COLLOQUIUM MATHEMATICUM

VOL. XXIV 1972 FASC. 2

ON UNIFORM SYMMETRIC DERIVATIVES

BY

N. K. KUNDU (ARAMBACH, WEST BENGAL)

Let f be a real function defined in an open interval I and let $I_0 = [a, b]$ be a closed subinterval of I. Let

$$\psi(x, h) = \frac{f(x+h)-f(x-h)}{2h}, \quad x \in I_0, x \pm h \in I, h \neq 0.$$

Then $\limsup_{h\to 0} \psi(x, h)$ and $\liminf_{h\to 0} \psi(x, h)$ are called the *upper* and the *lower symmetric derivative* of f at x and are denoted by $\bar{f}^{(1)}(x)$ and $\underline{f}^{(1)}(x)$, respectively [8], and $\lim_{h\to 0} \psi(x, h)$, if exists, is called the *symmetric derivative* [2] or *Schwarz derivative* [10] of f at x and is denoted by $f^{(1)}(x)$.

Let us suppose that $\bar{f}^{(1)}(x)$ is finite for each $x \in I_0$. Write

$$\varphi(x, h) = \frac{(x+h)-f(x-h)}{2h} - \bar{f}^{(1)}(x),$$

where h may be taken positive without loss of generality. Now, for each $x \in I_0$ and for each $\varepsilon > 0$, there is a $\delta(x) > 0$ such that $\varphi(x, h) < \varepsilon$ whenever $0 < h < \delta(x)$ and $x \pm h \in I$. It may happen that, for a fixed $\varepsilon > 0$, $\delta(x)$ has no positive lower bound in I_0. If, however, $\delta(x)$ has a positive lower bound in I_0 for each $\varepsilon > 0$, then $\bar{f}^{(1)}$ is said to be the *uniform upper symmetric derivative* of f in I_0.

$\xi \in I_0$ is said to be a *point* of uniform upper symmetric defferentiability of f if for each $\varepsilon > 0$ there is a neighbourhood of ξ in which $\delta(x)$ has a positive lower bound. So a point in every neighbourhood of which $\delta(x)$ has no positive lower bound for some sufficiently small $\varepsilon > 0$ is said to be a point of a non-uniform upper symmetric differentiability of f. It is clear that $\bar{f}^{(1)}$ is uniform or non-uniform at $\xi \in I_0$ according to as $\limsup_{(x,h)\to(\xi,0)} \varphi(x, h) = 0$ or > 0.

Letting

$$\overline{W}(\xi) = \limsup_{(x,h)\to(\xi,0)} \varphi(x, h),$$

we shall term $\overline{W}(\xi)$ as the *measure* of the non-uniformity of $\bar{f}^{(1)}$ at ξ and the function $\overline{W}(x)$ is defined as the measure function of the non-uniformity of $\bar{f}^{(1)}$. Similarly, for finite $\underline{f}^{(1)}$, we define

$$\underline{W}(\xi) = \liminf_{(x,h)\to(\xi,0)} \left(\frac{f(x+h)-f(x-h)}{2h} - \underline{f}^{(1)}(x)\right).$$

It is clear that if the symmetric derivative $f^{(1)}$ exists finitely in I_0, then a point $\xi \in I_0$ is a point of uniform symmetric (or Schwarz) differentiability of f if and only if $\bar{f}^{(1)}$ and $\underline{f}^{(1)}$ are both uniform at ξ [6].

Recently, Mukhopadhyay studied some properties of symmetric derivative by help of the notion of uniform symmetric differentiability [6, 7]. He showed that for a continuous and symmetric differentiable function f, if ξ is a point of uniform symmetric differentiability of f, then the symmetric derivative $f^{(1)}$ is continuous at ξ, and that the uniform symmetric differentiability of f in an interval implies the uniform differentiability of f therein [6]. He also proved that for a continuous and symmetric differentiable function f, the set of points of the non-uniform symmetric differentiability of f is of the first category [7], and that if $f^{(1)}$ is continuous, then f is uniformly symmetric differentiable [9]. He also raised the question whether the uniform symmetric differentiability of f implies the continuity of f. Swetits showed that under certain conditions the uniform symmetric differentiability of f implies the continuity of f' [11]. In the present paper, these results are sharpened and some further consequences are studied by help of the notions of the uniform upper and lower symmetric differentiability of f. It may be of interest to note that Manna obtained analogous generalization of the concept of uniform differentiability by considering Dini derivates [5].

THEOREM 1. *The function $\overline{W}_{(x)}$ is upper semi-continuous on the interval $I_0 = [a, b]$.*

Proof. Let $\xi \in I_0$ and let $\varepsilon > 0$ be arbitrary. Since

$$\overline{W}(\xi) = \limsup_{(x,h)\to(\xi,0)} \varphi(x, h),$$

there exists a neighbourhood D of ξ and a positive number δ such that

$$\text{(1)} \qquad \varphi(x, h) < \overline{W}(\xi) + \frac{\varepsilon}{2} \qquad \text{for all } x \in D \text{ and } 0 < h < \delta.$$

Suppose that there is $x' \in D$ such that $\overline{W}(x)' \geqslant \overline{W}(\xi) + \varepsilon$. Then $\overline{W}(x') > \overline{W}(\xi) + \varepsilon/2$. This implies that there are points $x \in D$ and h, $0 < h < \delta$, such that

$$\text{(2)} \qquad \varphi(x, h) > \overline{W}(\xi) + \frac{\varepsilon}{2}.$$

Since (1) and (2) are contradictory, we conclude that

$$\overline{W}(x) < \overline{W}(\xi) + \varepsilon \quad \text{for all } x \in D.$$

Hence $\overline{W}(x)$ is upper semi-continuous at ξ. This completes the proof.

COROLLARY 1. *If $\overline{W}(x)$ is unbounded from above on the closed interval I_0, then there exists at least one point ξ, where $\overline{W}(\xi) = \infty$.*

COROLLARY 2. *The set of points, where $\overline{W}(x) = \infty$, is closed.*

THEOREM 2. *A necessary and sufficient condition that $\varphi(x, h)$ is bounded from above for all x in $[a, b]$ and for all h, where $0 < |h| < \delta$ for some sufficiently small δ, is that $\overline{W}(x) \neq \infty$ in $[a, b]$.*

Proof. If $\varphi(x, h)$ is bounded from above, $\overline{W}(x)$ is also bounded from above. So we may suppose that $\overline{W}(x) < \infty$ for all $x \in [a, b]$. Then, for every $x_0 \in [a, b]$, there is a neighbourhood Dx_0 of x_0 and a $\delta(x_0)$ such that $\varphi(x, h) < \overline{W}(x_0) + \varepsilon$ for all $x \in Dx_0$ and for all $h \neq 0$, $|h| < \delta(x_0)$. From the family of neighbourhoods $\{Dx_0 \colon x_0 \in [a, b]\}$ we can choose a finite number, say $Dx_1, Dx_2, \ldots, Dx_n$ such that

$$[a, b] \subset \bigcup_{r=1}^{n} Dx_r.$$

Let $\delta = \min[\delta(x_1), \delta(x_2), \ldots, \delta(x_n)]$ and $k = \max[\overline{W}(x_1), \overline{W}(x_2), \ldots, W(x_n)]$. Then $\varphi(x, h) < k + \varepsilon$ for all $x \in [a, b]$ and for all h, $0 < |h| < \delta$.

THEOREM 3. *If f is continuous in a neighbourhood of ξ and if $\overline{W}(\xi) = 0$, then $\bar{f}^{(1)}$ is lower semi-continuous at ξ.*

Proof. Since $\overline{W}(\xi) = 0$, corresponding to $\varepsilon > 0$ there is a $\delta > 0$ such that

$$(1) \qquad \varphi(x, h) < \frac{\varepsilon}{3} \text{ for all } x \in (\xi - \delta, \xi + \delta) \text{ and for all } h,\ 0 < |h| < \delta.$$

Then we may suppose that f is continuous in $(\xi - \delta, \xi + \delta)$. Since $\limsup_{h \to 0} \varphi(\xi, h) = 0$, there is h_1, $0 < |h_1| < \frac{1}{2}\delta$, such that

$$(2) \qquad |\varphi(\xi, h_1)| < \frac{\varepsilon}{3}.$$

And since f is continuous in $(\xi - \delta, \xi + \delta)$, the function ψ, defined by

$$\psi(x) = \frac{f(x + h_1) - f(x - h_1)}{2h_1},$$

is also continuous at ξ and so there is a δ_0, $0 < \delta_0 < \delta$, such that

$$(3) \qquad |\psi(x) - \psi(\xi)| < \frac{\varepsilon}{3} \quad \text{for all } x \in (\xi - \delta_0, \xi + \delta_0).$$

From (1), (2) and (3) we infer that

$$\bar{f}^{(1)}(x)-\bar{f}^{(1)}(\xi) = \bar{f}^{(1)}(x)-\frac{f(x+h_1)-f(x-h_1)}{2h_1}+\frac{f(x+h_1)-f(x-h_1)}{2h_1}+$$

$$+\frac{f(\xi+h_1)-f(\xi-h_1)}{2h_1}-\bar{f}^{(1)}(\xi)-\frac{f(\xi+h_1)-f(\xi-h_1)}{2h_1}$$

$$= -\varphi(x, h_1)+\varphi(\xi, h_1)+\psi(x)-\psi(\xi)$$

$$> -\frac{\varepsilon}{3}-\frac{\varepsilon}{3}-\frac{\varepsilon}{3} = -\varepsilon \quad \text{for all } x\in(\xi-\delta_0, \xi+\delta_0).$$

Hence $f^{(1)}$ is lower semi-continuous at ξ, which completes the proof.

COROLLARY. *If f is continuous and $f^{(1)}$ exists in some neighbourhood of ξ and if ξ is a point of uniform symmetric differentiability of f, then $f^{(1)}$ is continuous at ξ.*

Note. This result is proved in [6].

THEOREM 4. *Let f be continuous in some neighbourhood of ξ. Then a necessary and sufficient condition that $\bar{f}^{(1)}$ be continuous at ξ is that*

$$\lim_{(x,h)\to(\xi,0)}\left\{\frac{f(x+h)-f(x-h)}{2h}-\bar{f}^{(1)}(x)\right\} = 0.$$

Proof. Let f be continuous in $(\xi-\delta, \xi+\delta)$. Let us suppose that $\bar{f}^{(1)}$ is continuous at ξ. We shall first show that for each $x\in[\xi-\frac{1}{2}\delta, \xi+\frac{1}{2}\delta]$ and for each h, $0<|h|<\frac{1}{2}\delta$, one of the following must be true:

(i) $$D^+f(x+\vartheta h)\leqslant\frac{f(x+h)-f(x-h)}{2h}\leqslant D_-f(x+\vartheta h),$$

(ii) $$D_+f(x+\vartheta h)\geqslant\frac{f(x+h)-f(x-h)}{2h}\geqslant D^-f(x+\vartheta h),$$

where $-1<\vartheta<1$.

Let $c\in[\xi-\frac{1}{2}\delta, \xi+\frac{1}{2}\delta]$, $0<|h|<\frac{1}{2}\delta$, and let

$$\psi(x) = f(x)-\frac{f(c+h)-f(c-h)}{2h}x.$$

Then ψ is continuous in $[c-h, c+h]$. Also $\psi(c-h) = \psi(c+h)$. Let M and m be the upper and the lower bounds of ψ in $[c-h, c+h]$. If $M=m$, then ψ is constant in $[c-h, c+h]$ and hence the conclusion remains valid. So we suppose that at least one of M and m is different from $\psi(c-h)$. If $M\neq\psi(c-h)$, then there is a ϑ, $-1<\vartheta<1$, such that $\psi(c+\vartheta h) = M$ and hence

$$D^+\psi(c+\vartheta h)\leqslant 0\leqslant D_-\psi(c+\vartheta h),$$

i.e., $$D^+f(c+\vartheta h)\leqslant\frac{f(c+h)-f(c-h)}{2h}\leqslant D_-f(c+\vartheta h).$$

Similarly, if $m \neq \psi(c-h)$, then, for some ϑ $(-1<\vartheta<1)$

$$D_+f(c+\vartheta h) \geqslant \frac{f(c+h)-f(c-h)}{2h} \geqslant D^-f(c+\vartheta h).$$

Thus, for each $x \in [\xi-\frac{1}{2}\delta, \xi+\frac{1}{2}\delta]$ and for each h, $0<|h|<\frac{1}{2}\delta$, at least one of the following must be true:

$$\text{(i)}' \; D^+f(x\vartheta+h)-\bar{f}^{(1)}(x) \leqslant \frac{f(x+h)-f(x-h)}{2h} -\bar{f}^{(1)}(x) \leqslant D_-f(x+\vartheta h)-\bar{f}^{(1)}(x),$$

$$\text{(ii)}' \; D_+f(x+\vartheta h)-\bar{f}^{(1)}(x) \geqslant \frac{f(x+h)-f(x-h)}{2h} -\bar{f}^{(1)}(x) \geqslant D^-f(x+\vartheta h)-\bar{f}^{(1)}(x).$$

Since the function $\bar{f}^{(1)}(x)$ is continuous at ξ, it follows from [3] that D^+f, D_+f, D^-f and D_-f are also continuous at ξ and $\bar{f}^{(1)}(\xi) = D^+f(\xi) = D_+f(\xi) = D^-f(\xi) = D_-f(\xi)$. Hence letting $x \to \xi$, $h \to 0$, we get

$$\lim_{(x,h)\to(\xi,0)} \left\{\frac{f(x+h)-f(x-h)}{2h} -\bar{f}^{(1)}(x)\right\} = 0.$$

To prove the converse, suppose

$$\lim_{(x,h)\to(\xi,0)} \left\{\frac{f(x+h)-f(x-h)}{2h} -\bar{f}^{(1)}(x)\right\} = 0.$$

Then, for every $\varepsilon>0$, there is a neighbourhood D_1 of ξ and a δ_1, $0<\delta_1<\frac{1}{2}\delta$, such that $|\varphi(x,h)|<\varepsilon/3$, whenever $x\in D_1$ and $0<|h|<\delta_1$. Fix h_1, $0<|h_1|<\delta_1$. Then the function ψ, where

$$\psi(x) = \frac{f(x+h_1)-f(x-h_1)}{2h_1},$$

is continuous at ξ and hence there is a neighbourhood D_2 of ξ such that $|\psi(x)-\psi(\xi)|<\varepsilon/3$ whenever $x\in D_2$. Hence, for $x \in D = D_1 \cap D_2$, we get as in Theorem 3

$$|\bar{f}^{(1)}(x)-\bar{f}^{(1)}(\xi)| \leqslant |\varphi(x,h_1)|+|\varphi(\xi,h_1)|+|\psi(x)-\psi(\xi)| < \varepsilon$$

showing that $\bar{f}^{(1)}$ is continuous at ξ. This completes the proof.

COROLLARY 1. *Let f be continuous and $f^{(1)}$ exist in some neighbourhood of ξ. Then ξ is a point of uniform symmetric differentiability of f if and only if ξ is a point of continuity of $f^{(1)}$.*

It is known that if f is continuous and $f^{(1)}$ exists in some neighbourhood of ξ and if ξ is a point of uniform symmetric differentiability of f, then $f'(\xi)$ exists [7]. So from this corollary we get

COROLLARY 2. *If f is continuous and $f^{(1)}$ exists in some neighbourhood of ξ and if $f^{(1)}$ is continuous at ξ, then $f'(\xi)$ exists.*

Note. This result is obtained by Aull [1].

THEOREM 5. *If $f^{(1)}$ exists and is bounded in some neighbourhood of ξ and if ξ is a point of uniform symmetric differentiability of f, then f is continuous in some neighbourhood of ξ.*

Proof. Let $\varepsilon > 0$ be arbitrary. Since ξ is a point of uniform symmetric differentiability of f, there is a $\delta > 0$ such that

$$(1) \qquad |\varphi(x, h)| < \varepsilon \quad \text{for all } x \in (\xi - \delta, \xi + \delta) \text{ and for all } h,\ 0 < |h| < \delta.$$

We may suppose that $f^{(1)}$ is bounded in $(\xi - \delta, \xi + \delta)$. So there is $M > 0$ such that

$$(2) \qquad |f^{(1)}(x)| < M \quad \text{for all } x \in (\xi - \delta, \xi + \delta).$$

If possible, suppose that there is no neighbourhood of ξ in which f is continuous. So, there is a point $\xi_1 \in (\xi - \frac{1}{2}\delta, \xi + \frac{1}{2}\delta)$ and a positive number ε_1 such that the relation

$$(3) \qquad |f(x_n) - f(\xi_1)| > \varepsilon_1$$

holds for a sequence of points $\{x_n\}$ such that $x_n \to \xi_1$ as $n \to \infty$. We may suppose $\{x_n\} \subset (\xi - \frac{1}{2}\delta, \xi + \frac{1}{2}\delta)$. Now, from (2) and (3), we have

$$(4) \qquad \left|\varphi\left(\frac{z_n + \xi_1}{2}, \frac{x_n - \xi_1}{2}\right)\right| = \left|\frac{f(x_n) - f(\xi_1)}{x_n - \xi_1} - f^{(1)}\left(\frac{x_n + \xi_1}{2}\right)\right|$$

$$\geqslant \frac{\varepsilon_1}{|x_n - \xi_1|} - M \quad \text{for all } n.$$

Since x_n can be taken sufficiently near to ξ_1, (1) and (4) are contradictory. This completes the proof.

APPLICATION OF COROLLARIES 1 AND 2 OF THEOREM 4

COROLLARY. *If $f^{(1)}$ exists and is bounded in some neighbourhood of ξ and if ξ is a point of uniform symmetric differentiability of f, then $f^{(1)}$ is continuous at ξ and $f'(\xi)$ exists.*

THEOREM 6. *Let f be such that*

(i) *$f^{(1)}$ exists in some neighbourhood of ξ;*

(ii) *if*

$$\limsup_{x \to \xi} |f(x)| = \infty,$$

then either there exists $x_1 < \xi$ such that f is locally bounded in $[x_1, \xi)$ or there exists $x_2 > \xi$ such that f is locally bounded in $(\xi, x_2]$;

(iii) *ξ is a point of uniform symmetric differentiability of f.*

Then $f'(\xi)$ exists.

Proof. Suppose that ξ is a point of uniform symmetric differentiability of f. Then we assert that

$$\limsup_{x\to\xi} |f(x)| \neq \infty.$$

For, if possible, let

$$\limsup_{x\to\xi} |f(x)| = \infty.$$

Since ξ is a point of uniform symmetric differentiability of f, there is a δ, $0<\delta<1$, such that

$$(1) \qquad |\varphi(x, h)| = \left|\frac{f(x+h)-f(x-h)}{2h} - f^{(1)}(x)\right| < 1$$

for all $x\in(\xi-\delta, \xi+\delta)$ and for all h, $0<|h|<\delta$.

Let us suppose by Condition (ii) that f is locally bounded in $[x_1, \xi) \subset (\xi-\frac{1}{2}\delta, \xi)$.

Let $x_0 = (5\xi+x_1)/6$. Then f is bounded in $[x_1, x_0]$. So there is $M>0$ such that $|f(x)| \leqslant M$ for all $x\in[x_1, x_0]$.

Fix h_0, $0<|h_0|<(\xi-x_1)/3$, such that

$$|f(x_0+h_0)| > 1+M+|f^{(1)}(x_0)|.$$

Then

$$(2) \qquad |\varphi(x_0, h_0)| \geqslant \left|\frac{f(x_0+h_0)-f(x_0-h_0)}{2h_0}\right| - |f^{(1)}(x_0)| > 1.$$

Since (1) and (2) are contradictory, we conclude

$$\limsup_{x\to\xi} |f(x)| < \infty.$$

So there is a $\delta>0$ and a positive number M such that

$$|f(x)| < M \quad \text{for all } x\in(\xi-\delta, \xi+\delta).$$

Since ξ is a point of uniform symmetric differentiability of f, there is δ_0, $0<\delta_0<\frac{1}{2}\delta$, such that $|\varphi(x, h)|<1$ for all $x\in(\xi-\delta_0, \xi+\delta_0)$ and all h, $0<|h|<\delta_0$.

Fix h_1, $0<|h_1|<\delta_0$. Then, for all $x\in(\xi-\delta_0, \xi+\delta_0)$, we have

$$|f^{(1)}(x)| < 1+\left|\frac{f(x+h_1)-f(x-h_1)}{2h_1}\right| < 1+\frac{M}{|h_1|}.$$

Thus $f^{(1)}$ is bounded in $(\xi-\delta_0, \xi+\delta_0)$. Hence, by Theorem 5, f is continuous in a neighbourhood of ξ.

So, by Corollaries 1 and 2 of Theorem 4, $f'(\xi)$ exists.

Note. The above result sharpens a result of Swetits [11].

THEOREM 7. *If f is continuous on $I_0 = [a, b]$, then the set*

$$\{x\colon x \in I_0;\ \overline{W}(x) > \sup_{x \in I_0} [\bar{f}^{(1)}(x) - \bar{f}^{(1)}(x)]\}$$

is an F_σ-set of the first category.

Proof. Let

$$K = \sup_{x \in I_0} [\bar{f}^{(1)}(x) - \underline{f}^{(1)}(x)]$$

and let $\alpha > K$. Then since $\overline{W}(x)$ is upper semi-continuous, the set $S_\alpha = \{x\colon x \in I_0; \overline{W}(x) \geqslant \alpha\}$ in closed. Suppose, if possible, that the set S_α is not non-dense in $I_0 = [a, b]$. Then there exists a subinterval $[a', b'] \subset [a, b]$ in which the set S_α is every where dense and since the S_α is closed, $[a', b'] \subset S_\alpha$. Let ξ be any point of the interval (a', b'). Then $\overline{W}(\xi) \geqslant \alpha$. Choose $K < \alpha' < \alpha$ and a positive null sequence $\{\delta_n\}$. Then since $\overline{W}(\xi) > \alpha'$, there is a point ξ' in some neighbourhood of ξ contained in (a', b') and a number h_1, $0 < h_1 < \delta_1$, such that

$$\frac{f(\xi' + h_1) - f(\xi' - h_1)}{2h_1} - \bar{f}^{(1)}(\xi') > \alpha'.$$

Hence, there is an h_2, $0 < h_2 < h_1$, such that

$$\frac{f(\xi' + h_1) - f(\xi' - h_1)}{2h_1} - \frac{f(\xi' + h_2) - f(\xi' - h_2)}{2h_2} > \alpha'.$$

Since f is continuous on $I_0 = [a, b]$, both the functions

$$\frac{f(x + h_1) - f(x - h_1)}{2h_1} \quad \text{and} \quad \frac{f(x + h_2) - f(x - h_2)}{2h_2}$$

are continuous at ξ'. So there exists a small neighbourhood D_1 of ξ' contained in $(a,' b')$ such that

$$\frac{f(x + h_1) - f(x - h_1)}{2h_1} - \frac{f(x + h_2) - f(x - h_2)}{2h_2} > \alpha' \quad \text{for all } x \in D_1.$$

Again since $\overline{W}(\xi') \geqslant \alpha$ corresponding to the positive number $\delta_2' = \min[\delta_2, h_2]$, there exists a point $\xi'' \in D_1$ and a number h_3, $0 < h_3 < \delta_2'$, such that

$$\frac{f(\xi'' + h_3) - f(\xi'' - h_3)}{2h_3} - \bar{f}^{(1)}(\xi'') > \alpha'.$$

In a similar way, it is possible to find a neighbourhood $D_2 \subset D_1$ of ξ'' such that

$$\frac{f(x + h_3) - f(x - h_3)}{2h_3} - \frac{f(x + h_4) - f(x - h_4)}{2h_4} > \alpha' \quad \text{for all } x \in D_2$$

$$\text{and} \quad 0 < h_4 < h_3.$$

Proceeding in this way, we can select a decreasing sequence of neighbourhoods $\{D_n\}$ such that, for every $x \in D_n$,

$$\frac{f(x+h_{2n-1})-f(x-h_{2n-1})}{2h_{2n-1}} - \frac{f(x+h_{2n})-f(x-h_{2n})}{2h_{2n}} > a' \quad \text{for } x \in D_n,$$

$$0 < h_{2n} < h_{2n-1} < \delta_n' = \min[\delta_n, h_{2n-2}].$$

The neighbourhoods D_n can be so chosen that there exists a point η which belongs to each D_n and η is a point such that

$$\frac{f(\eta+h_{2n-1})-f(\eta-h_{2n-1})}{2h_{2n-1}} - \frac{f(\eta+h_{2n})-f(\eta-h_{2n})}{2h_{2n}} > a'$$

for all positive integer n.

Hence we have $[\bar{f}^{(1)}(\eta)-\underline{f}^{(1)}(\eta)] \geqslant a'$. Again since $\eta \in [a, b]$, $[\bar{f}^{(1)}(\eta)-$ $-\underline{f}^{(1)}(\eta)] \leqslant K < a'$.

This is a contradiction. Hence we conclude that the set S_a is closed and non-dense in $[a, b]$. Let us choose a sequence $\{a_n\}$, $a_n > K$ and $a_n \to K$. Then the set

$$\{x\colon x \in [a, b];\ \overline{W}(x) > K\} = \bigcup_{n=1}^{\infty} \{x\colon x \in [a, b];\ \overline{W}(x) \geqslant a_n\}$$

is an F_σ-set of the first category.

Corollary 1. *For a continuous function f, if $f^{(1)}$ exists, then the set of points where f is not uniformly symmetric differentiable, is a set of the first category.*

Note. This result is also proved in [7].

Since it is known that for a continuous function f, if $f^{(1)}$ exists, then f' also exists almost everywhere [2] and if, moreover, ξ is a point of uniform symmetric differentiability of f, then $f'(\xi)$ exists (Corollaries 1 and 2 of Theorem 4), we conclude from the above given corollary that, for a continuous function f, if $f^{(1)}$ exists everywhere in an interval, then the set of points where f' does not exist is of measure zero and of the first category.

Example 1. Let a function g be defined in $(0, 1]$ in the following way:

$$g(x) = \begin{cases} x & \text{if } x \in \bigcup_{n=1}^{\infty} \left(\frac{1}{2n}, \frac{1}{2n-1}\right], \\ 0 & \text{if } x \in \bigcup_{n=1}^{\infty} \left(\frac{1}{2n+1}, \frac{1}{2n}\right]. \end{cases}$$

Let

$$f(x) = \begin{cases} \int_0^x g(t)\,dt & \text{if } x \in (0, 1], \\ 0 & \text{if } \in [-1, 0]. \end{cases}$$

Then f is continuous in $[-1, 1]$. Also $f^{(1)}$ exists in $(-1, 1)$ and

$$f^{(1)}(x) = \begin{cases} 0 & \text{if } x \in (-1, 0] \cup \left\{ \bigcup_{n=1}^{\infty} \left(\frac{1}{2n+1}, \frac{1}{2n} \right) \right\}, \\ x & \text{if } x \in \bigcup_{n=1}^{\infty} \left(\frac{1}{2n}, \frac{1}{2n-1} \right), \\ \frac{x}{2} & \text{if } x \in \left\{ \frac{1}{n}; n = 2, 3, \ldots \right\}. \end{cases}$$

Clearly, $x = 0$ is a point of uniform symmetric differentiability of f. But f' does not exist at each of the points $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ and hence each of the points $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ is a point of non-uniform symmetric differentiability of f.

This example shows that if f is continuous and $f^{(1)}$ exists in a certain neighbourhood of a point ξ and if ξ is a point of uniform symmetric differentiability of f, then ξ need not be a point of uniform differentiability of f [4]; but if f' exists in a certain neighbourhood of ξ, then ξ must be a point of uniform differentiability of f.

Example 2. Let a function g be defined in $(0, 1]$ in the following way:

$$g(x) = \begin{cases} \frac{2^{2n+4}}{3} \left(x - \frac{1}{2^{n+1}} \right) & \text{if } x \in \left(\frac{1}{2^{n+1}}, \frac{3}{2^{n+2}} \right], \\ \frac{2^{2n+4}}{3} \left(\frac{1}{2^n} - x \right) & \text{if } x \in \left(\frac{3}{2^{n+2}}, \frac{1}{2^n} \right], \quad n = 0, 1, 2, \ldots \end{cases}$$

Let

$$f(x) = \begin{cases} g(x) & \text{if } x \in (0, 1], \\ g(-x) & \text{if } x \in [-1, 0), \\ 0 & \text{if } x = 0. \end{cases}$$

Then f is continuous in $[-1, 1]$ except at $x = 0$. Also $f^{(1)}$ exists in $(-1, 1)$ and $\limsup_{x \to 0} |f(x)| = \infty$. Finally, f is locally bounded in every deleted neighbourhood of 0 and hence f satisfies Conditions (i) and (ii) of Theorem 6. But $f'(0)$ does not exist which shows that Condition (iii) of Theorem 6 cannot be omitted.

I wish to express my sincere gratitude to Dr. S. N. Mukhopadhyay for his kind help and suggestions in the preparation of the paper.

REFERENCES

[1] C. E. Aull, *The first symmetric derivative*, The American Mathematical Monthly 74 (1967), p. 708-711.

[2] A. Khintchine, *Recherches sur la structure des fonctions mesurables*, Fundamenta Mathematicae 9 (1927), p. 212-279.

[3] N. K. Kundu, *On some properties of symmetric derivatives of measurable functions*, submitted for publication in the American Mathematical Monthly.

[4] B. K. Lahiri, *On non-uniform differentiability*, The American Mathematical Monthly 67 (1960), p. 649-652.

[5] N. C. Manna, *On uniform Dini derivatives*, Colloquium Mathematicum 21 (1970), p. 95-100.

[6] S. N. Mukhopadhyay, *On Schwarz differentiability, I*, Proceedings of the National Academy of Sciences, India, 36 (1966), Section A.

[7] —, *On Schwarz differentiability II*, Revue Roumaine de Mathématiques Pures et Appliquées 9 (1964), p. 859-862.

[8] —, *On Schwarz differentiability, IV*, Acta Mathematica Academiae Scientiarum Hungaricae 17 (1966), p. 129-136.

[9] —, *On Schwarz differentiability, V*, The American Mathematical Monthly 74 (1967), p. 542-544.

[10] I. P. Natanson, *Theory of functions of a real variable*, Vol. II, New York 1960, p. 36.

[11] John Swetits, *A note on Schwarz differentiability*, The American Mathematical Monthly 75 (1968), p. 1093-1095.

DEPARTMENT OF MATHEMATICS
NETAJI COLLEGE, ARAMBAGH

Reçu par la Rédaction le 27. 6. 1970;
en version modifiée le 18. 1. 1971