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On the localization property of square partial sums for multiple Fourier series

by

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To A. Zygmund on the 50th anniversary of his first mathematical publication

Abstract. It seems that localization and convergence of multiple Fourier series are related to the Sobolev spaces W_p^1. This paper establishes the existence of such a relation regarding the square partial sums. It is shown that for $f \epsilon W_p^1$, $p \geqslant n-1$, this sort of localization holds for the n-torus. For each $p < n-1$ there is an $f \epsilon W_p^1$ for which localization fails. Examples are given of an everywhere differentiable periodic function of 2 variables for which localization by square partial sums fails and of a function in W_2^1 for which localization by rectangular partial sums fails.

1. In the study of Fourier series, a primary feature is the localization property, which has been known to hold in the case of functions of one variable since Riemann. That localization does not generally hold for functions of several variables has also been known for a long time. Our purpose is to obtain precise information regarding the functions of n variables which have this property.

Tonelli, [2], observed that, for $n = 2$, localization holds for those functions now known as the functions whose partial derivatives (in the distribution sense) are measures; this includes the Sobolev space W_1^1. An example by Torrigiani, [3], shows that a condition given by Tonelli, which guarantees convergence at a point, and holds almost everywhere for $n = 2$ for functions in W_1^1, may hold nowhere for $n = 3$.

In a recent paper, Igari [1], settles the localization problem for the square $(C, 1)$ partial sums of a multiple Fourier series. He shows that this sort of localization holds for $f \epsilon L^p$, $p \geqslant n-1$, and fails to hold for $p < n-1$. For the square partial sums themselves — not the averages — he points out that there are continuous functions for which localization fails.

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In the present paper, we return to the main issue. We show that localization holds for square partial sums of the multiple Fourier series for all functions in the Sobolev space W_p^1, $p \geqslant n-1$. For the converse, we show that for every $p < n-1$ there is a function in W_p^1 for which this sort of localization fails. For $n = 2$ we construct an everywhere differentiable function for which localization fails. For rectangular partial sums localization fails for $p = n-1$ but we do not know the situation for larger values of p.

Thus, the solution of the localization problem is found to lie among the properties of the first derivative.

2. Let T_n be the n-dimensional cube which consists of those points $x = (x_1, x_2, \ldots, x_n)$ in n-space with $-\pi \leqslant x_i < \pi$, $i = 1, 2, \ldots, n$, and let $\mathring{T}_n$ be the interior of T_n. As usual, we denote by $W_p^1(\mathring{T}_n)$ (or simply by W_p^1) the Sobolev space of those functions in $L^p(\mathring{T}_n)$ whose partial derivatives in the sense of distributions are functions and belong to $L^p(\mathring{T}_n)$. We shall only be concerned with the class $\mathring{W}_p^1$ which consists of those functions f in W_p^1 with $f(-\pi, \bar{t}_i) = f(\pi, \bar{t}_i)$ for a.e. $\bar{t}_i$, $i = 1, 2, \ldots, n$. This class may be described in several ways. Perhaps the simplest is the completion of the periodic and continuously differentiable functions with respect to the W_p^1 norm. We are of course interested in the operators

$$L_j(x, f) = \frac{1}{\pi^n} \int_{T_n} f(x+t) D_{(j,\ldots,j)}(t)\, dt,$$

where $D_{(j,\ldots,j)}$ is the appropriate Dirichlet kernel. For a function $f \in \mathring{W}_p^1$, we shall use f_i, $i = 1, 2, \ldots, n$, to denote its partial derivative with respect to x_i. The norm $\|f\|_{1,p}$, for $f \in \mathring{W}_p^1$, denotes

$$\|f\|_{1,p} = \|f\|_p + \sum_{i=1}^{n} \|f_i\|_p,$$

where $\|\cdot\|_p$ is the L^p norm.

3. We show that localization holds for $f \in \mathring{W}_p^1$, $p \geqslant n-1$. We first state a trivial lemma.

LEMMA 1. *Let Q consist of those functions in $\mathring{W}_p^1$ which are piecewise linear. Then the localization property for square partial sums holds for all functions in Q.*

Let $B(0, r)$ be the open cube of center $(0, \ldots, 0)$ and side $2r > 0$, and let $\bar{B}(0, r)$ be its closure.

LEMMA 2. *If $p \geqslant n-1$ and $0 < \varepsilon < \delta < \pi$, there is a constant $A = A(\varepsilon, \delta) > 0$ such that*

$$\sup_{j, |x| < \varepsilon} |L_j(x, f)| \leqslant A \|f\|_{1,p}$$

for any $f \in \mathring{W}_p^1$ with $f = 0$ almost everywhere on $B(0, \delta)$.

Proof. For any positive integer j, $x \in \bar{B}(0, \varepsilon)$, and f satisfying the condition of the lemma, we have

$$(*) \quad |L_j(x, f)| = \frac{1}{\pi^n} \Big| \int_{T_n} f(x+t) D_{(j,\ldots,j)}(t)\, dt \Big|$$

$$\leqslant \frac{1}{\pi^n} \Big\{ \Big| \underbrace{\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi}}_{n-1} \int_{b \leqslant |t_1| \leqslant \pi} f(x+t) D_{(j,\ldots,j)}(t)\, dt_1 \ldots dt_n \Big|$$

$$+ \Big| \underbrace{\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi}}_{n-2} \int_{b \leqslant |t_2| \leqslant \pi} \int_{-b}^{b} f(x+t) D_{(j,\ldots,j)}(t)\, dt_1 \ldots dt_n \Big|$$

$$+ \ldots + \Big| \int_{b \leqslant |t_n| \leqslant \pi} \underbrace{\int_{-b}^{b} \cdots \int_{-b}^{b}}_{n-1} f(x+t) D_{(j,\ldots,j)}(t)\, dt_1 \ldots dt_n \Big| \Big\},$$

where $b = (\delta - \varepsilon)$.

For convenience we let $\bar{j} = (\underbrace{j, \ldots, j}_{n-1})$, $y_i = (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots y_n)$ and $d\bar{t}_i = dt_1 \ldots dt_{i-1} dt_{i+1} \ldots dt_n$. Consider, for the first integral on the right hand side of $(*)$,

$$\Big| \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \int_{b}^{\pi} f(x_1 + t_1, \overline{x_1 + t_1}) D_j(t_1)\, dt_1 D_{\bar{j}}(\bar{t}_1)\, d\bar{t}_1 \Big|$$

$$\leqslant \int_{\pi}^{\pi} \cdots \int_{\pi}^{\pi} \Big| \int_{b}^{\pi} f(x_1 + t_1, \overline{x_1 + t_1}) \frac{\sin(j + \frac{1}{2}) t_1}{2 \sin \frac{1}{2} t_1}\, dt_1 \Big| \cdot |D_{\bar{j}}(\bar{t}_1)|\, d\bar{t}_1.$$

Since the partial derivative of f (in the distribution sense) is a function, $f(x_1 + t_1, \overline{x_1 + t_1})$ is an absolutely continuous function of t_1 for almost all values of $\bar{t}_1$. Also, it follows from Fubini theorem that there is $\mathring{t}_1$ with $b \leqslant \mathring{t}_1 \leqslant \pi$ such that

$$(\Delta) \qquad \int_{T_{n-1}} |f(\mathring{t}_1, \bar{t}_1)|^p\, d\bar{t}_1 \leqslant \frac{1}{\pi - b} \|f\|_p^p.$$

We write, for almost all $\bar{t}_1$,

$$\Big|\int_b^\pi f(x_1+t_1, \overline{x_1+t_1}) D_j(t_1)\, dt_1\Big|$$

$$\leqslant \Big|\int_b^\pi \{f(x_1+t_1, \overline{x_1+t_1}) - f(\mathring{t}_1, \overline{x_1+t_1})\} D_j(t_1)\, dt_1\Big|$$

$$+ |f(\mathring{t}_1, t_1)| \cdot \Big|\int_b^\pi D_j(t_1)\, dt_1\Big|$$

$$\leqslant \Big|\int_b^{\mathring{t}_1} \{f(x_1+t_1, \overline{x_1+t_1}) - f(\mathring{t}_1, \overline{x_1+t_1})\} D_j(t_1)\, dt_1\Big|$$

$$+ \Big|\int_{\mathring{t}_1}^\pi \{f(x_1+t_1, \overline{x_1+t_1}) - f(\mathring{t}_1, \overline{x_1+t_1})\} D_j(t_1)\, dt_1\Big| + A(j+\tfrac{1}{2})^{-1} |f(\mathring{t}_1, \bar{t}_1)|$$

where $A > 0$ is independent of j and $x \in \bar{B}(0, \varepsilon)$. We note that in the preceding inequality, the last step follows by applying second mean-value theorem to the integral $\int_b^\pi D_j(t_1)\, dt_1$. For the next step we express $f(x_1+t_1, \overline{x_1+t_1}) - f(\mathring{t}_1, \overline{x_1+t_1})$ as the difference of its positive and negative variation starting from $\mathring{t}_1$ as follows

$$f(x_1+t_1, \overline{x_1+t_1}) - f(\mathring{t}_1, \overline{x_1+t_1}) = p(t_1) - n(t_1),$$

where $p(\mathring{t}_1) = n(\mathring{t}_1) = 0$. This can be done for almost all $\bar{t}_1$. Applying the second mean-value theorem, we have

$$\Big|\int_b^{\mathring{t}_1} \{f(x_1+t_1, \overline{x_1+t_1}) - f(\mathring{t}_1, \overline{x_1+t_1})\} D_j(t_1)\, dt_1\Big|$$

$$\leqslant \Big|\int_b^{\mathring{t}_1} p(t_1) \frac{\sin(j+\frac{1}{2})t_1}{2\sin\frac{1}{2}t_1}\, dt_1\Big| + \Big|\int_b^{\mathring{t}_1} n(t_1) \frac{\sin(j+\frac{1}{2})t_1}{2\sin\frac{1}{2}t_1}\, dt_1\Big|$$

$$= \Big|p(b)\int_b^\xi \frac{\sin(j+\frac{1}{2})t_1}{2\sin\frac{1}{2}t_1}\, dt_1 + p(\mathring{t}_1)\int_\xi^{\mathring{t}_1} \frac{\sin(j+\frac{1}{2})t_1}{2\sin\frac{1}{2}t_1}\, dt_1\Big|$$

$$+ \Big|n(b)\int_b^\eta \frac{\sin(j+\frac{1}{2})t_1}{2\sin\frac{1}{2}t_1}\, dt_1 + n(\mathring{t}_1)\int_\eta^{\mathring{t}_1} \frac{\sin(j+\frac{1}{2})t_1}{2\sin\frac{1}{2}t_1}\, dt_1\Big|$$

$$= |p(b)| \Big|\int_b^\xi \frac{\sin(j+\frac{1}{2})t_1}{2\sin\frac{1}{2}t_1}\, dt_1\Big| + |n(b)| \Big|\int_b^\eta \frac{\sin(j+\frac{1}{2})t_1}{2\sin\frac{1}{2}t_1}\, dt_1\Big|$$

$$\leqslant A(j+\tfrac{1}{2})^{-1}\{|p(b)| + |n(b)|\} \leqslant A(j+\tfrac{1}{2})^{-1} \int_\pi^\pi |f_1(x_1+t_1, \overline{x_1+t_1})|\, dt_1,$$

and

$$\Big|\int_{\mathring{t}_1}^\pi \{f(x_1+t_1, \overline{x_1+t_1}) - f(\mathring{t}_1, \overline{x_1+t_1})\} D_j(t_1)\, dt_1\Big|$$

$$\leqslant A(j+\tfrac{1}{2})^{-1} \int_{-\pi}^\pi |f_1(x_1+t_1, \overline{x_1+t_1})|\, dt_1.$$

It follows that (from Hölder's inequality and (Δ))

$$\Big|\int_\pi^\pi \ldots \int_\pi^\pi \int_b^\pi f(x_1+t_1, \overline{x_1+t_1}) D_j(t_1) D_{\bar{j}}(\bar{t}_1)\, dt_1\, d\bar{t}_1\Big|$$

$$\leqslant A(j+\tfrac{1}{2})^{-1} \Big\{2\int_\pi^\pi \ldots \int_\pi^\pi |f_1(x_1+t_1, \overline{x_1+t_1})| \cdot |D_{\bar{j}}(\bar{t}_1)|\, dt$$

$$+ \int_\pi^\pi \ldots \int_\pi^\pi |f(\mathring{t}_1, \bar{t}_1)|\, |D_{\bar{j}}(\bar{t}_1)|\, d\bar{t}_1\Big\}$$

$$\leqslant 2A(j+\tfrac{1}{2})^{-1} \Big\{(2\pi)^{1/q} \|f_1\|_p \Big[\int_{-\pi}^\pi \Big|\frac{\sin(j+\frac{1}{2})s}{2\sin\frac{1}{2}s}\Big|^q ds\Big]^{\frac{n-1}{q}}$$

$$+ \Big[\int_{T_{n-1}} |f(\mathring{t}_1, \bar{t}_1)|^p\, d\bar{t}_1\Big]^{1/p} \cdot \Big[\int_{-\pi}^\pi \Big|\frac{\sin(j+\frac{1}{2})s}{2\sin\frac{1}{2}s}\Big|^q ds\Big]^{\frac{n-1}{q}}\Big\}$$

$$\leqslant D(j+\tfrac{1}{2})^{-1} \|f\|_{1,p} (j+\tfrac{1}{2})^{\frac{q-1}{q}(n-1)} = D(j+\tfrac{1}{2})^{\frac{n-1}{p}-1} \cdot \|f\|_{1,p},$$

where D is independent of j and $x \in \bar{B}(0, \varepsilon)$, and $q = \dfrac{p}{p-1}$.

That similar inequalities hold for the other integrals that appear on the right-hand side of $(*)$ is evident. The claim of the lemma is now clear.

THEOREM 1. *If $p \geqslant n-1$, the square partial sums of the multiple Fourier series of a function f of n variables has the localization property for $f \in \mathring{W}_p^1$.*

Proof. It is sufficient to prove that for any pair of positive numbers ε, δ with $0 < \varepsilon < \delta$, and $f \in \mathring{W}_p^1$, with $f = 0$ almost everywhere on $B(0, \delta)$, $\lim\limits_{j\to\infty} L_j(x, f) = 0$ uniformly for $x \in \bar{B}(0, \varepsilon)$.

Let f be any such function. Choose A as in Lemma 2. For any $\eta > 0$ there is a $g \in Q$ with the same property as f and $\|f-g\|_{1,p} < \dfrac{\eta}{2A}$. By Lemma 2

$$\sup_{\substack{j\\|x|\leqslant\varepsilon}} |L_j(x, f)| \leqslant \sup_{\substack{j\\|x|\leqslant\varepsilon}} |L_j(x, f-g)| + \sup_{\substack{j\\|x|\leqslant\varepsilon}} |L_j(x, g)| < \eta + \sup_{\substack{j\\|x|\leqslant\varepsilon}} |L_j(\mathring{x}, g)|.$$

The theorem follows from Lemma 1.

4. We now show that if $p < n-1$ there is a function f of n variables, with $f \in \mathring{W}_p^1$, the square sums of whose multiple Fourier series do not have the localization property.

For each positive integer $j \geqslant N$, where N is suitably large, let $b = b_j = \dfrac{2\pi}{2j+1}$, $m = m_j = \inf[k\colon kb \geqslant \frac{1}{2}\pi]$ and $M = M_j = \sup[k\colon kb \leqslant \frac{2}{3}\pi]$. There are positive numbers α and β such that $\alpha j \leqslant M-m \leqslant \beta j$, for all $j \geqslant N$.

For each $j \geqslant N$ and k with $m \leqslant k \leqslant M-1$, let $I_k = \{(x_1, \ldots, x_n)\colon kb \leqslant x_1 \leqslant (k+1)b\}$, $0 \leqslant x_i \leqslant b$, $i = 2, \ldots, n\}$, and let J_k be the closed cube concentric with J_k with sides half the length of the sides of I_k. We define a function f_j on T_n by

$$f_j(x) = \begin{cases} 0 & \text{if } x \in T_n \sim \bigcup_{k=m}^{M-1} \mathring{I}_n, \\ \text{the sign of } D_{(j,\ldots,j)} \text{ on } J_k & \text{if } x \in J_k, \end{cases}$$

and elsewhere f_j is defined in the natural way so that f_i is quasilinear, of the same sign on each I_k and so that the partial derivatives of f_j are bounded in magnitude by $\dfrac{4}{b}$. If $\|f_j\|$ is the norm of f_j in W_p^1, an elementary calculation yields

$$\|f_j\| \leqslant 4\beta^{1/p}(n+1)b^{n/p-1} \leqslant db^{(n-1)/p-1}$$

where d is a positive constant.

Let E consist of those $f \in W_p^1$ which are 0 almost everywhere on the cube $-\dfrac{\pi}{2} \leqslant x_i \leqslant \dfrac{\pi}{2}$, $i = 1, \ldots, n$. E is a Banach space with norm induced from W_p^1. Consider the sequence of linear functionals

$$L_j(f) = \frac{1}{\pi^n}\int_{T_n} D_{(j,\ldots,j)}(t)f(t)\,dt = S_j(0,f), \qquad i \geqslant N.$$

Then

$$|L_j(f_j)| \geqslant \frac{1}{\pi^n}\sum_{k=m}^{M-1}\int_{J_k}|D_{(j,\ldots,j)}(t)|\,dt$$

$$= \frac{1}{\pi^n}\left\{\int_{\frac{b}{4}}^{\frac{3b}{4}}\frac{\sin(j+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,dt\right\}^{n-1}\sum_{k=m}^{M-1}\int_{kb+\frac{b}{4}}^{kb+\frac{3b}{4}}\left|\frac{\sin(j+\frac{1}{2})t}{2\sin\frac{1}{2}t}\right|dt$$

$$\geqslant \frac{\alpha j}{\pi^n}\left\{\int_{\frac{b}{4}}^{\frac{3b}{4}}\frac{\sin(j+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,dt\right\}^{n-1}\cdot\min_{m\leqslant k\leqslant M-1}\int_{kb+\frac{b}{4}}^{kb+\frac{3b}{4}}\left|\frac{\sin(j+\frac{1}{2})t}{2\sin\frac{1}{2}t}\right|dt$$

$$\geqslant c > 0, \quad \text{where } c \text{ is a constant.}$$

It follows from these estimates on $\|f_j\|$ and $|L_j(f_j)|$ that

$$\|L_j\| \geqslant \frac{|L_j(f_j)|}{\|f_j\|} \geqslant \frac{c}{d}b^{1-\frac{n-1}{p}}$$

and this tends to $+\infty$ as $j \to \infty$ whenever $p < n-1$. By the uniform boundedness principle there is an $f \in E$ such that $\limsup_{j\to\infty} S_j(0,f) = +\infty$.

THEOREM 2. *For every $p < n-1$, there is a function $f \in W_p^1$, of n variables, the square partial sums of whose multiple Fourier series do not have the localization property.*

5. Our purpose now is to give an example of an everywhere differentiable function of two variables such that localization does not hold for the square partial sums of its double Fourier series.

The construction depends on the following elementary lemma.

LEMMA 2. *For any a, b, with $0 < a < b < \pi$, any $m > 0$, $M > 0$, and positive integer j_0, there is a $j > j_0$ and a continuously differentiable f, whose support is in the vertical strip $a \leqslant x \leqslant b$, such that $|f(x,y)| \leqslant m$, for all (x,y), and $L_j(f) \geqslant M$.*

Proof. For each j sufficiently large let r and s, $r < s$, be positive integers such that $a < \dfrac{r\pi}{j} < \dfrac{s\pi}{j} < b$ and $s - r < \dfrac{j}{\log j}$.

Let S_j be the vertical strip $\dfrac{r\pi}{j} \leqslant x \leqslant \dfrac{s\pi}{j}$ and let f_j be the sign of the Dirichlet kernel D_{jj} on S_j, i.e.

$$f_j(x,y) = \begin{cases} 1 \text{ if } (x,y) \in S_j \text{ and } D_{jj}(x,y) \geqslant 0, \\ -1 \text{ if } (x,y) \in S_j \text{ and } D_{jj}(x,y) < 0, \\ 0 \text{ if } (x,y) \notin S_j. \end{cases}$$

Then

$$\int_{T_2} f_j(x,y)D_{jj}(x,y)\,dx\,dy = \int_{S_j} f_j(x,y)D_{jj}(x,y)\,dx\,dy$$

$$\geqslant \frac{\log j}{j}\cdot\frac{j}{\sqrt{\log j}} = \sqrt{\log j}.$$

By choosing j sufficiently large and slightly modifying f_j to a continuously differentiable f, we obtain the desired result.

The next lemma is also elementary and we omit the proof.

LEMMA 3. *Let $0 < a < b < c < d < \pi$ and suppose f is continuously differentiable with support in the strip $\sigma_1 = \{a \leqslant x \leqslant b\}$, j_0 and j_1 are positive integers, $m > 0$, $M > 0$, and $\varepsilon > 0$. There is a continuously differentiable g, whose support is in $\sigma_2 = \{c \leqslant x \leqslant d\}$, and a positive integer $j_2 > \max(j_0, j_1)$,* *such that* $|g(x,y)| \leqslant m$, *for every* (x,y), $\int\limits_{\sigma_2} g(x,y) D_{j_2 j_2}(x,y)\,dx\,dy \geqslant M$, $\left|\int\limits_{\sigma_2} g(x,y) D_{j_1 j_1}(x,y)\,dx\,dy\right| < \varepsilon$ *and* $\left|\int\limits_{\sigma_1} g(x,y) D_{j_2 j_2}(x,y)\,dx\,dy\right| < \varepsilon$.

We now define an everywhere differentiable f for which localization fails. Let $0 < a_1 < b_1 < \ldots < a_n < b_n < \ldots$, where $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = a < \pi$. For each n, let $m_n = (a-b_n)^2$. By Lemmas 2 and 3 we may obtain, for each n, a continuously differentiable function f_n whose support is in the strip $\sigma_n = \{a_n \leqslant x \leqslant b_n\}$ such that $|f_n(x,y)| < m_n$ for all (x,y), and $\int\limits_{\sigma_n} f_n(x,y) D_{j_n j_n}(x,y)\,dx\,dy > n-1$, for some j_n, where $\{j_n\}$ is increasing, and if $\tau_n = \bigcup\limits_{j\neq n} \sigma_j$, then $\left|\int\limits_{\tau_n} f_n(x,y) D_{j_n j_n}(x,y)\,dx\,dy\right| \leqslant 1$.

Let f be defined by $f(x,y) = f_n(x,y)$ for all $(x,y) \in \sigma_n$, $n = 1, 2, \ldots$, and $f(x,y) = 0$ otherwise. It is clear that

$$\limsup_{n\to\infty} \int f(x,y) D_{nn}(x,y)\,dx\,dy = +\infty,$$

and that f is everywhere differentiable.

THEOREM 3. *There is an everywhere differentiable function f, of 2 variables, the square partial sums of whose double Fourier series does not have the localization property.*

6. For $n = 2$, Tonelli actually showed that localization holds for rectangular partial sums for $f \in W_1^1$. We now note that this does not hold for $n > 2$. For $n = 3$, we give an example of a function $f \in W_2^1$, for which this sort of localization does not hold. The function f is of the form $f(x_1, x_2, x_3) = g(x_1) h(x_2, x_3)$, where $h(x_2, x_3)$ is in W_2^1 and is such that the sequence of square partial sums, $\{s_{jj}(h, (0,0))\}$, of h at $(0,0)$ are unbounded, and where $g(x_1)$ is infinitely differentiable, zero in a neighborhood of $x_1 = 0$, but not identically zero. Then, f is zero in a neighborhood of $(0,0,0)$, but there are increasing sequences $\{n_j\}$ and $\{m_j\}$ such that $\{s_{n_j n_j m_j}(f, (0,0,0))\}$ is unbounded.

We indicate an example of an $h(x_2, x_3)$ of the desired type. For each n, let I_n be a square of center $(0,0)$ and side $2k_n$, with sides parallel to the coordinate axes. Let h_n be continuous, zero off I_n, with $h_n(0,0) = \frac{1}{n}$, and linear on each of the 4 parts into which the lines $x_3 = x_2$ and $x_3 = -x_2$ divide I_n. By properly choosing the sequence $\{k_n\}$ it is easy to see that the function $h = \sum\limits_{n=1}^{\infty} h_n$ has the desired properties. A similar construction applies to each $n > 2$ for $f \in W_{n-1}^1$. Thus localization for square partial sums holds for $f \in W_{n-1}^1$ while localization for rectangular partial sums does not hold.

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