sequences $\{a_m\}_{m=1}^{\infty}$, $\{b_m\}_{m=1}^{\infty}$ such that $a_m \uparrow x$, $b_m \downarrow x$, $f/p(x)$ is strictly monotonic, where

$$p(x) = \{a_1, b_1, a_2, b_2, \ldots\}, \qquad \big(f/p(x)\big)'(x) = \lambda,$$

$$\max\big(f(x)-\varepsilon(x), f(x)-a_{m+1}(x)\big) < f(a_m) < \min\big(f(x)+\varepsilon(x), f(x)+a_{m+1}(x)\big),$$

$$\max\big(f(x)-\varepsilon(x), f(x)-a_{m+1}(x)\big) < f(b_m) < \min\big(f(x)+\varepsilon(x), f(x)+a_{m+1}(x)\big).$$

We put $\psi(x) = \big\{\{a_m\}_{m=1}^{\infty}, \{b_m\}_{m=1}^{\infty}\big\}$. Let $D_L(f, x) \cap D_R(f, x) = \{0\}$; let there exist no two sequences for which (i)–(iv) hold true. As there exist two sequences $\{a_m\}_{m=1}^{\infty}$, $\{b_m\}_{m=1}^{\infty}$, $a_m \uparrow x$, $b_m \downarrow x$ such that $(f/\{a_1, b_1, \ldots\})'(x) = 0$, there is x the point of $M_f \cup N_f$. This contradicts our propositions on the $D(f)$. Hence there exist sequences with conditions as in the case of $D_L(f, x) \cap D_R(f, x) \subsetneq \{0\}$. Therefore these sequences satisfy (i)–(iv). We denote the set of these two sequences by $\psi(x)$.

We put $A_{n+1} = \bigcup_{x \in H_n} \psi(x)$. It remains to construct the functions $\underline{\psi}(x)$, $\overline{\psi}(x)$ with (v) for every point of H_n. Let x be any point of H_n. For $n = 1$ the construction of $\underline{\psi}(x)$, $\overline{\psi}(x)$ is easy. Suppose that $n > 1$, $y \in H_{n+1}$ such that $y \in p(x)$. Therefore there exist a sequence $\{a_m\}_{m=1}^{\infty} \in A_{n+1}$ and a natural number n_0 such that $y = a_{n_0}$ and $a_m \to x$. There exist real numbers γ_n, δ_n such that

a) $f(x)-\varepsilon(x) < \delta_m < f(a_m) < \gamma_m < f(x)+\varepsilon(x)$,

b) $\left|\dfrac{f(a_m)-\gamma_m}{a_m - x}\right| < \dfrac{1}{m}$, $\left|\dfrac{f(a_m)-\delta_m}{a_m - x}\right| < \dfrac{1}{m}$.

We put $\overline{\psi}(x)(a_m) = \gamma_m$, $\underline{\psi}(x)(a_m) = \delta_m$. Between a_k and a_{k+1} we define $\underline{\psi}(x)$ and $\overline{\psi}(x)$ continuous and monotonic. Then $\big(\underline{\psi}(x)\big)'(x) = \big(\overline{\psi}(x)\big)'(x) = \big(f/p(x)\big)'(x)$.

We have chosen the set A_{n+1} and the functions φ_{n+1}, a_{n+1}, $\underline{\psi}_{n+1}$, $\overline{\psi}_{n+1}$ from the conditions (i)–(v) for $n+1$.

We put $H = \bigcup_{i=0}^{\infty} H_i$. H is obviously bilaterally dense-in-itself. $H = H' \cup H''$, where H' is the set of points x such that $f/p(x)$ is increasing and $H'' = H - H'$. It is easy to prove that either H' or H'' contains a non-empty bilaterally dense-in-itself subset B. Then f/B is monotonic and differentiable; this completes the proof of Theorem 1.

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On shape and fundamental deformation retracts II

by

M. Moszyńska (Warszawa)

According to Fox (see [2]), two spaces X, Y are of the same homotopy type iff they are both imbeddable in some space Z as its deformation retracts.

The main result of this note is the following: For two compact metric spaces X, Y to be of the same shape it is necessary and sufficient that both X and Y be imbeddable in some compactum Z as its fundamental deformation retracts (Theorem 3.3).

We shall refer to 3.3 as the Fox Theorem for shapes.

The proof is based on some statements concerning maps of ANR-sequences. For the particular case of usual maps, these statements have been proved by the author in [6].

For terminology and notation, see [6].

1. Mapping cylinder for an arbitrary map of inverse sequences. The notion of mapping cylinder introduced in [6] § 3 for usual maps of inverse systems can be extended — in the case of inverse sequences — to arbitrary maps.

Take two inverse sequences of topological spaces, $\boldsymbol{X} = (X_n, p_n^{n+1})$, $\boldsymbol{Y} = (Y_n, q_n^{n+1})$, and a map $\boldsymbol{f} = (\varphi, f_n)\colon \boldsymbol{X} \to \boldsymbol{Y}$. By definition (see [3] or [6]), all the diagrams

$$\begin{tikzcd} X_{\varphi(n)} \arrow[d, "f_n"'] & X_{\varphi(n+1)} \arrow[l, "p^{\varphi(n+1)}_{\varphi(n)}"'] \arrow[d, "f_{n+1}"] \\ Y_n & Y_{n+1} \arrow[l, "q_n^{n+1}"] \end{tikzcd}$$

commute up to homotopy.

Thus, there exist homotopies $k_n^{n+1}\colon X_{\varphi(n+1)} \times I \to Y_n$ such that

$$k_n^{n+1}(x, 0) = f_n p_{\varphi(n)}^{\varphi(n+1)}(x), \qquad k_n^{n+1}(x, 1) = q_n^{n+1} f_{n+1}(x)$$
$$\text{for} \quad x \in X_{\varphi(n+1)},\ n = 1, 2, \ldots$$

Let C_{f_n} be the mapping cylinder of f_n. Define

$$r_n^{n+1}\colon \quad C_{f_{n+1}} \to C_{f_n}$$

as follows [1]:

$$r_n^{n+1}(z) \underset{\mathrm{Df}}{=} \begin{cases} [p_{\varphi(n)}^{\varphi(n+1)}(x),\, 2t] & \text{for } \ z=[x,t],\ (x,t)\in X_{\varphi(n+1)}\times\langle 0,\tfrac12\rangle,\\ [k_n^{n+1}(x,2t-1)] & \text{for } \ z=[x,t],\ (x,t)\in X_{\varphi(n+1)}\times\langle \tfrac12,1\rangle,\\ [q_n^{n+1}(y)] & \text{for } \ z=[y],\ y\in Y_{n+1}.\end{cases}$$

Verify the continuity of r_n^{n+1}:
if $z=[x,t]$ and $(x,t)\in X_{\varphi(n+1)}\times(\frac12)$, then

$$\begin{aligned}[p_{\varphi(n)}^{\varphi(n+1)}(x),\, 2t] &= [p_{\varphi(n)}^{\varphi(n+1)}(x),\, 1] = [f_n p_{\varphi(n)}^{\varphi(n+1)}(x)]\\ &= [k_n^{n+1}(x,0)] = [k_n^{n+1}(x,2t-1)];\end{aligned}$$

if $z=[x,t]$ and $(x,t)\in X_{\varphi(n+1)}\times(1)$, then

$$z=[f_{n+1}(x)] \quad\text{and}\quad [k_n^{n+1}(x,2t-1)] = [k_n^{n+1}(x,1)] = [q_n^{n+1}f_{n+1}(x)];$$

so r_n^{n+1} is continuous.
Let us write

$$C_f \underset{\mathrm{Df}}{=} (C_{f_n}, r_n^{n+1}).$$

The inverse sequence C_f will be referred to as a *mapping cylinder* of the map $f\colon X\to Y$.
Notice that

1.1. *If X, Y are both ANR-sequences, then C_f is an ANR-sequence as well.* ∎

The maps

$$i_n\colon X_{\varphi(n)}\to C_{f_n}, \qquad j_n\colon Y_n\to C_{f_n}$$

defined by the formulae

$$i_n(x) \underset{\mathrm{Df}}{=} [x,0], \qquad j_n(y) \underset{\mathrm{Df}}{=} [y]$$

are topological imbeddings of $X_{\varphi(n)}$, Y_n into C_{f_n}. Take

$$\boldsymbol{i} = (\varphi, i_n)\colon \boldsymbol{X}\to\boldsymbol{C}_f, \qquad \boldsymbol{j} = (1_N, j_n)\colon \boldsymbol{Y}\to\boldsymbol{C}_f.$$

It is easy to show that $\boldsymbol{i},\boldsymbol{j}$ are both usual maps (as in the particular case of f being usual, [6], 4.1).
Let us prove

1.2. *The inverse sequence $\boldsymbol{Y}$ is a deformation retract of $\boldsymbol{C}_f$.* (More precisely, $\boldsymbol{Y}$ is isomorphic in the category of inverse sequences to a deformation retract of $\boldsymbol{C}_f$.)

[1] A similar construction has been made by W. Holsztyński (unpublished) for another aim.

Proof. Define $\boldsymbol{h}\colon \boldsymbol{C}_f\to\boldsymbol{Y}$ by the formulae

$$\boldsymbol{h} = (1, h_n), \qquad h_n(z) = \begin{cases} f_n(x) & \text{for } \ z=[x,t],\\ y & \text{for } \ z=[y].\end{cases}$$

Prove $\boldsymbol{h}$ to be a map of inverse sequences, i.e.

(1) $$h_n r_n^{n+1} \simeq q_n^{n+1} h_{n+1} \quad \text{for every } n.$$

We have

$$h_n r_n^{n+1}(z) = \begin{cases} f_n p_{\varphi(n)}^{\varphi(n+1)}(x) & \text{for } \ z=[x,t],\ 0\leqslant t\leqslant\tfrac12,\\ k_n^{n+1}(x,2t-1) & \text{for } \ z=[x,t],\ \tfrac12\leqslant t\leqslant 1,\\ q_n^{n+1}(y) & \text{for } \ z=[y],\end{cases}$$

$$q_n^{n+1}h_{n+1}(z) = \begin{cases} q_n^{n+1}f_{n+1}(x) & \text{for } \ z=[x,t],\\ q_n^{n+1}(y) & \text{for } \ z=[y].\end{cases}$$

Define $u_n\colon C_{f_{n+1}}\times I\to Y_n$ by the formula

$$u_n(z,s) = \begin{cases} k_n^{n+1}(x,s) & \text{for } \ z=[x,t],\ 0\leqslant t\leqslant\tfrac12,\\ k_n^{n+1}\big(x,\, s+(1-s)(2t-1)\big) & \text{for } \ z=[x,t],\ \tfrac12\leqslant t\leqslant 1,\\ q_n^{n+1}(y) & \text{for } \ z=[y].\end{cases}$$

Prove u_n to be continuous:
for $z=[x,\frac12]$, $k_n^{n+1}\big(x,\, s+(1-s)(2\cdot\frac12-1)\big) = k_n^{n+1}(x,s)$;
for
$z=[x,1]=[f_{n+1}(x)]$, $k_n^{n+1}\big(x,\, s+(1-s)(2\cdot 1-1)\big) = k_n^{n+1}(x,1) = q_n^{n+1}f_{n+1}(x)$.
Let us show that u_n is a homotopy joining the maps $h_n r_n^{n+1}$ and $q_n^{n+1}h_{n+1}$.
In fact,

$$u_n(z,0) = \left.\begin{cases} f_n p_{\varphi(n)}^{\varphi(n+1)}(x) & \text{for } \ z=[x,t],\ 0\leqslant t\leqslant\tfrac12\\ k_n^{n+1}(x,2t-1) & \text{for } \ z=[x,t],\ \tfrac12\leqslant t\leqslant 1\\ q_n^{n+1}(y) & \text{for } \ z=[y]\end{cases}\right\} = h_n r_n^{n+1}(z),$$

$$u_n(z,1) = \left.\begin{cases} q_n^{n+1}f_{n+1}(x) & \text{for } \ z=[x,t],\ 0\leqslant t\leqslant 1\\ q_n^{n+1}(y) & \text{for } \ z=[y].\end{cases}\right\} = q_n^{n+1}h_{n+1}(z).$$

Thus condition (1) is proved.
Now, it suffices to show that

(2) $$\boldsymbol{hj} \simeq \boldsymbol{1}_{\boldsymbol{Y}}$$

and

(3) $$\boldsymbol{jh} \simeq \boldsymbol{1}_{\boldsymbol{C}_f}.$$

We show a little more:

$$h_n j_n = 1_{Y_n} \quad \text{for } n = 1, 2, \ldots \tag{2'}$$

and

$$j_n h_n \simeq 1_{C_{f_n}} \quad \text{for } n = 1, 2, \ldots \tag{3'}$$

In fact,

$$h_n j_n(y) = h_n[y] = y \quad \text{for every } n,$$

which proves (2′); next,

$$j_n h_n(z) = \begin{cases} j_n f_n(x) & \text{for } z = [x, t] \\ j_n(y) & \text{for } z = [y] \end{cases} = \begin{cases} [x, 1] & \text{for } z = [x, t], \\ [y] & \text{for } z \neq [y]. \end{cases}$$

Obviously, the map $\xi_n\colon C_{f_n} \times I \to C_{f_n}$ defined by the formula

$$\xi_n(z, s) = \begin{cases} [x, 1-s+ts] & \text{for } z = [x, t], \\ [y] & \text{for } z = [y] \end{cases}$$

is a homotopy joining the maps $j_n h_n$ and $1_{C_{f_n}}$. Thus (3′) is satisfied. ∎

Notice that

1.3. $\boldsymbol{i} \simeq \boldsymbol{jf}$.

Proof. Let us show a little more:

$$i_n \simeq j_n f_n \quad \text{for every } n.$$

In fact, $i_n(x) = [x, 0]$, $j_n f_n(x) = [f_n(x)] = [x, 1]$, thus the map

$$\xi_n\colon X_{\varphi(n)} \times I \to C_{f_n}$$

defined by the formula

$$\xi_n(x, t) = [x, t]$$

is a homotopy joining the maps i_n and $j_n f_n$. ∎

By 1.2 and 1.3 we obtain (as for usual maps [6], 4.4).

1.4. *$\boldsymbol{f}$ is a homotopy equivalence iff $\boldsymbol{i}$ is a homotopy equivalence.*

Remark. Obviously $\boldsymbol{C_f}$ is not uniquely determined by $\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y}$; it depends on the choice of homotopies $k_n^{n+1}\colon X_{\varphi(n+1)} \times I \to Y_n$. However, by 1.2, the homotopy type of $\boldsymbol{C_f}$ (and thus the shape of its inverse limit) is independent of the choice of these homotopies.

2. Fox Theorem for ANR-sequences. By Proposition 2.3 of [6], the statement 1.4 implies the following

2.1. THEOREM. *Let $\boldsymbol{X}$ be an inclusion-ANR-sequence, $\boldsymbol{Y}$ — an arbitrary inverse sequence, and $\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y}$ — a cofinal map. Then $\boldsymbol{f}$ is a homotopy equivalence iff $\boldsymbol{X}$ is a deformation retract of $\boldsymbol{C_f}$ (up to isomorphism).* ∎

Let us prove

2.2. THEOREM. *Let $\boldsymbol{X}$ be an inclusion ANR-sequence and $\boldsymbol{Y}$ — an arbitrary ANR-sequence. Then, for $\boldsymbol{X}, \boldsymbol{Y}$ to be of the same homotopy type it is necessary and sufficient that both $\boldsymbol{X}$ and $\boldsymbol{Y}$ be deformation retracts (up to isomorphism) of some ANR-sequence $\boldsymbol{Z}$.*

Proof. The sufficiency is obvious. In order to prove the necessity, assume $\boldsymbol{X}, \boldsymbol{Y}$ to be of the same homotopy type. Then there is a homotopy equivalence $\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y}$. Let $\boldsymbol{Z} = \boldsymbol{C_f}$. By 1.1, $\boldsymbol{Z}$ is an ANR-sequence. By 1.2, $\boldsymbol{Y}$ is isomorphic to a deformation retract of $\boldsymbol{Z}$. By the statement 3.8 of [6], $\boldsymbol{f}$ can be assumed cofinal. By 2.1, since $\boldsymbol{f}$ is a homotopy equivalence and $\boldsymbol{X}$ is an inclusion ANR-sequence, $\boldsymbol{X}$ is isomorphic to a deformation retract of $\boldsymbol{Z}$. ∎

3. Fox Theorem for shapes. In § 8 of [6] we were concerned with some special kind of fundamental sequences. We considered a fundamental sequence $\underline{f}$ generated by a map f (see [1]) and a related map of inverse systems, $\bar{\boldsymbol{f}}$. In that case, $\varprojlim \boldsymbol{C_f}$ was proved to be an inclusion ANR-system associated with C_f (see [6], 6.1) and therefore the statement 7.4 of [6] could be applied to prove the Fox Theorem for this special case ([6], 8.1).

Now, we are going to extend the Fox Theorem for shapes to the general case. Unfortunately, the mapping cylinder of an arbitrary map $\boldsymbol{f}$ of inclusion ANR-sequences fails to be an inclusion ANR-sequence itself. To get over this difficulty we use the notion of shape retraction due to S. Mardešić, [3] (²).

A map $\boldsymbol{r}\colon \hat{\boldsymbol{X}} \to \boldsymbol{X}$ is said to be a *shape retraction* whenever there is a map $\boldsymbol{i}\colon \boldsymbol{X} \to \hat{\boldsymbol{X}}$ such that $\boldsymbol{ri} \simeq \boldsymbol{1}_{\boldsymbol{X}}$ (see [3]). Consequently, we define a *shape deformational retraction* as a shape retraction $\boldsymbol{r}$ which satisfies the additional condition $\boldsymbol{ir} \simeq \boldsymbol{1}_{\hat{\boldsymbol{X}}}$.

Notice that

3.1. *If $\boldsymbol{i}\colon \boldsymbol{X} \to \hat{\boldsymbol{X}}$ is an inclusion (in the sense of [6]), then $\boldsymbol{i}$ is associated with the inclusion of the inverse limits.*

Proof. Let $X = \varprojlim \boldsymbol{X}$, $\hat{X} = \varprojlim \hat{\boldsymbol{X}}$. By Proposition 2, Appendix, [6], if $\boldsymbol{i}$ is an inclusion, then its inverse limit $i\colon X \to \hat{X}$ is an inclusion. Since $i = \varprojlim \boldsymbol{i}$, the diagram

$$\begin{tikzcd} X_n & X \arrow[l, "p_n"'] \arrow[d, "i"] \\ \hat{X}_n \arrow[from=u, "i_n"'] & \hat{X} \arrow[l, "\hat{p}_n"] \end{tikzcd}$$

is commutative for every n.

(²) These two approaches to fundamental retracts have been introduced independently by S. Mardešić in [3] and by the present author in [6].

Thus, moreover, it commutes up to homotopy, i.e. $\boldsymbol{i}$ is associated with the inclusion i. ■

By 3.1, we get

3.2. *If* $\boldsymbol{r}$ *is a retraction (a deformational retraction), then* $\boldsymbol{r}$ *is a shape retraction (a shape deformational retraction).* ■

Now, let us establish the main result:

3.3. THEOREM. *For two compact metric spaces* X, Y *to be of the same shape it is necessary and sufficient that both* X *and* Y *be imbeddable in some compactum* Z *as its fundamental deformation retracts.*

Proof. The sufficiency is obvious. Let us prove the necessity. Take two compacta X, Y and let $\boldsymbol{X}$, $\boldsymbol{Y}$ be inclusion-ANR-sequences associated with X, Y respectively. If $\operatorname{Sh} X = \operatorname{Sh} Y$, then, by the theorem due to Mardešić and Segal ([5]), $\boldsymbol{X} \simeq \boldsymbol{Y}$. Thus, by 2.2, $\boldsymbol{X}$, $\boldsymbol{Y}$ are (up to isomorphism) deformation retracts of some ANR-sequence $\boldsymbol{Z}$. Hence, by 3.2, $\boldsymbol{X}$, $\boldsymbol{Y}$ are (up to isomorphism) shape deformation retracts of $\boldsymbol{Z}$. Let $Z \underset{\mathrm{Df}}{=} \varprojlim \boldsymbol{Z}$. By the Mardešić Theorem 11 [3], both X and Y can be imbedded in the compactum Z as its fundamental deformation retracts. ■

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Čech homology for movable compacta

by

R. H. Overton (*) (Sheboygan, Wisc.)

Introduction. This paper is devoted to an investigation of properties relating to movability of compacta. Movability is a shape invariant introduced, for metric compacta, by K. Borsuk [3] in 1969. Subsequently, S. Mardešić and J. Segal gave alternative definitions of shape and of movability, using inverse sequences of ANR's. These are equivalent in the metric case to Borsuk's definitions, and extend the concept to non-metric compacta ([12], [13], [14]).

The purpose of this paper is to use the ANR-sequence approach to answer a question in [3], as to whether the Čech homology sequence for a pair of movable metric compacta is necessarily exact. We formulate in a natural way a definition of *movable pairs* of metric compacta, and, using ANR-sequences, we show that such pairs do have exact homology (Theorem 1), but that pairs of movable compacta may not (Theorem 2). The construction of a counter-example yields, as well, a new example of a non-movable compactum. We further describe a method of obtaining a certain useful class of movable compacta.

The notation used here is that of [7], [12], and [13]. All topological spaces considered are Hausdorff compacta.

1. Movability of compacta and of pairs. Suppose $\boldsymbol{X} = \{X_\alpha, p_{\alpha\beta}\}_{\alpha \in I}$ is an inverse system of ANR's, where I is closure-finite; that is, each $\alpha \in I$ has but a finite number of predecessors. If $X = \varprojlim X_\alpha$, then $\boldsymbol{X}$ is said to be *associated* with the compactum X. If $\boldsymbol{Y} = \{Y_\gamma, q_{\gamma\delta}\}_{\gamma \in I'}$ is an ANR-system also, we define a *map of* ANR-*systems* $\boldsymbol{f}$: $\boldsymbol{X} \to \boldsymbol{Y}$ to be a pair consisting of an order-preserving function f: $I' \to I$ and a collection of maps f_γ: $X_{f(\gamma)} \to Y_\gamma$ such that for any $\delta > \gamma$ in I', $f_\gamma \circ p_{f(\gamma), f(\delta)} \simeq q_{\gamma\delta} \circ f_\delta$. The identity $\mathrm{id}_{\boldsymbol{X}}$: $\boldsymbol{X} \to \boldsymbol{X}$ is given by $\mathrm{id}(\alpha) = \alpha$, for all $\alpha \in I$, and $\mathrm{id}_\alpha = \mathrm{id}_{X_\alpha}$.

Every compactum X has an associated ANR-system of cardinality no greater than the weight of the topology on X ([13], § 5, Theorem 7).

(*) These results form a portion of the author's Ph. D. thesis, written under the supervision of Professor Jack Segal at the University of Washington.