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# DENSE DECOMPOSITIONS OF LOCALLY COMPACT GROUPS

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A subset $E$ of a topological space $X$ is called *non-meager* if it cannot be written as a countable union of sets which are nowhere dense in $X$. $E$ is condensation point dense in $X$ if the intersection of $E$ with every non-empty open set in $X$ is uncountable.

Aiming for a more symmetric decomposition of the reals $R$ than the rationals and irrationals, we ask: can $R$ be decomposed into two condensation point dense sets both of which are non-meager? An affirmative answer is provided by

THEOREM 1. *Every locally compact Abelian group $G$ which is not totally disconnected has a subgroup $H$ for which*

(i) *$H$ and its complement meet every neighborhood of $G$ is a non-meager set,*

(ii) *$H$ and its complement meet every neighborhood of $G$ which is measurable with respect to completed Haar measure in a non-measurable set.*

*In particular, $H$ decomposes $G$ into two non-measurable, non-meager, condensation point dense subsets of the same cardinality.*

Proof. First, we show $G$ has a proper dense subgroup. $G$ is topologically isomorphic with $R^n \times G'$, where $G'$ contains a compact open subgroup ([3], 24.30, p. 389). If $n > 0$ and $Q$ denotes the rationals, $Q^n \times G'$ is a proper dense subgroup of $G$. If $n = 0$, $G$ contains a compact open subgroup, so that the non-zero component $C$ of 0 in $G$ is a compact connected subgroup. It follows that $C$ is a divisible subgroup of $G$([3], 24.25, p. 385), and hence $G = C + A$ for some subgroup $A$ whose intersection with $C$ is 0 ([4], Theorem 2, p. 8). It therefore suffices to show $C$ has a proper dense subgroup $K$, since $K + A$ is then a proper dense subgroup of $G$.

Let $\Gamma$ denote the infinite discrete dual of $C$. Let $D$ denote the proper dense subgroup of the circle $T$ consisting of all $n$-th roots of unity, $n = 1, 2, \ldots$ For fixed $\gamma_0 \in \Gamma - \{0\}$, set $K = \{x \in C : \gamma_0(x) \in D\}$. $K$ is proper subgroup of $C$. For since $C$ is connected and $\gamma_0 \neq 0$, $\gamma_0(C) = T$. $K$ is also dense in $C$. By duality, it suffices to show that for each $x \in C$, $\gamma \in \Gamma$ and $\varepsilon > 0$, there is an $x' \in K$ with $|\gamma(x') - \gamma(x)| < \varepsilon$.

Suppose first that $\gamma_0$ and $\gamma$ are independent elements of $\Gamma$. Let $S$ denote the subgroup generated by $\{\gamma_0, \gamma\}$ and define $f: S \to T$ by $f(n\gamma_0 + m\gamma) = \xi^n \gamma(x)^m$, where $\xi \in D$ is fixed. Because $\gamma_0$ has infinite order, $f$ is a well-defined character on $S$ (cf. [6], 5.1.3, p. 98). Because $\Gamma$ is discrete, $f$ extends to some $\hat{x}' \in \hat{\Gamma}$, $x' \in C$. Since $\gamma_0(x') = \hat{x}'(\gamma_0) = f(\gamma_0) = \xi \in D$, $x' \in K$. But also $|\gamma(x') - \gamma(x)| = |f(\gamma) - \gamma(x)| = 0 < \varepsilon$.

On the other hand, suppose $\gamma$ and $\gamma_0$ satisfy a relation $n\gamma_0 + m\gamma = 0$, where $n\gamma_0, m\gamma \neq 0$. Since $D$ is dense in $T = \gamma_0(C)$, there is a sequence $\{x_k\}$ on $K$ with $\gamma_0(x_k) \to \gamma_0(x)$. Observe that $\overline{\gamma(x_k)^m} = \gamma_0(x_k)^n \to \gamma_0(x)^n = \overline{\gamma(x)^m}$. By taking a subsequence, we may assume $\{\gamma(x_k)\}$ converges to some $a \in T$. $[\bar{a}\gamma(x)]^m = \lim_k \overline{\gamma(x_k)^m} \gamma(x)^m = 1$, so that $\gamma(x) = a\beta$, where $\beta$ is an $m$-th root of unity. Since $m\gamma \neq 0$, we have $\gamma(C) = T$ and $\gamma(y) = \bar{\beta}$ for some $y \in C$. Notice $\gamma_0(y)^n = \overline{\gamma(y)^m} = \beta^m = 1$, so actually $y \in K$. Thus $x_k + y \in K$ and $\gamma(x_k + y) = \gamma(x_k)\beta \to a\beta = \gamma(x)$ as required.

Let $L$ denote a proper dense subgroup of $G$ and $\Lambda$ a non-zero countable subgroup of $G/L$. Embed $\Lambda$ in a countable divisible group $\Omega$, and extend this embedding to a group homomorphism $\vartheta: G/L \to \Omega$([6], 2.5.1, p. 44). Let $H = p^{-1}(\ker\vartheta)$, where $p: G \to G/L$ is projection. $H$ is a subgroup of $G$ containing $L$, and since

$$G/H \approx \frac{G/L}{H/L} = \frac{G/L}{\ker\vartheta} \approx \vartheta(G/L) \subset \Omega,$$

$H$ has countable index in $G$. $H$ is proper since $\Lambda \neq 0$ implies $\ker\vartheta \neq G/L$.

(i) Let $U = x_0 + V$ be a neighborhood in $G$, with $V$ a neighborhood of 0. Let $W \subset V$ be a symmetric open neighborhood of 0. Since $x_0 + H$ is dense in $G$, we may choose a $y_0 \in W \cap x_0 + H$. $x_0 = y_0 + h$ $(h \in H)$ and a computation shows that $U \cap H = h + [H \cap y_0 + V]$. Since $-y_0 \in -W = W \subset \operatorname{int} V$, $y_0 + V$ is a neighborhood of 0. Therefore to show $U \cap H$ is non-meager, we may assume $U$ is a neighborhood of 0.

Choose an open neighborhood $V$ of 0 for which $V - V \subset U$. $G = E + H$, where $E$ is a countable set. For each $x \in E$ choose a $y \in V \cap x + H$, and let $F$ denote the countable set so obtained. For $y \in F$, $h \in H$ and $y + h \in V$, $h \in V - y \subset V - V \subset U$. This means $V = V \cap [E + H] = V \cap [F + H] = \bigcup\{V \cap y + H : y \in F\} \subset \bigcup\{y + U \cap H : y \in F\}$. If $U \cap H$ is a meager in $G$, with $U \cap H$ the countable union of nowhere dense sets $E_i$, $i \in I$, then the open set $V \subset \bigcup\{y + E_i : y \in F, i \in I\}$ is meager in $G$. But this is impossible, since $G$ is a Baire space ([2], p. 249).

The complement $H^c$ is dense in $G$, since if $b \in H^c$, then $b + H \subset H^c$. Choosing $a \in -V \cap H^c$, we have $a + V \cap H \subset U \cap a + H \subset U \cap H^c$, so that $U \cap H^c$ is also category II in $G$ by the above.

(ii) Let $\mathscr{L}$ denote the completion of the $\sigma$-algebra generated by the open sets in $G$ with respect to Haar measure $m$. Let $U$ be a neighborhood in $G$ with $U \in \mathscr{L}$. Since $U \cap H = U \cap (U \cap H^c)^c$, it suffices to show $U \cap H \notin \mathscr{L}$. Since $\mathscr{L}$ is translation invariant, we may assume, exactly as above, that $U$ is a neighborhood of 0. If $V$ is an open neighborhood of 0 with $V - V \subset U$, and $D = \{y_n\} \subset V$ is chosen countable, so that $G = D + H$, then $V$ is contained in the union $\bigcup_n y_n + U \cap H$. If $U \cap H \in \mathscr{L}$, then $m(U \cap H) > 0$, since

$$0 < m(V) \leqslant \sum_n m(y_n + U \cap H) = \sum_n m(U \cap H).$$

But this means $U \cap H + U \cap H \subset H$ has non-empty interior ([1], Theorem 1, p. 648), so that $H^c$ is not dense. Contradiction.

Finally, $H$ and $H^c$ are condensation points dense since every non-meager set in $G$ is uncountable. They have the same cardinality because $H$ has countable index in $G$: for $a \in H^c$, card $H^c \leqslant$ card $G =$ card $G/H$ card $H \leqslant \aleph_0$ card $H =$ card $H =$ card $a + H \leqslant$ card $H^c$.

Since every non-meager subset of $G$ is non-meager in any containing subspace, we observe

COROLLARY 1. *$H$ and its complement are Baire spaces.*

COROLLARY 2. *Every non-empty open set in such a group is the disjoint union of two non-measurable sets.*

Theorem 1 is not generic to groups which are not 0-dimensional, but it may characterize those which are non-discrete. For we have

THEOREM 2. *The conclusion of Theorem* 1 *holds if $G$ is a non-discrete LCA group which is either*: (i) *separable,* (ii) *compact,* (iii) *torsion free and divisible or* (iv) *compactly generated.*

Proof. As the proof of Theorem 1 reveals, it suffices in each case to construct a proper dense subgroup of $G$. We may assume $G$ is 0-dimensional. If $G$ is separable, the group generated by a countable dense set is proper since $G$ is uncountable ([3] 4.26, p. 31). If $G$ is compact, its dual $\Gamma$ is a discrete torsion group, and hence the direct sum of its $p$-primary components $\Gamma_p$, $p$ prime ([4], p. 5). Thus $G \approx \prod_p \hat{\Gamma}_p$. If $\Gamma_p$ is non-zero for infinitely many $p$, $\bigoplus_p \hat{\Gamma}_p$ is a proper dense subgroup of $G$. If only finitely many $\Gamma_p$'s are non-zero, at least one $\Gamma_{p_0}$ is infinite, since $G$ is non-discrete. Refer now to ([3], 25.22, p. 412). $\Gamma_{p_0}$ contains a subgroup $B$ isomorphic to a direct sum $\bigoplus_{i \in I} Z/p^{n_i}Z$ whose annihilator, Ann $B$, is a compact, pure subgroup of $\hat{\Gamma}_{p_0}$. Ann $B$ is an algebraic direct summand of $\hat{\Gamma}_{p_0}$ ([3], 25.21, p. 410), so that $\Gamma_{p_0} = H + \text{Ann } B$ for a subgroup $H$ whose intersection with Ann $B$ is 0. If Ann $B \neq 0$, Ann $B$ is topologically isomorphic with $\Delta_p^a$, the direct product of the $p$-adic integers with itself $a \neq 0$ number of times. Since $\Delta_p$ is monothetic and non-discrete ([3], 10.6, p. 111), it has a proper dense subgroup. It follows that $\Gamma_{p_0}$ has a proper dense subgroup. If Ann $B = 0$, $\Gamma_{p_0} = B \approx \bigoplus_i Z/p^{n_i}Z$, so that $\hat{\Gamma}_{p_0} \approx \prod_{i \in I} Z/p^{n_i}Z$. $I$ is infinite because $\Gamma_{p_0}$ is, and $\bigoplus_i Z/p^{n_i}Z$ provides a proper dense subgroup of $\hat{\Gamma}_{p_0}$. In either case, then, $G \approx \prod_{p \neq p_0} \hat{\Gamma}_p \times \hat{\Gamma}_{p_0}$ has a proper dense subgroup.

If $G$ is generated by a compact neighborhood $V$ of 0, then $G$ contains a closed subgroup $H$ such that $H \cap V = 0$ and $G/H$ is compact ([6], 2.4.2, p. 41). Choose a neighborhood $U$ of 0 such that $U - U \subset V$. Since $G$ is non-discrete and the quotient map $p : G \to G/H$ injects $U$ into $G/H$, $G/H$ is a non-discrete compact group. By the above, it has a proper dense subgroup $K$. Plainly, $p^{-1}(K)$ is a proper dense subgroup of $G$.

If $G$ is torsion-free, divisible and 0-dimensional, $G$ is topologically isomorphic to the product of a direct sum of copies of the rationals and a group $E$ which is the minimal divisible extension of a group of the form $\prod_p \Delta_p^{a_p}$, $p$ prime, $a_p$ a cardinal number (cf. [3], 25.33, p. 421]). If $\Omega_p$ denotes the $p$-adic number field and $\Omega_p^{a_p\prime}$ denotes the minimal divisible extension of $\Omega_p^{a_p}$, then $E$ is the local direct product of the groups $\Omega_p^{a_p\prime}$ relative to the compact open subgroups $\Delta_p^{a_p}$; that is, $E = \{\{x_p\} \in \prod_p \Omega_p^{a_p\prime} : x_p \in \Delta_p^{a_p}$ for all but finitely many $p\}$ ([3], 25.32 (d), p. 420). If the number of non-zero $a_p$'s is infinite, $\bigoplus_p \Omega_p^{a_p\prime}$ is a proper dense subgroup of $E$. If not, $E = \prod_p \Omega_p^{a_p\prime}$, and $E$ will have a proper dense subgroup if some $\Omega_p^{a_p\prime}$ does, $a_p' \neq 0$. This follows essentially because $\Omega_p^{a_p\prime} = \bigcup_{k=-\infty}^{\infty} \Lambda_k^{a_p}$ ([3], 25.32 (c), p. 420), where the $\Lambda_k$ are defined as in ([3], 10.4, p. 110), because each $\Lambda_k$ is monothetic and compact ([3], p. 111) and because $\Lambda_k \subset \Lambda_m$ if $k \geqslant m$. In either case, it follows that $G$ has a proper dense subgroup.

Other results are possible. For example, Theorem 1 holds if $G$ is a torsion group one of whose $p$-primary components $G_p$ contains a proper dense subgroup. For the structure theorem ([5], 3.21, p. 494) implies that $G$ is topologically isomorphic to $\{\{x_p\} \in \prod_p G_p : x_p \in K_p$ for all but finitely many $p\}$, where $\{K_p \subset G_p\}$ is a family of open subgroups. In particular, if $H_{p_0}$ is a proper dense subgroup of some $G_{p_0}$, $\bigoplus_{p \neq p_0} G_p \oplus H_{p_0}$ is a proper dense subgroup of $G$. Theorem 1 also holds if $G$ contains a precompact divisible subgroup $H$. For a computation shows that $\bar{H}$ is divisible, and hence a compact, infinite algebraic direct summand of $G$([4], p. 8). Since Theorem 2 implies $\bar{H}$ has a proper dense subgroup, $G$ does also. Again Theorem 1 holds if $H = \{x \in G : nx \to 0\}$ is infinite and precompact. For if $G$ is 0-dimensional, $H$ is a closed, pure subgroup of $G$; hence an infinite compact direct summand ([3], p. 410). We are lead to conjecture that Theorem 1 holds for all non-discrete LCA groups. Of course in view of the argument in Theorem 1, the problem is whether every non-discrete LCA group has a proper dense subgroup (**P 764**). Finishing off the remaining 0-dimensional cases will undoubtedly involve a structure theory for non-compact 0-dimensional groups.

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