $F(y)$ in the inner integral. But Hölder's inequality gives

$$\left|\int [F(y) - P(z, y)] \Phi_\alpha[\lambda(y-z)]\, dz\right|$$
$$\leqslant c\Big[\int\limits_{|y-z|\leqslant 1/\lambda} |F(y) - P(z, y)|^p\, dy\Big]^{1/p} \lambda^{-n/p'} \leqslant cN(F, z)\lambda^{-n-m}.$$

Thus, substituting above, we obtain

$$\left|\int F(y) \left(\frac{\partial}{\partial y}\right)^\alpha \eta(y)\, dy\right| \leqslant c \int N(F, z) |\eta(z)|\, dz,$$

whence the assertion of the lemma follows.

THEOREM 5. *Let F be a function in $\mathcal{N}_m^{p,q}$ and let $t > 0$. Then F can be decomposed as the sum of two functions F_1 and F_2 with the following properties*

(i) *F_1 vanishes outside the open set $\mathcal{O}$ where $N(F, x) > t$, and the measure $|\mathcal{O}|$ of $\mathcal{O}$ is finite.*

(ii) *$N(F_1, x) \leqslant ct$ in the complement $\mathcal{C}$ of $\mathcal{O}$, where the constant c depends only on m.*

(iii) *$\int_{\mathcal{C}} N(F_1, x)^q\, dx \leqslant c^q |\mathcal{O}| t^q$, where c depends only on m.*

(iv) *$\int_{\mathcal{C}} dx \int |F_1(y)|^r |x-y|^{-(n+mr)}\, dy \leqslant c^r |\mathcal{O}| t^r$ $1 \leqslant r \leqslant q$, where c depends only on m.*

(v) *If $\delta(y)$ denotes the distance between y and $\mathcal{C}$ then $\int |F_1(y)|^r \delta(y)^{-mr}\, dy \leqslant c^r t^r |\mathcal{O}|$ for $1 \leqslant r \leqslant q$, with c depending only on m.*

(vi) *$N(F_2, x) \leqslant ct$ and $\int N(F_2, x)^p\, dx \leqslant c^p [|\mathcal{O}| t^p + \int_{\mathcal{C}} N(F, x)^p\, dx]$ where c depends only on m.*

Proof. Since $N(F, x)$ is finite and integrable to the power p in the complement of a set of finite measure, the set $\mathcal{O}$ where $N(F, x) > t > 0$ has finite measure. Furthermore, $N(F, x)$ is lower semicontinuous and consequently $\mathcal{O}$ is open. Now define F_2 to be the Whitney extension of the restriction of F to the complement C of $\mathcal{O}$. Specifically, let $\Sigma\eta_j = 1$ be a partition of unity in $\mathcal{O}$ where the functions $\eta_j \geqslant 0$ have the following properties: $\eta_j \in C_0^\infty$; for each x, $x \in \mathcal{O}$, there exist at most 2^n functions η_j whose supports contain the point x; if d_j and δ_j denote the diameter of the support of η_j and its distance to $\mathcal{C}$ respectively then

$$c^{-1} \leqslant \frac{d_j}{\delta_j} \leqslant c, \quad \text{for all } j$$

and

$$\left(\frac{\partial}{\partial x}\right)^{\alpha}\eta_j(x)\leqslant c\delta_j^{-|\alpha|},\quad \text{for all } j,\ |\alpha|\leqslant m \tag{3}$$

where c depends on m but not on $\mathcal{O}$. That such partitions of unity exist is well known. Let now x_j be a point of $\mathcal{C}$ at distance δ_j from the support of η_j and let $F_2(x)=F(x)$ for x in $\mathcal{C}$ and

$$F_2(x)=\sum_j \eta_j(x)P(x_j,x)$$

for x in $\mathcal{O}$. Since for each x there are at most 2^n non vanishing terms in the preceding series, and the same is true for any derivative of these terms, it is plain that $F_2(x)$ is infinitely differentiable in $\mathcal{O}$. Furthermore,

$$\left(\frac{\partial}{\partial x}\right)^{\alpha}F_2(x)=\sum_j\sum_{\beta+\gamma=\alpha}\frac{\alpha!}{\beta!\,\gamma!}\left(\frac{\partial}{\partial x}\right)^{\beta}\eta_j(x)\left(\frac{\partial}{\partial x}\right)^{\gamma}P(x_j,x).$$

If $\bar{x}\in\mathcal{C}$, $x\in\mathcal{O}$ and x' is the point in $\mathcal{C}$ closest to x, then

$$\begin{aligned}\left(\frac{\partial}{\partial x}\right)^{\alpha}F_2(x)-\left(\frac{\partial}{\partial x}\right)^{\alpha}P(\bar{x},x)\\ =\sum_j\sum_{\beta+\gamma=\alpha}\frac{\alpha!}{\beta!\,\gamma!}\left(\frac{\partial}{\partial x}\right)^{\beta}\eta_j(x)\left[\left(\frac{\partial}{\partial x}\right)^{\gamma}P(x_j,x)-\left(\frac{\partial}{\partial x}\right)^{\gamma}P(x',x)\right]+\\ +\left[\left(\frac{\partial}{\partial x}\right)^{\alpha}P(x',x)-\left(\frac{\partial}{\partial x}\right)^{\alpha}P(\bar{x},x)\right].\end{aligned}$$

Now, for x in the support of η_j we have

$$|x'-x|\leqslant|x_j-x|\leqslant\delta_j+d_j\leqslant(1+c)\,\delta_j,$$
$$|\bar{x}-x|\geqslant\delta_j.$$

Thus, from (3), Lemma 3, and the fact that for each x the sums above have at most 2^n non vanishing terms we find that, for $|\alpha|\leqslant m$

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha}F_2(x)-\left(\frac{\partial}{\partial x}\right)^{\alpha}P(\bar{x},x)\right|\leqslant ct\,|x-\bar{x}|^{m-|\alpha|} \tag{4}$$

with c depending on m only. Let now $b_\alpha(x)=\left(\frac{\partial}{\partial x}\right)^{\alpha}F_2(x)$ for x in $\mathcal{O}$, and $b_\alpha(x)=a_\alpha(x)$, where $a_\alpha(x)$ is as in Lemma 5, if $x\in\mathcal{C}$. Then (4) can be rewritten as

$$\left|b_\alpha(x)-\sum\frac{1}{\beta!}(x-\bar{x})^{\beta}b_{\alpha+\beta}(\bar{x})\right|\leqslant ct\,|x-\bar{x}|^{m-|\alpha|}. \tag{5}$$

Now since $N(F,x)\leqslant t$ in $\mathcal{C}$, Lemma 5 shows that this holds also for x in $\mathcal{C}$. This shows that $b_\alpha(x)$ is continuous for $|\alpha|\leqslant m-1$ and that $b_\alpha(x)$, $|\alpha|=m-1$ satisfies a Lipschitz condition with constant ct at all points of $\mathcal{C}$. Now for $x\in\mathcal{O}$ and $|\alpha|=m$, (5) gives $|b_\alpha(x)|\leqslant ct$, and thus $b_\alpha(x)$, $|\alpha|=m-1$ satisfies a Lipschitz condition with constant ct for all x. Consider now the function $b_0(x)=b_\alpha(x)$, $\alpha=(0,0,\ldots,0)$. Then $b_0(x)$ is infinitely differentiable in $\mathcal{O}$ and (5) shows that $b_\alpha(x)=\left(\frac{\partial}{\partial x}\right)^{\alpha}b_0(x)$ for $x\in\mathcal{C}$. The same holds true for $x\in\mathcal{O}$ on account of the definition of b_α, namely $b_\alpha(x)=\left(\frac{\partial}{\partial x}\right)^{\alpha}F_2(x)=\left(\frac{\partial}{\partial x}\right)^{\alpha}b_0$. Furthermore, Lemma 2 permits us to assert that $b_0(x)=a_0(x)=F_2(x)$ almost everywhere in $\mathcal{C}$. Thus we conclude that $F_2(x)$ coincides almost everywhere with a function $b_0(x)$ which has continuous derivatives up to order $m-1$, and whose derivatives of order $m-1$ satisfy a uniform Lipschitz condition with constant ct.

Next let us show that $N(F_2,x)=N(b_0,x)$ is dominated by ct. For $x\in\mathcal{C}$ this follows from (4) with $|\alpha|=0$. If $x\in\mathcal{O}$ we set

$$b_0(x+y)-\sum_{|\alpha|<m-1}\frac{1}{\alpha!}y^{\alpha}b_\alpha(x)+(m-1)\sum_{|\alpha|=m-1}\frac{y^{\alpha}}{\alpha!}\int_0^1 b_\alpha(x+sy)(1-s)^{m-2}ds.$$

But, for $|\alpha|=m-1$ we have

$$|b_\alpha(x+sy)-b_\alpha(x)|\leqslant cts\,|y|$$

whence substituting above we find that

$$\left|b_0(x+y)-\sum_{|\alpha|\leqslant m-1}\frac{1}{\alpha!}b_\alpha(x)y^{\alpha}\right|\leqslant ct\,|y|^{m}$$

whence the desired conclusion follows from the fact that $F_2(x)=b_0(x)$ almost everywhere.

Now let us turn to the function F_1. That (i) holds is clear. As for (ii) not only do we have $N(F_1,x)\leqslant ct$ for x in $\mathcal{C}$, but actually

$$\left[\int_{|y-x|\leqslant\varrho}|F_1(y)|^{q}dy\right]^{1/q}\leqslant ct\varrho^{m+n/q},\qquad x\in\mathcal{C}. \tag{6}$$

This becomes evident if we observe that according to (4) and (5) the polynomials which we have to subtract from F and F_2 in order to calculate $N(F, x)$ and $N(F_2, x)$, coincide. Thus, since $N(F_2, x) \leqslant ct$ and $N(F, x) \leqslant t$

$$\Big[\int_{|y-x|\leqslant\varrho} |F_1(y)|^q dy\Big]^{1/q}$$
$$\leqslant \Big[\int_{|y-x|\leqslant\varrho} |F(y)-P(x,y)|^q dy\Big]^{1/q} + \Big[\int_{|y-x|\leqslant\varrho} |F_2(y)-P(x,y)|^q dy\Big]^{1/q}$$
$$\leqslant (1+c)t\varrho^{m+n/q}.$$

Next let us prove (v). As is well known the open set $\mathcal{O}$ can be obtained as a union of cubes Q_j with disjoint interiors and with the property that if δ_j and d_j denote the distance between Q_j and $\mathscr{C}$ and the diameter of Q_j respectively, then

$$c^{-1} \leqslant \frac{d_j}{\delta_j} \leqslant c, \quad \text{for all } j.$$

Let χ_j denote the characteristic function of Q_j and $\delta(y)$ the distance between y and $\mathscr{C}$. Then if $1 \leqslant r \leqslant q$

$$\int F_1(y)^r \delta(y)^{-mr} dy = \sum \int F_1(y)^r \chi_j(y)\,\delta(y)^{-mr} dy.$$

But for y in Q_j we have $\delta(y) \geqslant \delta_j$ and, consequently, the sum on the right is dominated by

$$\sum \int F_1(y)^r \chi_j(y)\,\delta_j^{-mr} dy.$$

If x_j is a point in $\mathscr{C}$ at a distance δ_j from Q_j, setting $\varrho = d_j + \delta_j$, we have

$$\int F_1(y)^r \chi_j(y)\,dy \leqslant \int_{|y-x_j|<\varrho} F_1(y)^r dy$$

and according to (6)

$$\int_{|y-x_j|\leqslant\varrho} F_1(y)^r dy \leqslant c^r \Big[\int_{|y-x_j|\leqslant\varrho} F_1(y)^q dy\Big]^{r/q} \varrho^{n(1-r/q)} \leqslant c^r t^r \varrho^{n+mr}.$$

But $\varrho = d_j + \delta_j \leqslant (1+c)\,\delta_j$. Thus

$$\int F_1(y)^r \chi_j(y)\,\delta_j^{-mr} dy \leqslant c^r t^r \delta_j^n = c^r t^r |Q_j|$$

where $|Q_j|$ stands for the measure of Q_j. Substituting above we obtain (v).

Next let us prove (iv). We have

$$\int_{\mathscr{C}} dx \int |F_1(y)|^r |x-y|^{-(n+mr)} dy = \int_{\mathcal{O}} |F_1(y)|^r \int_{\mathscr{C}} |x-y|^{-(n+mr)} dx.$$

But for $y \in \mathcal{O}$ the inner integral is finite and is dominated by $c\delta(y)^{-mr}$ and the desired result follows from (v).

In order to prove (iii) let us observe that, since

$$\varrho^{-n-mq} \int_{|x-y|\leqslant\varrho} F_1(y)^q dy \leqslant \int F_1(y)^q |x-y|^{-n-mq} dy,$$

we have

$$N(F_1, x)^q \leqslant \int F_1(y)^q |x-y|^{-n-mq} dy$$

whence (iii) follows from (iv) by integration.

Finally, let us prove (vi). As we saw earlier, $N(F_2, x) \leqslant ct$. Thus

$$\int_{\mathcal{O}} N(F_2, x)^p dx \leqslant c^p t^p |\mathcal{O}|.$$

Furthermore,

$$\int_{\mathscr{C}} N(F_2, x)^p dx \leqslant 2^{p-1} \int_{\mathscr{C}} N(F_1, x)^p dx + 2^{p-1} \int_{\mathscr{C}} N(F, x)^p dx.$$

But $N(F_1, x) \leqslant ct$ in $\mathscr{C}$. Thus, according to (iii), the first term on the right above is dominated by

$$(ct)^{p-q} \int_{\mathscr{C}} N(F_1, x)^q dx \leqslant c^p t^p |\mathcal{O}|,$$

and this combined with the preceding estimates gives (vi).

Lemma 7. *Let F be locally integrable and suppose that its derivatives F_α of order m are functions in L^p. Suppose that $|F_\alpha(x)| \leqslant h(x)$, $|\alpha| = m$, where $h(x)$ is also in L^p. If ν is a unit vector let*

$$h_1(x, \nu) = \sup_{t>0} \frac{1}{t} \int_0^t h(x+\nu s)\,ds$$

and

$$h^*(x) = \Big[\int h(x+\nu)^q d\sigma_\nu\Big]^{1/q},$$

where $d\sigma_\nu$ is the surface area element of the unit sphere $\{\nu\}$.

Then h^* belongs to L^p, $\|h^*\|_p \leqslant c\|h\|_p$, where c depends on p, and $N(F, x) \leqslant ch^*(x)$ almost everywhere, where c depends only on m.

Proof. Let us start discussing the function h^*. First observe that $h_1(x, \nu)$ is precisely the one-dimensional Hardy–Littlewood maximal function of the restriction of h to lines s parallel to the unit vector ν. Thus we have

$$\int h_1(x, \nu)^p dx \leqslant c \int h(x)^p dx,$$

with c depending on p. But then

$$h^*(x)^p = \Big[\int h_1(x, \nu)^q d\sigma_\nu\Big]^{p/q} \leqslant c \int h_1(x, \nu)^p d\sigma_\nu$$

whence the desired result about h^* follows by integration.

Assume now that F is infinitely differentiable. Let $P(x, y)$ denote the sum of terms of degree less than or equal to $m-1$ of the Taylor expansion of $F(y)$ at x. Then, if $y = |y|\nu$,

$$(7)\qquad |F(y)-P(x, y)| \leqslant c|y|^m \int_0^1 h(x+sy)(1-s)^{m-1}ds \leqslant c|y|^m h_1(x, \nu)$$

and

$$(8)\qquad \int_{|y|\leqslant\varrho} |F(y)-P(x, y)|^q dy \leqslant c^q \varrho^{mq+n} \int h_1(x, \nu)^q d\sigma_\nu \leqslant c^q h^*(x)^q \varrho^{mq+n}$$

which is the desired conclusion in this case.

In the general case let $\eta(x) \in C_0^\infty$ be non negative and have integral equal to 1. Let $\eta_\lambda = \lambda^n \eta(\lambda x)$, let F_λ be the convolution $F*\eta_\lambda$ of F and η_λ, and $h_\lambda = h*\eta_\lambda$. Then $F_\lambda(y)$ converges to $F(y)$ almost everywhere and $P_\lambda(x, y)$ converges to $P(x, y)$ for almost all x and all y. Furthermore, according to the maximal theorem of Hardy and Littlewood the function $g(x) = \sup_\lambda h_\lambda(x)$ is locally integrable, and we can pass to the limit in

$$|F_\lambda(y)-P_\lambda(x, y)| \leqslant c|y|^m \int_0^1 h_\lambda(x+sy)(1-s)^{m-1}ds$$

whenever

$$\int_0^1 g(x+sy)(1-s)^{m-1}ds$$

is finite, i.e. (7) holds for almost all x and y. Thus (8) holds for almost all x, and the lemma is established.

LEMMA 8. *If f is a function in L^p and k a positive integer, there exists a locally integrable function F such that $\Delta^k F = f$ and $\left\|\left(\frac{\partial}{\partial x}\right)^\alpha F\right\|_p \leqslant c\|f\|_p$ for all α, $|\alpha| = 2k$, with c independent of f.*

Proof. Let $\eta\hat{\ }(x)$ be an infinitely differentiable function with compact support which equals 1 near the origin and $\eta_\lambda\hat{\ }(x) = \eta\hat{\ }(\lambda^{-1}x)-\eta\hat{\ }(\lambda x)$. Let $\eta(x)$ and $\eta_\lambda(x) = \lambda^n\eta(\lambda x)-\lambda^{-n}\eta(\lambda^{-1}x)$ be the inverse Fourier transforms of $\eta\hat{\ }$ and $\eta_\lambda\hat{\ }$ respectively. Then if $f_\lambda = f*\eta_\lambda$ is the convolution of f and η_λ, $f_\lambda\hat{\ }$ vanishes near the origin and near infinity. Let $F_\lambda\hat{\ }$ be the distribution vanishing near the origin defined by $(-4\pi^2)^k|x|^{2k}F_\lambda\hat{\ } = f_\lambda\hat{\ }$. Then its inverse Fourier transform F_λ is infinitely differentiable and $\Delta^k F_\lambda = f_\lambda$. Furthermore, $[(\partial/\partial x)^\alpha F_\lambda]\hat{\ }$ vanishes near the origin and

$$\left[\left(\frac{\partial}{\partial x}\right)^\alpha F_\lambda\right]\hat{\ } = x^\alpha|x|^{-2k}f_\lambda\hat{\ }.$$

As is well known, $x^\alpha|x|^{-2k}$, $|\alpha| = 2k$ is a multiplier in L^p. Thus

$$\left\|\left(\frac{\partial}{\partial x}\right)^\alpha F_\lambda\right\|_p \leqslant c\|f_\lambda\|_p \leqslant 2c\|f\|_p\|\eta\|_1,$$

the last inequality being a consequence of Young's theorem on convolutions. Thus we can select a sequence λ_j tending to infinity so that $\left(\frac{\partial}{\partial x}\right)^\alpha F_{\lambda_j}$ converges weakly to a limit g_α with $\|g_\alpha\|_p \leqslant 2c\|f\|_p\|\eta\|_1$. Furthermore, inspection of their Fourier transforms shows that $\left(\frac{\partial}{\partial x}\right)^\alpha F_\lambda = g_\alpha * \eta_\lambda$. Let now $h(x) = \sup_\lambda[\eta_\lambda * \sum_\alpha |g_\alpha|]$. Then the maximal theorem of Hardy and Littlewood asserts that $h \in L^p$ and $\|h\|_p \leqslant c\sum_\alpha\|g_\alpha\|_p \leqslant c\|f\|_p$. Thus if h^* is defined in terms of h as in Lemma 7, and $h^*(x_0) < \infty$, we have, by Lemma 7,

$$\int_{|y-x|<\varrho} |F_\lambda(y)-P_\lambda(x_0, y)|^q dy \leqslant ch^*(x_0)\varrho^{n+mq}.$$

Let now $F_\lambda^\sim(y) = F_\lambda(y)-P_\lambda(x_0, y)$. Then since $P_\lambda(x_0, y)$ is a polynomial in y of degree less than or equal to $2k-1$, we have that

$$\left(\frac{\partial}{\partial x}\right)^\alpha F_\lambda^\sim = \left(\frac{\partial}{\partial x}\right)^\alpha F_\lambda$$

for $|\alpha| = 2k$. Now take a subsequence λ_j of the one above so that not only

$$\left(\frac{\partial}{\partial x}\right)^\alpha F_{\lambda_j}^\sim = \left(\frac{\partial}{\partial x}\right)^\alpha F_{\lambda_j}$$

converges weakly to g_α, but also F_λ converges weakly on bounded sets to a limit F. Evidently $\left(\frac{\partial}{\partial x}\right)^\alpha F = g_\alpha$. Furthermore

$$\Delta^k F = \lim \Delta^k F_{\lambda_j} = \lim f_{\lambda_j},$$

where the limits here are weak limits in L^p. But

$$f_\lambda(x) = \lambda^n \int f(y)\eta[\lambda(x-y)]dy - \lambda^{-n}\int f(y)\eta[\lambda^{-1}(x-y)]dy$$

and since, as readily seen,

$$\|\lambda^{-n}\eta[\lambda^{-1}x]\|_p \leqslant c\lambda^{n(1/p-1)}$$

the second integral tends to zero and $f_\lambda(x)$ converges to $f(x)$ almost everywhere. Thus f_{λ_j} can have no limit other than f and we conclude that $\Delta^k F = f$. This completes the proof of the lemma.

4. In this section we will discuss the spaces $\mathcal{M}_m^{p,q}$.

THEOREM 6. *The space $\mathcal{M}_m^{p,q}$ is contained in L^p_{-m-s}, $s = n(p-q)/pq$.*

Proof. If $f \in \mathcal{M}_m^{p,q}$ and m is even, then $f = \Delta^{m/2}F$ where $F \in \mathcal{N}_m^{p,q}$. Decompose F as in Theorem 5. Since the set $\mathcal{O}$ there has finite measure, the function $\delta(y)$ in (v) is bounded, and (v) implies that F_1 is in L^q. On the other hand, according to (vi), $N(F_2, x)$ is in L^p, thus it follows from Theorem 4 that all derivatives of order m of F_2 are in L^p. In particular $\Delta^{m/2}F_2 \in L^p$. Consequently, $f = \Delta^{m/2}F_1 + \Delta^{m/2}F_2 \in L^p + L^q_{-m}$ and according to Theorem 6 in [2] $L^q_{-m} \subset L^p_{-m-s}$. The proof in the case m odd is analogous to the preceding one and is left to the reader.

LEMMA 9. *Let F be in $\mathcal{N}_m^{p,q}$ and $\Delta^l F = 0$ for some $l \geq 1$. Then F is a polynomial of degree less than or equal to $m-1$.*

Proof. Since $F \in \mathcal{N}_m^{p,q}$, according to the preceding theorem, F is also a tempered distribution. But $\Delta^l F = 0$ and, therefore, $|x|^{2l}F^\wedge = 0$. Thus $F^\wedge$ is supported at the origin and F is a polynomial. Let $N(F, x_0) < \infty$, and $F(y) = \sum a_\alpha y^\alpha$. Suppose that $k \geq m$ is the degree of $F(y)$. Then, if $P(x_0, y)$ is the polynomial associated with F as in (1) and $\varrho \geq 2|x_0|$, we have

$$N(F, x_0)\varrho^{(m-k)q} \geq \varrho^{-n-kq}\int_{|y-x_0|<\varrho} |F(y) - P(x_0, y)|^q dy$$

$$\geq \varrho^{-n-kq}\int_{|y|<\varrho/2} \Big|\sum a_\alpha y^\alpha\Big|^q dy + o(1)$$

$$= 2^{-n-kq}\int_{|z|<1} \Big|\sum_{|\alpha|=k} a_\alpha z^\alpha\Big|^q dz + o(1).$$

Thus if $k > m$, letting $\varrho \to \infty$ we find that

$$\int_{|z|<1} \Big|\sum_{|\alpha|=k} a_\alpha z^\alpha\Big|^q dz = 0$$

which implies that $a_\alpha = 0$ for $|\alpha| = k$ contradicting the assumption that $F(y)$ is of degree k. If $k = m$, on the other hand, letting $\varrho \to \infty$ we find that

$$\int_{|z|=1} \Big|\sum_{|\alpha|=m} a_\alpha z^\alpha\Big|^q dz \leq N(F, x_0)2^{n+kq}.$$

But the function $N(F, x)$ is integrable to the power p in the complement of a set of finite measure. Thus $N(F, x_0)$ can be taken arbitrarily small and we conclude again that

$$\int \Big|\sum_{|\alpha|=m} a_\alpha z^\alpha\Big|^q dz = 0$$

which implies that $F(y)$ is of degree $m-1$ at most, as we wished to show.

THEOREM 7. *Let f be in $\mathcal{M}_m^{p,q}$, then $M(f, x)$ is well defined, that is, if m is even, F_1 and F_2 are in $\mathcal{N}_m^{p,q}$ and are such that $\Delta^{m/2}F_1 = \Delta^{m/2}F_2 = f$, then $N(F_1, x) = N(F_2, x)$; if m is odd, F_1 and F_2 are locally integrable, such that $\Delta^{(m+1)/2}F_1 = \Delta^{(m+1)/2}F_2 = f$, and $\frac{\partial F_1}{\partial x_j}$, $\frac{\partial F_2}{\partial x_j}$ are in $\mathcal{N}_m^{p,q}$, then*

$$N\left(\frac{\partial F_1}{\partial x_i}, x\right) = N\left(\frac{\partial F_2}{\partial x_i}, x\right).$$

Proof. Suppose first that m is even. Then, according to the preceding lemma, $F_1 - F_2$ is a polynomial of degree $m-1$, and from Lemma 4 we see readily that $N(F_1, x) = N(F_2, x)$.

If, on the other hand, m is odd then, since $\Delta^{(m+1)/2}(F_1 - F_2) = 0$, we also have $\Delta^{(m+1)/2}\left(\frac{\partial F_1}{\partial x_j} - \frac{\partial F_2}{\partial x_j}\right) = 0$ and the preceding lemma asserts that $\frac{\partial F_1}{\partial x_j} - \frac{\partial F_2}{\partial x_j}$ is a polynomial of degree at most $m-1$, which combined with Lemma 4 implies that $N\left(\frac{\partial F_1}{\partial x_j}, x\right) = N\left(\frac{\partial F_2}{\partial x_j}, x\right)$.

5. We are now ready to prove our main results.

Proof of Theorem 1. First we will show that $\left|\left(\frac{\partial}{\partial x}\right)^\beta k(x)\right| = |k_\beta(x)| \leq c|x|^{-n-|\beta|}$, $0 \leq |\beta| \leq m$. For $|\beta| = m$ this is part of our assumptions. Suppose that our assertion holds for $|\beta| = j \geq 1$. Let $|\gamma| = j-1$, and let ∇k_γ denote the gradient of k_γ. Then $|\nabla k_\gamma(x)| \leq c|x|^{-n-j}$ and expressing $k_\gamma(\varrho x) - k_\gamma(x)$, $\varrho \geq 1$ as the line integral of ∇k_γ along the segment joining x and ϱx we see that $k_\gamma(\varrho x)$ has a limit as $\varrho \to \infty$ for each $x \neq 0$. On the other hand, if $|x_1| = |x_2| = \varrho$, expressing again $k_\gamma(x_1) - k_\gamma(x_2)$ as the line integral of ∇k_γ along the arc of a circle with center at the origin, and joining x_1 and x_2 we see that $k_\gamma(x_1) - k_\gamma(x_2) \to 0$ as $\varrho \to \infty$. Consequently, $k_\gamma(x)$ has a limit as $|x| \to \infty$. Let now $\eta \in C_0^\infty$ and $\int \eta(x)dx = 1$. Then, since $k^\wedge$ is bounded, $K\eta_\gamma$ is square integrable and for $|x|$ sufficiently large we have

$$(K\eta_\gamma)(x) = \int k(x-y)\eta_\gamma(y)dy = \int k_\gamma(x-y)\eta(y)dy.$$

Thus, if $a = \lim_{|x|\to\infty} k_\gamma(y)$, the last integeral tends to a as $|x| \to \infty$, and therefore we must have $a = 0$. Consequently

$$|k_\gamma(x)| \leq |x|\int_1^\infty |\nabla k_\gamma(\varrho x)|d\varrho \leq c|x|^{-n-|\gamma|}\int_1^\infty \varrho^{-n-j}d\varrho \leq c|x|^{-n-|\gamma|}.$$

In particular we have

$$\left|\frac{\partial}{\partial x_j}k(x)\right| \leqslant c|x|^{-n-1}$$

and this, as readily seen, implies that

$$\int_{|x|>2|y|} |k(x-y)-k(x)|\,dx \leqslant c.$$

Consequently (see [3], Theorem 2 for example), K is bounded with respect to the norm of L^p, and from the very definition of the norm of L^p_u (see [2]) it follows that K is also bounded with respect to the norm of all these spaces.

Let now $f \in \mathcal{M}^{p,q}_m$ and let $t > 0$ be given. Let $F \in \mathcal{N}^{p,q}_m$ be such that $\Delta^{m/2}F = f$. Decompose F as in Theorem 5, and let $f_1 = \Delta^{m/2}F_1$ and $f_2 = \Delta^{m/2}F_2$. The function F_1 belongs to L^q and therefore (see [2], theorems 5 and 6) also to L^p_{-m-n}. Thus $\bar{K}F_1$ is well defined. Furthermore, since $\bar{K}$ concides with K in C_0^∞ and K is continuous with respect to the norm of L^q, $\bar{K}F_1$ is a function in L^q. But, for $g \in C_0^\infty$ we have $\Delta^{m/2}Kg = K\Delta^{m/2}g$, and thus, the continuity of $\bar{K}$ in L^p_{-n-m}, and the continuity of the embedding of L^q and L^q_{-m} in L^p_{-n-m}, imply that $\Delta^{m/2}\bar{K}F_1 = \bar{K}\Delta^{m/2}F_1 = Kf_1$. Now let us estimate $N(\bar{K}F_1, x)$ assuming that $x \in \mathscr{C}$ and

$$\psi(x) = \int |F_1(y)|\,|x-y|^{-n-m}dy < \infty \tag{9}$$

which, according to (iv) in Theorem 5, holds for almost all x in $\mathscr{C}$. Without loss of generality, we may further assume that $x = 0$. Let $\varphi(y)$ be infinitely differentiable, equal to 1 in $|y| \leqslant \varrho$ and equal to zero in $|y| \geqslant 2\varrho$. Then if $|x| \leqslant \varrho/2$, x is not contained in the support of $F_1(1-\varphi)$, and, as readily verified, $\bar{K}[F_1(1-\varphi)](x)$ is given by the integral

$$\int k(x-y)F_1(y)[1-\varphi(y)]\,dy.$$

Expanding $k(x-y)$ in powers of y and substituting in this integral we obtain

$$\int k(x-y)F_1(y)[1-\varphi(y)]\,dy = \sum_{|\alpha|<m}\frac{x^\alpha}{\alpha!}\int k_\alpha(-y)F_1(y)\,dy - $$
$$-\sum_{|\alpha|<m}\frac{x^\alpha}{\alpha!}\int k_\alpha(-y)F_1(y)\varphi(y)\,dy + $$
$$+\sum_{|\alpha|=m}\frac{x^\alpha}{\alpha!}\int k_\alpha(\theta x-y)F_1(y)[1-\varphi(y)]\,dy,$$

where $0 \leqslant \theta = \theta(x,y) \leqslant 1$. But $|k_\alpha(-y)| \leqslant c|y|^{-n-|\alpha|}$, and since F_1 is in L^q and vanishes outside a set of finite measure, (9) with $x = 0$ implies that all the integrals in the preceding expression are absolutely convergent. The first sum there is a polynomial $P(x)$ of degree $m-1$ at most. Each term in the second sum is dominated by

$$c|x^\alpha|\int_{|y|\leqslant 2\varrho}|F_1(y)|\,|y|^{-n-|\alpha|}dy \leqslant c\varrho^m\int |F_1(y)|\,|y|^{-n-m}dy$$
$$\leqslant c\varrho^m\psi(0).$$

Finally, since $1-\varphi(y)$ has support in $|y| \geqslant \varrho$ and $|\theta x| \leqslant \frac{1}{2}\varrho$, we have $k_\alpha(\theta x-y) \leqslant c|y|^{-n-m}$, and consequently, the terms in the third sum are also dominated by $c\varrho^m\psi(0)$. Collecting estimates we see that if $|x| \leqslant \varrho/2$ then

$$|\bar{K}[F_1(1-\varphi)](x) - P(x)| \leqslant c\varrho^m\psi(0). \tag{10}$$

On the other hand for $\bar{K}F_1\varphi$ we have

$$\int |\bar{K}(F_1\varphi)|^q dx \leqslant c\int |F_1(y)\varphi(y)|^q dy \leqslant c\int_{|y|\leqslant 2\varrho}|F_1(y)|^q dy.$$

As it was pointed out in showing (6) in the proof of Theorem 5, the last integral is dominated by $cN(F_1,0)^q\varrho^{n+mq}$. Combining this with (10) we obtain

$$\int_{|x|\leqslant\varrho/2}|\bar{K}F_1(x)-P(x)|^q dx \leqslant c[\psi(0)+N(F_1,0)]^q\varrho^{n+mq}.$$

Thus we have shown that

$$N(\bar{K}F_1, x) \leqslant c[\psi(x)+N(F_1,x)]. \tag{11}$$

According to (iii) and (iv) in Theorem 5, $\psi(x)$ and $N(F_1,x)^q$ are integrable in the complement of $\mathcal{O}$. Thus since $\mathcal{O}$ has finite measure and $1 < q \leqslant p$, both $\psi(x)^p$ and $N(F_1,x)^p$ are integrable outside a set of finite measure, and the same holds for $N(\bar{K}F_1, x)$. Thus $KF_1 \in \mathcal{N}^{p,q}_m$ and $\bar{K}f_1 = \Delta^{m/2}\bar{K}F_1 \in \mathcal{M}^{p,q}_m$. Furthermore, according to (iii) and (iv) in Theorem 5, the measure of the set of points where

$$M(\bar{K}f_1, x) = N(\bar{K}F_1, x) > t/2$$

does not exceed $c|\mathcal{O}| = c\mu(f,t)$.

Now let us turn our attention to $\bar{K}f_2$. On account of (vi) in Theorem 5 and of Theorem 4, we can assert that $f_2 = \Delta^{m/2}F_2$ is in L^p and that

$$\|f_2\|_p^p \leqslant c\Big[|\mathcal{O}|t^p + \int_{\mathscr{C}} N(F,x)^p dx\Big].$$

Since $M(f,x) = N(F,x)$ and $\mathcal{O}$ is precisely the set where $M(f,x) = N(F,x) > t$ we have

$$p\int_0^t s^{p-1}\mu(f,s)\,ds = t^p|\mathcal{O}| - \int_0^t s^p\,d\mu(f,s)$$
$$= t^p|\mathcal{O}| + \int_{\mathscr{C}} N(F,x)^p dx,$$

and consequently

$$\|f_2\|_p^p \leqslant cp \int_0^t s^{p-1} d\mu(f, s).$$

Since f_2 is in L^p so is $\bar{K}f_2$, and Lemmas 7 and 8 applied to $\bar{K}f_2$ show that there exists a locally integrable function G such that $\Delta^{m/2}G = \bar{K}f_2$ and $N(G, x) \in L^p$, $\|N(G, x)\|_p^p \leqslant c\|f_2\|_p^p$. Therefore, $\bar{K}f_2 \in \mathscr{M}_m^{p,q}$ and $M(\bar{K}f_2, x) \in L^p$ with

$$\|M(\bar{K}f_2, x)\|_p^p \leqslant c\|f\|_p^p \leqslant c\int_0^t s^{p-1}\mu(f, s)\,ds. \tag{12}$$

Thus we have shown that $g = \bar{K}f$ belongs to $\mathscr{M}_m^{p,q}$. Furthermore the measure $\mu(g, t)$ of the set where $M(g, x) > t$ is less than or equal to the sum of the measures of the sets where $M(\bar{K}f_1, x) > t/2$ and $M(\bar{K}f_2, x) > t/2$, respectively. Thus, combining (12) with the estimate for the measure of the set where $M(\bar{K}f_1, x) > t/2$ we obtained previously, we find that

$$\mu(g, t) \leqslant c\Big[\mu(f, t) + t^{-p}\int_0^t s^{p-1}\mu(f, s)\,ds\Big].$$

Now, $\mu(f, s)$ is a decreasing function of s and consequently

$$\mu(f, t) \leqslant pt^{-p}\int_0^t s^{p-1}\mu(f, s)\,ds,$$

and the preceding inequality can be written as

$$\mu(g, t) \leqslant ct^{-p}\int_0^t s^{p-1}\mu(f, s)\,ds.$$

Thus our theorem is established.

Proof of Theorem 2. Let us start defining $M^*(f, x)$. This is done simply by setting

$$M^*(f, x) = \sup_K M(\bar{K}f, x),$$

where the supremum is taken over all operators K satisfying (iii). Thus we merely have to show that $M^*(f, x)$ is finite outside a set of finite measure and satisfies (ii). This we do by examining the proof of Theorem 1, and show that under the hypothesis (iii) the same estimates hold for $M^*(f, x)$. First observe that the constant in (11) depends only on the norm of $\bar{K}$ as an operator in L^q and on the bounds for $|x|^{n+|\alpha|}k_\alpha(x)$, $0 \leqslant |\alpha| \leqslant m$. Now

$$\int |k(x)|\,dx \leqslant \int \sup_{|y|\geqslant|x|} |k(y)|\,dx \geqslant 1,$$

which implies that the norm of K in L^q is less than or equal to 1. On the other hand, the argument showing that $|x|^{n+|\alpha|}|k_\alpha(x)|$ is bounded also shows that there exist a finite common bound for all K satisfying (iii). Thus (11) holds for all K with a fixed constant c, and since $M(\bar{K}f_1, x) = N(\bar{K}F_1, x)$ we have

$$\sup_K M(\bar{K}f_1, x) \leqslant c[\psi(x) + N(F_1, x)].$$

Before we discuss $\bar{K}f_2$ let us make some preliminary observations. For each α, $|\alpha| = m$, let K_α be the operator on C_0^∞ defined by $(K_\alpha f)\hat{} = |x|^{-m}x^\alpha \hat{f}$. Since $x^\alpha|x|^{-m}$ is a multiplier in L^p, K_α can be extended continuously to L^p. Now if $f \in C_0^\infty$ we evidently have $\Big[\Delta^{m/2}K_\alpha - \Big(\frac{\partial}{\partial x}\Big)^\alpha\Big]f = 0$. Since K_α is continuous with respect to the norm of L^p and C_0^∞ is dense in L^p, this also holds for $f \in L^p$. Let now f and $f^{(\alpha)}$ be in L^p and such that $\Delta^{m/2}f^{(\alpha)} = \Big(\frac{\partial}{\partial x}\Big)^\alpha f$. Then

$$\Delta^{m/2}[K_\alpha f - f^{(\alpha)}] = \Delta^{m/2}K_\alpha f - \Big(\frac{\partial}{\partial x}\Big)^\alpha f = 0$$

therefore the Fourier transform of $K_\alpha f - f^{(\alpha)}$ is supported at the origin and $K_\alpha f - f^{(\alpha)}$ is a polynomial. But $K_\alpha f - f^{(\alpha)}$ is in L^p and this polynomial must therefore be qual to zero. Thus, if $\Delta^{m/2}f^{(\alpha)} = \Big(\frac{\partial}{\partial x}\Big)^\alpha f$ then $K_\alpha f = f^{(\alpha)}$.

Let us turn now to the function f_2 which we shall henceforth denote simply by f. Let $f_\alpha = \Big(\frac{\partial}{\partial x}\Big)^\alpha F_2$, $|\alpha| = m$, $g = \bar{K}f$. Since $N(F_2, x)$ belongs to L^p, according to Theorem 4, the functions f_α and f also belong to L^p and $|f_\alpha(x)| \leqslant cN(F_2, x)$. Since $\bar{K}$ is continuous in L^p, $g = \bar{K}f$ also belongs to L^p. Now, according to Lemma 8, there exists a function G such that $g_\alpha = \Big(\frac{\partial}{\partial x}\Big)^\alpha G$, $|\alpha| = m$, is in L^p and $\Delta^{m/2}G = g$. Since $\Delta^{m/2}F_2 = f$, we have $\Delta^{m/2}f_\alpha = \Big(\frac{\partial}{\partial x}\Big)^\alpha f$ for all $|\alpha| = m$, and similarly $\Delta^{m/2}g_\alpha = \Big(\frac{\partial}{\partial x}\Big)^\alpha g$. Thus, as we saw above, $g_\alpha = K_\alpha g$ and $f_\alpha = K_\alpha f$. Now as is readily seen, K_α and $\bar{K}$ commute and therefore we have

$$\bar{K}f_\alpha = \bar{K}K_\alpha f = K_\alpha\bar{K}f = K_\alpha g = g_\alpha.$$

Let now $h(x)$ be the Hardy–Littlewood maximal function of $N(F_2, x)$. Then

$$v(\varrho, x) = \int_{|y|\leqslant\varrho} N(F_2, x-y)\,dy \leqslant \omega\varrho^n h(x)$$

where ω denotes the volume of the unit sphere. Furthermore, if $\psi(\varrho) = \sup_{|y|\geqslant\varrho} |k(y)|$, $\psi(\varrho)$ is non increasing and

$$|g_a(x)| = |\overline{K}f_a(x)| \leqslant \int N(F_2, x-y)|k(y)|\,dy \leqslant \int N(F_2, x-y)\psi(|y|)\,dy$$

$$= \int_0^\infty \psi(\varrho)\,dv(\varrho, x) = -\int_0^\infty v(\varrho, x)\,d\psi(\varrho) \leqslant -h(x)\int_0^\infty \omega\varrho^n\,d\psi(\varrho)$$

$$= nh(x)\,\omega\int_0^\infty \varrho^{n-1}\psi(\varrho)\,d\varrho \leqslant nh(x),$$

and, according to Lemma 7,

$$N(G, x) = M(g, x) = M(\overline{K}f_2, x) \leqslant ch^*(x)$$

where c is independent of $\overline{K}$. Thus

$$\sup_{K} M(\overline{K}f_2, x) \leqslant ch^*(x)$$

where $\|h^*\|_p \leqslant c\|h\|_p \leqslant c\|N(F_2, x)\|_p$. From here on the proof proceeds as that of Theorem 1.

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Isomorphic characterizations of inner product spaces by orthogonal series with vector valued coefficients

by

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Abstract. A Banach space X is isomorphic to a Hilbert space if and only if one of the following conditions holds for all sequences (x_n) in X

(a) if $\sum_{n=0}^\infty \|x_n\|^2 < +\infty$, then

$$\int_0^{2\pi} \Big\|x_0 + \sum_{k=1}^\infty (x_{2k-1}\sin kt + x_{2k}\cos kt)\Big\|^2 dt < +\infty,$$

(b) if $\int_0^{2\pi} \|x_0 + \sum_{k=1}^\infty (x_{2k-1}\sin kt + x_{2k}\cos kt)\|^2 dt < +\infty$, then $\sum_{n=0}^\infty \|x_n\|^2 dt < +\infty$,

(c) $\sum_{n=1}^\infty \|x_n\|^2 < +\infty$ if and only if $\int_0^1 \|\sum_{n=1}^\infty x_n r_n(t)\|^2 dt < +\infty$.

Here (r_n) denotes the Rademacher system of functions.

1. Introduction. In the present paper we prove the following

THEOREM 1.1. *A real or complex Banach space X is isomorphic to a Hilbert space if and only if one of the following conditions holds for all sequences (x_n) in X*

(a) *if* $\sum_{n=0}^\infty \|x_n\|^2 < +\infty$, *then*

$$\int_0^{2\pi} \Big\|x_0 + \sum_{k=1}^\infty (x_{2k-1}\sin kt + x_{2k}\cos kt)\Big\|^2 dt < +\infty,$$

(b) *if* $\int_0^{2\pi} \Big\|x_0 + \sum_{k=1}^\infty (x_{2k-1}\sin kt + x_{2k}\cos kt)\Big\|^2 dt < +\infty,$

then $\sum_{n=0}^\infty \|x_n\|^2 < +\infty,$

(c) $\sum_{n=1}^\infty \|x_n\|^2 < +\infty$ *if and only if* $\int_0^1 \Big\|\sum_{n=1}^\infty x_n r_n(t)\Big\|^2 dt < +\infty$.

Here (r_n) denotes the Rademacher system defined by

$$r^n(t) = \operatorname{sign}\sin 2^n\pi t \quad \text{for} \quad t\in[0,1]\ (n = 1, 2, \ldots).$$