and then, as in theorem 18, we can develop a 5 term exact sequence of long sequences and commutative ladders.

$$0 \to \underline{\pi}(X, A, x) \to I(X, A, x) \to I(X, A, x) \to S^3\underline{\underline{\pi}}(X, A, x) \to S^3\underline{\pi}(X, A, x) \to 0$$

where if C is a graded module then S^3C is that graded module with $(S^3C)_n = C_{n-3}$. $\underline{\underline{\pi}}(X, A, x)$ is exact (see [3]) and it is easy to show that $I(X, A, x)$ is exact. Using this set up it is possible to prove that if (X, A, x) is a movable pointed pair of compacta then $\underline{\pi}(X, A, x)$ is exact. The concept of movable compactum was defined by K. Borsuk in [2].

21. APPENDIX. *For each* $n \geqslant 0$, $\underline{\pi}_n(X, x)$ *is the inverse limit* L *of the system* $\{\pi_n(\mathrm{inc}(U, U')); \pi_n(U, x) \to \pi_n(U', x)\}_{U\subset U', U, U' \in \mathrm{Nhd}(X)}$ *where for* $U \subset U'$ *both neighbourhoods of* X $\mathrm{inc}(U, U')$ *is the inclusion mapping* $U \subset U'$.

Proof. If f is a continuous mapping from (S^n, p_0) to (U, x) denote its homotopy class by $[f] \in \pi_n(U, x)$, then L is the set of lists $\{[a_U]\}_{U \in \mathrm{Nhd}(X)}$ where for each $U \in \mathrm{Nhd}(X)$, $[a_U] \in \pi_n(U, x)$ and if $U \subset U'$, $U, U' \in \mathrm{Nhd}(X)$, $\pi_n(\mathrm{inc}(U, U'))([a_U]) = [a_{U'}]$.

If $\{U_n\}_{n\geqslant 0}$ is a nested sequence of neighbourhoods of X such that $\bigcap_{n\geqslant 0} U_n = X$ there is a morphism

$$\Psi;\ L \to \underline{\pi}_n(X, x), \qquad \{[a_U]\} \to \langle\{a_{U_n}\}\rangle$$

which has as 2 sided inverse the morphism

$$\Phi;\ \underline{\pi}_n(X, x) \to L, \qquad \langle\{a_n\}\rangle \to \{[b_U]\}$$

where b_U is defined as follows. Given $U \in \mathrm{Nhd}(X)$ there is an $N(U) \in J^+$ such that a_n is homotopic to a_{n+1} in U, for all $n \geqslant N(U)$, define $b_U = a_{N(U)}$. Q.E.D.

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The realization of dimension function d_2 (*)

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K. Nagami and J. H. Roberts [6] introduced the metric-dependent dimension function d_2 and posed the following question, which we will call the Realization Question. Let (X, ϱ) be a metric space with $d_2(X, \varrho) < \dim X$ and let k be an integer with $d_2(X, \varrho) \leqslant k \leqslant \dim X$. Does there exist a topologically equivalent metric σ for X with $d_2(X, \sigma) = k$? For each Cantor n-manifold (K_n, ϱ) with $n \geqslant 3$, Nagami and Roberts described a subset (X_n, ϱ) with the property that $d_2(X_n, \varrho) = [n/2]$ and $\dim X_n \geqslant n-1$. This paper answers the above question in the affirmative for these spaces (X_n, ϱ) where $K_n = I^n$ (n-cube). The question remains unanswered for arbitrary metric spaces.

DEFINITION. Let (X, ϱ) be a non-empty metric space and let n be a non-negative integer. $d_2(X, \varrho) \leqslant n$ if (X, ϱ) satisfies the condition: For any collection $\mathcal{C} = \{(C_i, C_i'): i = 1, \ldots, n+1\}$ of $n+1$ pairs of closed sets with $\varrho(C_i, C_i') > 0$ for each $i = 1, \ldots, n+1$, there exist closed sets B_i, $i = 1, \ldots, n+1$, such that (i) B_i separates X between C_i and C_i' for each $i = 1, \ldots, n+1$ and (ii) $\bigcap_{i=1}^{n+1} B_i = \emptyset$.

If $d_2(X, \varrho) \leqslant n$ and the statement $d_2(X, \varrho) \leqslant n-1$ is false, we set $d_2(X, \varrho) = n$. The empty set $\emptyset$ has $d_2(\emptyset) = -1$.

DEFINITION. Let X be a topological space, $g\colon X \times X \to R$ a real valued function, and let A and B be two subsets of X. Let

$$g(A, B) = \inf\{|g(x, y)|:\ x \in A,\ y \in B\}.$$

This real number $g(A, B)$ will be called the *g-distance between A and B*.

DEFINITION. Let I^n denote the Euclidean n-cube, let $p, q \in I^n$ and let $A \subseteq I^n$. We define $\mathrm{Join}(p, q)$ to be the collection of all the points

(*) This work is taken from the author's doctoral dissertation at Duke University. I would like to thank Dr. J. H. Roberts for his guidance in the preparation of this paper.

$x \in I^n$ such that there exists a real number $\lambda \in [0, 1]$ with $x = \lambda p + (1-\lambda) q$. We define $\mathrm{Join}(p, A)$ by the following:

$$\mathrm{Join}(p, A) = \bigcup_{q \in A} \mathrm{Join}(p, q).$$

DEFINITION. Let X be a topological space, let m and k be any two non-negative integers and let $g\colon X \times X \to R$ be a real valued function. X is said to have *property* $\mathcal{T}(m, k, g)$ if given any collection of m pairs of closed subsets of X, $\mathcal{C} = \{(H_i, K_i)\colon 1 \leqslant i \leqslant m\}$, such that there exists a real number $\varepsilon > 0$ with $g(H_i, K_i) > \varepsilon$ for all i, then there exists a collection of closed sets $\mathcal{B} = \{B_i\colon 1 \leqslant i \leqslant m\}$ such that B_i separates X between H_i and K_i and order $\mathcal{B} \leqslant k$.

DEFINITION. Let X be a topological space. Suppose $\mathcal{C} = \{(C_i, C_i')\colon i = 1, \ldots, n\}$ is a collection of n pairs of closed subsets of X such that if B_i is a closed set separating X between C_i and C_i' for all $i = 1, \ldots, n$ then $\bigcap_{i=1}^{n} B_i \neq \emptyset$. Then $\mathcal{C}$ will be called an *n-defining system for X*.

If S is a subset of a topological space X we will write $\mathrm{Cl}(X, S)$, $\mathrm{Int}(X, S)$ and $\mathrm{Bdry}(X, S)$ for the topological closure, interior and boundary respectively of S in X. If the result is unambiguous we will write $\mathrm{Cl}(S)$ for $\mathrm{Cl}(X, S)$ and similary for Bdry and Int.

For a proof of the following lemma see [3].

LEMMA 1. *Let (X, ϱ) be a metric space, $f\colon X \to [0, 1]$ a continuous function with values in the unit interval, and for all $x, y \in X$ let*

$$\sigma(x, y) = \varrho(x, y) + |f(x) - f(y)|.$$

Then σ is a metric on X which is topologically equivalent to ϱ.

The following theorem has been proved by K. Morita [5].

LEMMA 2. *Let X be a normal topological space, let $\mathcal{G} = \{G_a\colon a \in A\}$ be a locally finite collection of open sets, and let $\mathcal{F} = \{F_a\colon a \in A\}$ be a collection of closed sets such that order $\mathcal{F} \leqslant n$ for some non-negative integer n and $F_a \subset G_a$ for all $a \in A$. Then there exists a collection of open sets $\mathcal{W} = \{W_a\colon a \in A\}$ such that order $\mathcal{W} \leqslant n$ and $F_a \subset W_a \subset \mathrm{Cl}(W_a) \subset G_a$ for all $a \in A$.*

The following lemma is proved in [5], p. 42.

LEMMA 3. *Let X be a completely normal topological space, let B, E, H and K be closed subsets of X with $H \cap K = \emptyset$, such that B separates $E \cap H$ from $E \cap K$ in X. Then there exists a closed set D such that D separates H from K in X and $(D \cap E) \subset B$.*

The same argument that is used to prove Theorem 1 in [7] may be used to prove the following theorem.

THEOREM 1. *Let X be a topological space, $g\colon X \times X \to R$ a real valued function, and $f\colon X \to [0, 1]$ a continuous function with values in the unit interval. For $x, y \in X$ let*

$$h(x, y) = g(x, y) + |f(x) - f(y)|.$$

If X has property $\mathcal{T}(m, k, g)$ for every non-negative integer m then X has property $\mathcal{T}(m, k+1, h)$ for every non-negative integer m.

We will apply Theorem 1 to prove the following preliminary theorem.

THEOREM 2. *Let X be a topological space and let m and r be any two non-negative integers. For each $j = 1, \ldots, m$ let $f_j\colon X \to [0, 1]$ be a continuous function with values in the unit interval. Let $\mathcal{C} = \{(C_i, C_i')\colon i = 1, \ldots, r\}$ be a collection of r pairs of closed subsets of X with the property that there exists a real number $\varepsilon > 0$ such that for every $i = 1, \ldots, r$*

$$\sum_{j=1}^{m} |f_j(x) - f_j(y)| \geqslant \varepsilon$$

for $x \in C_i$ and $y \in C_i'$. Then there exists a collection of closed subsets of X, $\mathcal{B} = \{B_i\colon i = 1, \ldots, r\}$, such that B_i separates X between C_i and C_i' for each $i = 1, \ldots, r$ and order $\mathcal{B} \leqslant m$.

Proof. The proof is by induction on the number of functions. Suppose $m = 1$. Then for each $i = 1, \ldots, r$ if $x \in C_i$ and $y \in C_i'$ we have $|f_1(x) - f_1(y)| \geqslant \varepsilon$, that is $\mathrm{Cl}(f_1(C_i))$ and $\mathrm{Cl}(f_1(C_i'))$ are disjoint closed subsets of the unit interval so there exists a collection $\mathcal{B}^* = \{B_i^*\colon i = 1, \ldots, r\}$ of closed sets such that B_i^* separates $\mathrm{Cl}(f_1(C_i))$ from $\mathrm{Cl}(f_1(C_i'))$ in the unit interval and order $\mathcal{B}^* \leqslant 1$, since the covering dimension of the unit interval is 1. Define $B_i = f_1^{-1}(B_i^*)$ for each $i = 1, \ldots, r$. Then order $\{B_i\colon i = 1, \ldots, r\} \leqslant 1$ and B_i is a closed subset of X separating X between C_i and C_i' for each $i = 1, \ldots, r$.

Suppose Theorem 2 is true for any collection of m continuous functions. Let $f_1, \ldots, f_{m+1}$ be any $m+1$ continuous functions. For all $x, y \in X$ let

$$g(x, y) = \sum_{j=1}^{m} |f_j(x) - f_j(y)|$$

and let

$$h(x, y) = g(x, y) + |f_{m+1}(x) - f_{m+1}(y)|$$

and let $f(x) = f_{m+1}(x)$ for all $x \in X$. By the induction hypothesis X has property $\mathcal{T}(k, m, g)$ for every integer k. Now f is a continuous function so by Theorem 1, X has property $\mathcal{T}(k, m+1, h)$ for every integer k.

THEOREM 3. *Let (K, γ) be a compact metric space, and let $p \in K$. If $X \subseteq K - \{p\}$ and if there are continuous functions $f_j\colon (K - \{p\}) \to [0, 1]$, $j = 1, 2, \ldots, m$ then*

$$\sigma(x, y) = \gamma(x, y) + \sum_{j=1}^{m} |f_j(x) - f_j(y)|$$

is a metric on $K-\{p\}$ which is topologically equivalent to γ and

$$d_2(X, \sigma) \leqslant \max\{m, d_2(X, \gamma)\}.$$

Proof. By Lemma 1, σ is a metric on $K-\{p\}$ which is topologically equivalent to γ. Let $h = d_2(X, \gamma)$ and let $n = \max\{m, h\}$. We show that $d_2(X, \sigma) \leqslant n$. Let

$$\mathcal{C} = \{(C_i, C_i'): i = 1, \ldots, n+1\}$$

be a collection of $n+1$ pairs of closed subsets of X with $\sigma(C_i, C_i') > \varepsilon$ for all i and some real number $\varepsilon > 0$. Let $E = \{x \in X: \gamma(x, p) \leqslant \varepsilon/8\}$, $J = \mathrm{Bdry}(E)$, $F = (K-E) \cup J$, $Y = K-\{p\}$, $I_1 = \{1, \ldots, n+1\}$, $I_2 = \{1, \ldots, h+1\}$ and let $I_3 = \{h+2, \ldots, n+1\}$.

For each $i \in I_1$ define

$$D_i = \{y \in Y: \sigma(y, C_i) \leqslant \varepsilon/8\},$$
$$D_i' = \{y \in Y: \sigma(y, C_i') \leqslant \varepsilon/8\}.$$

For each $x \in D_i \cap E$ and $y \in D_i' \cap E$ we have $\sigma(x, y) \geqslant 3\varepsilon/4$ and $\gamma(x, y) \leqslant \varepsilon/4$ so that

$$\sum_{j=1}^{m} |f_j(x) - f_j(y)| > \varepsilon/2.$$

Thus we can apply Theorem 2 and conclude that there exists a collection $\mathcal{B} = \{B_i: i \in I_1\}$ of closed subsets of Y such that B_i separates Y between $D_i \cap E$ and $D_i' \cap E$ and order $\mathcal{B} \leqslant m$.

By Lemma 2 there exists a collection $\mathcal{W} = \{W_i: i \in I_1\}$ of closed subsets of Y such that $B_i \subseteq \mathrm{Int}(W_i)$, W_i separates Y between $D_i \cap E$ and $D_i' \cap E$ and, order $\mathcal{W} \leqslant m$. Hence we can write

$$Y - W_i = U_i \cup V_i \quad \text{where} \quad U_i \cap V_i = \emptyset,\ (D_i \cap E) \subseteq U_i \ \text{and}\ (D_i' \cap E) \subseteq V_i.$$

Since J is a compact set we have that there exist real numbers β_1, β_2 and β_3, such that for all $i \in I_1$

$$\gamma(J \cap U_i, J \cap V_i) \geqslant \beta_1 > 0,$$
$$(1) \qquad \gamma(F \cap D_i, J \cap V_i) \geqslant \beta_2 > 0,$$
$$\gamma(F \cap D_i', J \cap U_i) \geqslant \beta_3 > 0.$$

Since F is a compact subset of Y, each function f_i is uniformly continuous on F, hence there exists a real number β_4 such that for $i \in I_1$

$$(2) \qquad \gamma\big((F \cap D_i), (F \cap D_i')\big) \geqslant \beta_4 > 0.$$

For each $i \in I_1$ we define

$$G_i = (F \cap D_i) \cup (U_i \cap J) \quad \text{and} \quad G_i' = (F \cap D_i') \cup (V_i \cap J).$$

From (1) and (2) above it follows that for all $i \in I_1$

$$(3) \qquad \gamma(G_i, G_i') \geqslant \min\{\beta_1, \beta_2, \beta_3, \beta_4\} > 0.$$

For each $i \in I_1$ we let

$$H_i = \mathrm{Cl}_X(G_i \cap X) \quad \text{and} \quad H_i' = \mathrm{Cl}_X(G_i' \cap X).$$

By (3) above, $\gamma(H_i, H_i') > 0$ for all $i \in I_1$.

We apply the hypothesis that $d_2(X, \gamma) = h$ to the first $h+1$ pairs of closed sets $\{(H_i, H_i'): i \in I_2\}$, and conclude that there exists a collection of closed subsets of X, $\mathcal{R} = \{R_i: i \in I_2\}$ where for each $i \in I_2$ R_i separates X between H_i and H_i'. We will write $X - R_i = K_i \cup T_i$ where $H_i \subseteq K_i$, $H_i' \subseteq T_i$, K_i and T_i are disjoint open subsets of X and order $\mathcal{R} \leqslant h$.

For each $i \in I_2$ let

$$Z_i = \big((W_i \cap E) \cup (R_i \cap F)\big) \cap X,$$
$$P_i = \big((U_i \cap E) \cup (K_i \cap (F-J))\big) \cap X,$$
$$Q_i = \big((V_i \cap E) \cup (T_i \cap (F-J))\big) \cap X.$$

Since P_i and Q_i are disjoint open sets and $C_i \subseteq P_i$ and $C_i' \subseteq Q_i$, and $X - Z_i \doteq P_i \cup Q_i$ we conclude that for each $i \in I_2$ Z_i is a closed subset of X separating X between C_i and C_i'.

The remainder of the proof is divided into 2 cases; $n = h$ and $n = m$. If $n = h$ then $\mathcal{Z} = \{Z_i: i \in I_2\}$ is the desired collection of separating sets. Since order $\mathcal{W} \leqslant m \leqslant n$ and order $\{F \cap R_i: i \in I_2\} \leqslant n$ and $(J \cap R_i) \subseteq W_i$ we have that order $\mathcal{Z} \leqslant n$.

If $n = m$ we have found separating sets for the first $h+1$ pairs of closed sets $\{(C_i, C_i'): i \in I_2\}$. We will now find separating sets for the remaining $(n+1)-(h+1) = n-h$ pairs of closed sets $\{(C_i, C_i'): i \in I_3\}$.

For each $i \in I_3$ we have a closed set W_i which separates X between $C_i \cap E$ and $C_i' \cap E$, such that order $\mathcal{W} \leqslant m = n$. For each $i \in I_3$ we apply Lemma 3 and conclude that there exists a closed set Z_i that separates X between C_i and C_i' such that $(Z_i \cap E) \subseteq W_i$. Let $\mathcal{Z} = \{Z_i: i \in I_1\}$. It remains to show that order $\mathcal{Z} \leqslant m$. Let $x \in X$. If $x \in E$ then order $(\mathcal{Z}, x) \leqslant m$ since order $\mathcal{W} \leqslant m$. If $x \in F$ then x is an element of at most h of the first $h+1$ closed sets $\{Z_i: i \in I_2\}$. There are $m-h$ sets remaining so that order $(\mathcal{Z}, x) \leqslant h + (m-h) = m$. This completes the proof of the theorem.

THEOREM 4. *For any integer $n \geqslant 1$ let $\{A_i: i \geqslant 1\}$ be any countable collection of closed subsets of the Euclidean n-cube (I^n, γ) such that if $i \neq j$ then $A_i \cap A_j = \emptyset$ and such that at least two of these closed sets are non-empty. Let $X = I^n - \bigcup_{i \geqslant 1} A_i$. Suppose $d_2(X, \gamma) = k$ and $\dim X = m$. If r is any integer such that $k \leqslant r \leqslant m$ then there exists a metric σ_r on X such that σ_r is topologically equivalent to γ and such that $d_2(X, \sigma_r) = r$.*

Proof. We will express the n-cube I^n as $\{(x_1, \ldots, x_n): -1 \leqslant x_i \leqslant 1, i = 1, \ldots, n\}$. We assume $k < m$, otherwise there is nothing to prove. Thus $k \leqslant n-1$ so that $\bigcup_{i \geqslant 1} A_i$ must be dense in I^n. Hence we can find a point $p \in \mathrm{Int}(I^n)$ with $p \in A_i$ for some i. Now $\mathrm{Int}(I^n) - A_i$ is an open non-empty subset of $\mathrm{Int}(I^n)$ since at least two of the elements of the set $\{A_j: j \geqslant 1\}$ are assumed to be non-empty. Similarly there exists a point q and an integer $j \neq i$ such that $q \in (\mathrm{Int}(I^n) \cap A_j)$ and $q \notin A_i$.

Since A_i is closed we can construct an $(n-1)$-cube B, disjoint from A_i, with center q and lying in the $(n-1)$-plane perpendicular to $\mathrm{Join}(p, q)$ at the point q.

Let $\{(R_i, S_i): i = 1, \ldots, n-1\}$ be the collection of pairs of opposite faces of B, and let D be the pyramid with base B, with apex the point p.

Since the line segment $\mathrm{Join}(p, q)$ is in the interior of I^n, the $(n-1)$-cell B may be taken small enough so that $D \subseteq \mathrm{Int}(I^n)$ and we will assume this has been done.

Let $\{(H_i, T_i): i = 1, \ldots, n-1\}$ be the $n-1$ pairs of opposite faces of D where

$$H_i = \mathrm{Join}(p, R_i) \quad \text{and} \quad T_i = \mathrm{Join}(p, S_i).$$

Let $B_i = H_i \cap X$ and $C_i = T_i \cap X$ for each $i = 1, \ldots, n-1$.

By an argument similar to that given in [6], p. 418 it can be shown that the collection $\{(B_i, C_i): i = 1, \ldots, n-1\}$ is an $(n-1)$-defining system for X. Now let $Y = I^n - \{p\}$. Then for each $i = 1, \ldots, n-1$ B_i and C_i are closed sets in Y and $B_i \cap C_i = \emptyset$. By Urysohn's Lemma there exist $n-1$ continuous functions, $f_1, \ldots, f_{n-1}$; $f_i\colon Y \to [0, 1]$ such that $f_i(B_i) = 1$ and $f_i(C_i) = 0$ for each $i = 1, \ldots, n-1$. For each $r = 1, \ldots, n-1$ we define

$$\sigma_r(x, y) = \varrho(x, y) + \sum_{j=1}^{r} |f_j(x) - f_j(y)|$$

for $x, y \in X$. By Theorem 3 we know that $d_2(X, \sigma_r) \leqslant \max\{r, k\}$. But for any integer r, $1 \leqslant r \leqslant n-1$,

$$\mathcal{C}_r = \{(B_i, C_i): i = 1, \ldots, r\}$$

is a collection of r pairs of closed subsets of X with $\sigma_r(B_i, C_i) \geqslant 1$ for all $i = 1, \ldots, r$. Thus $d_2(X, \sigma_r) \geqslant r$ and the proof of the theorem is complete.

For each $n \geqslant 3$, K. Nagami and J. H. Roberts [6] have described a subset (X_n, ϱ) of the n-cube (I^n, ϱ) with the property that $d_2(X_n, \varrho) = [n/2]$ and $\dim X_n = n-1$. In the following theorem (X_n, p) will refer to these spaces described by Nagami and Roberts.

THEOREM 5. *For any $n \geqslant 3$ and any integer r such that $[n/2] \leqslant r \leqslant n-1$, there exists a metric σ_r on (X_n, ϱ) such that σ_r is topologically equivalent to ϱ and such that $d_2(X_n, \sigma_r) = r$.*

Proof. From the definition of these spaces in [6] it can be seen that for each $n \geqslant 3$ X_n is the complement of a disjoint union of closed subsets of I^n, thus satisfying the hypothesis of Theorem 4.

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