257. *A linear form result in the geometry of numbers*, Math. Mag. 45 (1972), pp. 152–154.
258. *On some exponential sums related to Kloosterman sums*, Acta Arith. 21 (1972), pp. 65–69.
259. *On rational functions representing all residues* mod p, J. London Math. Soc. 5 (1972), pp. 166–168.
260. *On Salié's sum*, Glasgow Math. J. 14 (1972), pp. 25–26.
261. *On the representation of positive integers as sums of three cubes of positive rational numbers*, Mathematika 18 (1971), pp. 98–99.
262. *A finite evaluation of a special exponential sum*, Proc. Camb. Phil. Soc. 71 (1972), pp. 75–78.
263. *Systems of congruences*, Canad. Math. Bull., to appear.
264. *Rational functions representing all residues* mod p, *II*, Proc. Amer. Math. Soc. 35 (1972), pp. 411–412.
265. *The sign of the Bernouilli numbers*, Amer. Math. Monthly (Math. Notes section), 80 (1973), pp. 547–548.
266. *Some exponential sums*, Proc. of International Conference on Number Theory (Moscow, 1971), Trudy Mat. Inst. Steklov 132 (1973), pp. 30–34.
267. *On the exponential sum* $\sum_{x=1}^{X} e\big(2\pi i(ax+bg^x)/p\big)$, Mathematika 19 (1972), pp. 84–87.
268. *A new type of exponential series*, Quart. J. Math. Oxford (2), 23 (1972), pp. 373–374.
269. *Cubic polynomials with the same residues* mod p, Berichte aus dem Mathematischen Forschungsinstitut Oberwolfach, 5 (1971), pp. 87–88.
270. *Note on the product of n inhomogeneous linear forms*, J. Number Theory 4 (1972), pp. 405–407.

Books

Diophantine Equations (Academic Press, 1969), xii+312 pp.

The pamphlets *Three lectures on Fermat's last theorem* and *A chapter in the theory of numbers* (items 26 and 120) are being reprinted by the VEB Deutscher Verlag der Wissenschaften with an introduction by O. Neumann under the title: *Two papers on number theory*.

A Russian translation of *Reflections of a Mathematician* (item 167) has appeared under the title: *Размышления Математика* (Издательство Знание, Москва, 1971).

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Waring's problem in quadratic number fields. Addendum

by

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I am grateful to Professor P. T. Bateman for pointing out to me that there is some overlap between the results of [2] and those contained in [1] and [5]. In particular [2; Theorem 8] is a special case of [5; Theorem 10].

However, some of the results of [5] can be improved. Thus in the ring of Gaussian integers, it has been shown [3], that $g(3)\leqslant 4$, i.e. that every Gaussian integer is the sum of at most four cubes of Gaussian integers. It is easily seen that $g(3)\geqslant 3$ in this case, but which of the values 3 or 4 is the correct one is not known.

For fourth powers, we consider, again in the ring of Gaussian integers, two quantities $g(4)$ and $v(4)$, respectively the least number of fourth powers required to represent any member of J_4 as their sum, or as their sum or difference. In [4] it is shown that $g(4)\leqslant 18$, and in [5] that $g(4)\leqslant 14$ and $v(4)\leqslant 10$. We now show that $g(4)\leqslant 10$ and $v(4)\leqslant 8$. We have the identity

$$120x-131 = (2x+1)^4+(x-2+2i)^4+(x-2-2i)^4+\{(2+i)x\}^4+\\ +\{(2-i)x\}^4+\{(1+i)(x+1)\}^4,$$

and so if $\nu\equiv -11 \pmod{120}$, ν can be represented as the sum of six fourth powers. To conclude the proof that $g(4)\leqslant 10$, we observe that if $\nu\in J_4$ then $\nu\equiv 0$ or $\pm 1 \pmod 3$ and $\nu\equiv 0, \pm 1, \pm 2, \pm 3$ or $4 \pmod 8$ and it is easily seen that for any such ν it is possible to choose $\alpha, \beta_1, \beta_2, \beta_3, \beta_4, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \delta_1, \delta_2, \delta_3$, and δ_4 to satisfy

$$\nu-\alpha^4 \equiv 1 \pmod 3,$$
$$\nu-\beta_1^4-\beta_2^4-\beta_3^4-\beta_4^4 \equiv 5 \pmod 8,$$
$$\nu-\gamma_1^4-\gamma_2^4-\gamma_3^4-\gamma_4^4 \equiv -1 \pmod{1+2i},$$
$$\nu-\delta_1^4-\delta_2^4-\delta_3^4-\delta_4^4 \equiv -1 \pmod{1-2i}.$$

Now choose ξ_1 congruent to $\alpha \bmod 3$, to $\beta_1 \bmod 2$, to $\gamma_1 \bmod 1+2i$ and to $\delta_1 \bmod 1-2i$, and similarly for ξ_2, ξ_3, and ξ_4 and then

$$\nu - \xi_1^4 - \xi_2^4 - \xi_3^4 - \xi_4^4 \equiv -11 \pmod{120},$$

which concludes the proof.

We also have the identity

$$120(1+i)x = (x-2+2i)^4 - (x+2-2i)^4 + \{(1+i)x-1\}^4 - \{(1+i)x+1\}^4.$$

Now it is easily seen that for $\nu \in J_4$, the congruences

$$\nu - \alpha^4 \equiv 0 \pmod 3,$$
$$\nu - \beta_1^4 - \beta_2^4 - \beta_3^4 + \beta_4^4 \equiv 0 \pmod{1+2i},$$
$$\nu - \gamma_1^4 - \gamma_2^4 - \gamma_3^4 + \gamma_4^4 \equiv 0 \pmod{1-2i},$$
$$\nu - \delta_1^4 - \delta_2^4 - \delta_3^4 + \delta_4^4 \equiv 0 \pmod{8+8i}$$

can all be satisfied unless $\nu \equiv 10 \pmod{8+8i}$. Thus all such ν can be expressed as the sum or difference of at most eight fourth powers, exactly as before. Finally, if $\nu \equiv 10 \pmod{8+8i}$ then $-\nu \not\equiv 10 \pmod{8+8i}$ and so $-\nu$ can be so expressed. Thus $v(4) \leqslant 8$.

References

[1] Paul T. Bateman and Rosemarie Stemmler, *Waring's Problem for Algebraic Number Fields and primes of the form $(p^r-1)/(p^d-1)$*, Illinois J. Math. 6 (1962), pp. 142–156.

[2] J. H. E. Cohn, *Waring's problem in quadratic number fields*, Acta Arith. 20 (1972), pp. 1–16.

[3] — *Sums of cubes of Gaussian Integers*, Proc. Amer. Math. Soc. 29 (1971), p. 426.

[4] Ivan Niven, *Sums of fourth powers of Gaussian Integers*, Bull. Amer. Math. Soc. 47 (1941), pp. 923–926.

[5] Rosemarie Stemmler, *The easier Waring problem in algebraic number fields*, Acta Arith. 6 (1961), pp. 447–468.

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Sur les polynômes à coefficients entiérs et de discriminant donné

par

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Désignons par $\|f\| = \max_{0 \leqslant i \leqslant n} |a_i|$ la hauteur d'un polynôme $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n \in \mathbf{Z}[x]$. Appelons les polynômes $f(x)$ et $f^*(x) \in \mathbf{Z}[x]$ équivalents si l'on a $f^*(x) = f(x+a)$ avec un $a \in \mathbf{Z}$. Dans ce cas pour leur discriminant on obtient $D(f^*) = D(f)$. Donc, s'il y a un polynôme $f(x) \in \mathbf{Z}[x]$ ayant le discriminant D, alors il y en a une infinité. Dans notre travail nous démontrons le théorème suivant:

Théorème. *Soit $D \geqslant 1$ un nombre fixé arbitraire et considérons un polynôme normé $f(x) \in \mathbf{Z}[x]$ tel que $0 < |D(f)| \leqslant D$. Il existe des constantes $c_1(D)$, $c_2(D)$ calculables explicitement et dépendant seulement de D, telles que $\deg f \leqslant c_1(D)$ et $\|f^*\| \leqslant c_2(D)$, où f^* est un polynôme équivalent à f.*

Par conséquent, il n'existe qu'un nombre fini de polynômes normés $f(x) \in \mathbf{Z}[x]$, non équivalents deux à deux, de discriminant $0 < |D(f)| \leqslant D$ et on peut, par un nombre fini d'opérations, déterminer un tel système des polynômes $f(x)$.

Si $f(x) \in \mathbf{Z}[x]$ est un polynôme normé de degré $n \geqslant 2$, alors $|D(f)| \leqslant (2n\|f\|)^{2n-1}$. Inversement, de notre théorème, il résulte que $n \leqslant c_1(|D(f)|)$ et $\|f^*\| \leqslant c_2(|D(f)|)$ avec les constantes c_1 et c_2 précédentes, où f^* est un polynôme équivalent à f.

Considérons ensuite quelques corollaires de notre théorème.

Corollaire 1. *Soient donnés les nombres $D \geqslant 1$ et $N \geqslant 1$. Il n'existe qu'un nombre fini de polynômes normés $f(x) \in \mathbf{Z}[x]$ tels que $0 < |D(f)| \leqslant D$ et $|f(0)| \leqslant N$ et ces polynômes peuvent être déterminés.*

Ce corollaire reste vrai aussi dans le cas où nous fixons un autre coefficient des polynômes $f(x)$ au lieu de $f(0)$. Cette proposition est une généralisation des théorèmes analogues de Nagell [8], [10] qui concernent les polynômes normés à coefficients entiers de degré $\leqslant 4$ et les nombres algébriques entiers de degré $\leqslant 4$.

Désignons par

$$\Delta(f) = \min_{i \neq j} |\alpha_i - \alpha_j|$$