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Spaces of ANR's. II

by

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Abstract. It was shown by K. Borsuk [2] that the set of all ANR's lying in a compactum X can be metrized in such a way that the resulting space, denoted by 2_h^X, is separable and complete, and reflects the homotopy character of the ANR's in X, in the sense that if $A \in 2_h^X$, then all ANR's sufficiently close to A in 2_h^X are homotopically equivalent to A. In a previous paper [1], the authors considered a number of topological properties of these hyperspaces and proved, in particular, that every two homotopically equivalent connected ANR's in the 2-sphere S^2 can be joined by an arc in $2_h^{S^2}$. In the present note, this result is improved by showing that the space $2_h^{S^2}$ is in fact locally connected. (It was shown in [1] that in general 2_h^X need not be locally connected, even if X is an absolute retract.)

It was shown by K. Borsuk [2] that the set of all ANR's lying in a finite dimensional compactum X can be metrized in such a way that the resulting space, 2_h^X, is separable and complete, and reflects the homotopy character of the ANR's in X (in the sense that if A is any ANR in X, then all ANR's sufficiently close to A in 2_h^X are homotopically equivalent to A). In a previous paper [1], the authors studied a number of topological properties of these hyperspaces; in particular, an example was given of a 2-dimensional absolute retract X such that 2_h^X is not locally connected, and it was shown that if $\mathcal{C}$ denotes the (open and closed) subspace of $2_h^{S^2}$ consisting of all connected ANR's in S^2, then every component of $\mathcal{C}$ is arcwise connected. It is the aim of the present paper to improve this latter result by showing that $2_h^{S^2}$ is in fact locally connected.

1. Definitions and notations. If X is a compactum with metric ϱ and A and B are closed subsets of X, we will, following Borsuk, denote the Hausdorff distance between A and B by $\varrho_s(A, B)$, and will let $\varrho_c(A, B)$ denote the greatest lower bound of the set of all positive numbers ε such that each of A and B can be mapped into the other by an ε map (i.e., a continuous function which moves no point a distance ε or more). The homotopy metric, ϱ_h, which determines the space 2_h^X, is defined only in case A and B are locally contractible; in this case, $\varrho_h(A, B) = \varrho_c(A, B) + \psi(A, B)$, where $\psi(A, B)$ is a non-negative function, defined in [2], whose precise nature does not concern us here. We will, however, need the relation $\varrho_h(A, B) \geqslant \varrho_c(A, B)$, and will make repeated use of the fact that, for finite dimensional X, the metric ϱ_h is characterized by "homotopic convergence", as described below.

A subset of X of diameter less than ε will be called an *ε-set*, and for each $A \subset X$, we will let $s(A, \delta, \varepsilon)$ denote the statement "every δ-subset of A is contractible to a point in an ε-subset of A". A sequence $\{A_i\}$ of subsets of X is said to *converge homotopically* to a set A provided that (i) $\{A_i\}\xrightarrow[\varrho_s]{}A$ and (ii) for every $\varepsilon > 0$ there is a $\delta > 0$ such that $s(A_i, \delta, \varepsilon)$ holds for all i. It is shown in [2] that if X is finite dimensional, $\{A_i\}$ is a sequence of ANR's in X and $A \subset X$, then $\{A_i\}\xrightarrow[\varrho_h]{}A$ if and only if $\{A_i\}$ converges homotopically to A.

2. We begin with a few elementary results of a general nature, the object of which is to show that, for a finite dimensional compactum X, the local connectivity of 2^X_h is dependent only on that of its subspace of connected ANR's.

2.1. LEMMA. *Suppose X is a finite dimensional compactum and A and B are disjoint* ANR*'s lying in X. Then a sequence $\{M_i\}$ of elements of 2^X_h converges homotopically to $A \cup B$ if and only if for almost all i, M_i can be written as the union of two disjoint closed sets A_i, B_i such that $\{A_i\}\xrightarrow[\varrho_h]{}A$ and $\{B_i\}\xrightarrow[\varrho_h]{}B$.*

Proof. (1) Suppose that for each i, $M_i = A_i \cup B_i$, where A_i and B_i are closed and disjoint, and $\{A_i\}\xrightarrow[\varrho_h]{}A$, $\{B_i\}\xrightarrow[\varrho_h]{}B$. There is a positive number δ_0 such that for each i, $\varrho(A_i, B_i) > \delta_0$, where $\varrho(A_i, B_i)$ denotes the distance, in the usual sense, between A_i and B_i in X. Since $\{A_i\}\xrightarrow[\varrho_h]{}A$ and $\{B_i\}\xrightarrow[\varrho_h]{}B$, for every $\varepsilon > 0$, there is a positive number $\delta < \delta_0$ such that $s(A_i, \delta, \varepsilon)$ and $s(B_i, \delta, \varepsilon)$ hold for every i. For each i, every δ-subset of $M_i = A_i \cup B_i$ is contained either in A_i or in B_i, since $\delta < \delta_0 < \varrho(A_i, B_i)$, so every δ-subset of M_i is contractible to a point in an ε-subset of M_i. Thus $s(M_i, \delta, \varepsilon)$ holds for every i, and since it is clear that $\{M_i\}\xrightarrow[\varrho_s]{}A \cup B$, it follows that $\{M_i\}\xrightarrow[\varrho_h]{}A \cup B$.

(2) Suppose, conversely, that $\{M_i\}\xrightarrow[\varrho_h]{}A \cup B$. Let U and V be disjoint open subsets of X with $A \subset U$, $B \subset V$. Since $\{M_i\}\xrightarrow[\varrho_s]{}A \cup B \subset U \cup V$, there is an i_0 such that for $i > i_0$, $M_i \subset U \cup V$. Hence for $i > i_0$, $M_i = A_i \cup B_i$, where $A_i = M_i \cap U$ and $B_i = M_i \cap V$; it is clear that A_i and B_i are disjoint and closed, and that $\{A_i\}\xrightarrow[\varrho_s]{}A$ and $\{B_i\}\xrightarrow[\varrho_s]{}B$. Suppose $\varepsilon > 0$ and let δ be a positive number such that $s(M_i, \delta, \varepsilon)$ holds for every i. For $i > i_0$, M_i is the union of the disjoint closed sets A_i and B_i, and hence if a subset of A_i (resp., B_i) is contractible in a subset K of M_i,

then $K \subset A_i$ (resp., $K \subset B_i$). It follows that $s(A_i, \delta, \varepsilon)$ and $s(B_i, \delta, \varepsilon)$ hold for all $i > i_0$, and thus $\{A_i\}\xrightarrow[\varrho_h]{}A$ and $\{B_i\}\xrightarrow[\varrho_h]{}B$.

If $\mathcal{A}$ and $\mathcal{B}$ are collections of sets, we will let $\mathcal{A} \oplus \mathcal{B}$ denote the collection of all sets of the form $A \cup B$, with $A \in \mathcal{A}$, $B \in \mathcal{B}$. Two collections $\mathcal{A}$ and $\mathcal{B}$ will be said to be **-disjoint* if no element of $\mathcal{A}$ intersects any element of $\mathcal{B}$.

2.2. LEMMA. *Suppose X is a finite dimensional compactum and $\mathcal{A}$ and $\mathcal{B}$ are *-disjoint subsets of 2^X_h. Then (1) if $\mathcal{A}$ and $\mathcal{B}$ are open in 2^X_h, so is $\mathcal{A} \oplus \mathcal{B}$ and (2) if $\mathcal{A}$ and $\mathcal{B}$ are connected, so is $\mathcal{A} \oplus \mathcal{B}$.*

Proof. (1) Suppose $\mathcal{A}$ and $\mathcal{B}$ are open in 2^X_h and let $A \cup B$ be an element of $\mathcal{A} \oplus \mathcal{B}$, with $A \in \mathcal{A}$, $B \in \mathcal{B}$. Since $\mathcal{A}$ and $\mathcal{B}$ are *-disjoint, $A \cap B = \emptyset$. It follows from Lemma 2.1 that if $\{M_i\}\xrightarrow[\varrho_h]{}A \cup B$, then for almost all i, $M_i = A_i \cup B_i$, with $\{A_i\}\xrightarrow[\varrho_h]{}A$ and $\{B_i\}\xrightarrow[\varrho_h]{}B$; since $\mathcal{A}$ and $\mathcal{B}$ are open in 2^X_h and $A \in \mathcal{A}$, $B \in \mathcal{B}$, for almost all i, $A_i \in \mathcal{A}$ and $B_i \in \mathcal{B}$, so $M_i \in \mathcal{A} \oplus \mathcal{B}$.

(2) Suppose $\mathcal{A}$ and $\mathcal{B}$ are connected. Let f denote the natural function from $\mathcal{A} \times \mathcal{B}$ onto $\mathcal{A} \oplus \mathcal{B}$ defined by $f(A, B) = A \cup B$. If $(A_i, B_i) \to (A, B)$ in $\mathcal{A} \times \mathcal{B}$, then $\{A_i\}\xrightarrow[\varrho_h]{}A$ and $\{B_i\}\xrightarrow[\varrho_h]{}B$ and hence by Lemma 2.1, $\{A_i \cup B_i\}\xrightarrow[\varrho_h]{}A \cup B$. Thus f is continuous and hence $\mathcal{A} \oplus \mathcal{B}$, as a continuous image of the connected set $\mathcal{A} \times \mathcal{B}$, is connected.

2.3. LEMMA. *Suppose X is a finite dimensional compactum and $\mathcal{U}$ is an open subset of 2^X_h. If $A \cup B \in \mathcal{U}$, where A and B are closed and disjoint, then there exist two *-disjoint open subsets $\mathcal{A}$ and $\mathcal{B}$ of 2^X_h such that $A \in \mathcal{A}$, $B \in \mathcal{B}$, and $\mathcal{A} \oplus \mathcal{B} \subset \mathcal{U}$.*

Proof. Let $\{\varepsilon_i\}$ be a decreasing sequence of positive numbers converging to 0, with $\varepsilon_1 < \frac{1}{2}\varrho(A, B)$. For each i, let $\mathcal{A}_i$ and $\mathcal{B}_i$ denote the ε_i-neighborhoods in 2^X_h of A and B, respectively; since $\varepsilon_i \leqslant \varepsilon_1 < \frac{1}{2}\varrho(A, B)$, $\mathcal{A}_i$ and $\mathcal{B}_i$ are *-disjoint. If for each i, $A_i \in \mathcal{A}_i$ and $B_i \in \mathcal{B}_i$, then $\{A_i\}\xrightarrow[\varrho_h]{}A$ and $\{B_i\}\xrightarrow[\varrho_h]{}B$, so by Lemma 2.1, $\{A_i \cup B_i\}\xrightarrow[\varrho_h]{}A \cup B \in \mathcal{U}$. It follows that for some i, $A_i \cup B_i \in \mathcal{U}$ for every $A_i \in \mathcal{A}_i$, $B_i \in \mathcal{B}_i$, and hence $\mathcal{A}_i \oplus \mathcal{B}_i \subset \mathcal{U}$.

Note that, by induction, Lemma 2.2 holds for any finite sequence $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ of *-disjoint subsets of 2^X_h and Lemma 2.3 holds for any finite sequence $A_1, A_2, \ldots, A_n$ of disjoint elements of 2^X_h.

2.4. LEMMA. *Suppose X is a finite dimensional compactum and let $\mathcal{C}$ denote the subspace of 2^X_h consisting of all connected* ANR*'s in X. If $\mathcal{C}$ is locally connected, so is 2^X_h.*

Proof. Suppose $\mathcal{U}$ is an open subset of 2^X_h and $A \in \mathcal{U}$. Let $A_1, A_2, \ldots, A_n$ be the components of A. By Lemma 2.3, there exist *-disjoint open subsets $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ of 2^X_h such that $A_i \in \mathcal{A}_i$, for $1 \leqslant i \leqslant n$, and such that $\mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \ldots \oplus \mathcal{A}_n \subset \mathfrak{U}$. Since $\mathfrak{C}$ is locally connected (and open in 2_h^X), for each i, there is a connected open subset $\mathfrak{V}_i$ of 2_h^X with $A_i \in \mathfrak{V}_i \subset \mathcal{A}_i$. Let $\mathfrak{V} = \mathfrak{V}_1 \oplus \mathfrak{V}_2 \oplus \ldots \oplus \mathfrak{V}_n$. Then $A \in \mathfrak{V} \subset \mathfrak{U}$ and by Lemma 2.2, $\mathfrak{V}$ is open and connected. It follows that 2_h^X is locally connected.

3. We now turn to the main object of this note, to show that the space $2_h^{S^2}$ is locally connected. As in [1], we will take S^2 to be polyhedral in E^3, and by an *annulus* we will mean any continuum in S^2 whose boundary consists of a finite number of disjoint simple closed curves.

3.1. Lemma. *If C is a connected* ANR *properly contained in S^2 and $\varepsilon > 0$, then there exist a $\delta > 0$ and a neighborhood $\mathfrak{U}$ of C in $2_h^{S^2}$ such that if $A, B \in \mathfrak{U}$, $B \subset A$, and A is an annulus, then every two points of* $\operatorname{Bd} A$ *which can be joined by a δ-arc in $S^2 - B$ can be joined by an ε-arc in* $\operatorname{Bd} A$.

Proof. Since $C \neq S^2$, it may be assumed that C is contained in the interior of a (planar) 2-simplex σ of S^2. Let U_0 be a neighborhood of C in S^2 such that $\overline{U}_0 \subset \operatorname{Int} \sigma$, and let $\eta = \varrho(\overline{U}_0, S^2 - \sigma)$. Let $\mathfrak{U}_0$ denote the set of all ANR's lying in U_0; it is easily seen that $\mathfrak{U}_0$ is open in $2_h^{S^2}$. Note that (1) if $A \in \mathfrak{U}_0$, then $A \subset \operatorname{Int} \sigma$ and $\varrho(A, S^2 - \sigma) > \eta$.

Let $x_1, \ldots, x_m$ be points of $S^2 - C$ such that every complementary domain of C contains exactly one x_i. By Theorem 4.5 of [1], there is a neighborhood $\mathfrak{U}_1$ of C in $2_h^{S^2}$ such that $\mathfrak{U}_1 \subset \mathfrak{U}_0$ and (2) if $A \in \mathfrak{U}_1$, then every complementary domain of A contains exactly one of the points $x_1, \ldots, x_m$.

It follows easily from the definition of homotopic convergence that there exist a positive number $\delta < \min(\eta, \frac{1}{8}\varepsilon)$ and a neighborhood $\mathfrak{U}_2$ of C in $2_h^{S^2}$ such that $\mathfrak{U}_2 \subset \mathfrak{U}_1$ and (3) $s(A, \delta, \frac{1}{8}\varepsilon)$ is true for every $A \in \mathfrak{U}_2$.

Let $\mathfrak{U}$ denote the intersection of $\mathfrak{U}_2$ and the $\frac{1}{2}\delta$-neighborhood of C in $2_h^{S^2}$, and suppose $A, B \in \mathfrak{U}$, $B \subset A$ and A is an annulus. Let p and q be points of $\operatorname{Bd} A$ such that there is a δ-arc, α, from p to q in $S^2 - B$. Note that $\alpha \subset \operatorname{Int} \sigma$ since, by (1), $\varrho(A, S^2 - \sigma) > \eta$, and $\operatorname{dia} \alpha < \delta < \eta$.

If each of D_1, D_2 is a complementary domain of A whose boundary intersects α, then $D_1 \cup \alpha \cup D_2$ is a connected subset of $S^2 - B$ and hence lies in some complementary domain of B. By (2), no complementary domain of B contains two of the points $x_1, \ldots, x_m$ and each of D_1, D_2 contains one of them; it follows that $D_1 \cap D_2 \neq \emptyset$, so $D_1 = D_2$. Hence $\alpha \cap \operatorname{Bd} A$ is contained in a single boundary curve, J, of A.

Using Lemma 4.1 of [1] and standard polyhedral approximation theorems for the plane, it can be shown that there is a homeomorphism h of S^2 onto itself taking A, B, α, J onto sets A', B', α', J', respectively, such that $A', B' \in \mathfrak{U}$, α' is a δ-arc and $\alpha' \cap J'$ is finite, and such that the inverse under h of any set of diameter less than $\frac{3}{4}\varepsilon$ has diameter less than ε. Thus it may be assumed that $\alpha \cap J$ is finite, provided it is then shown that there is a $\frac{3}{4}\varepsilon$-arc from p to q in J.

It will be shown first that if $\alpha \cap J = \{p, q\}$, then one of the arcs of J from p to q lies in the $\frac{1}{4}\varepsilon$-neighborhood of α. There are two cases to be considered.

(1) Suppose $\alpha - \{p, q\} \subset S^2 - A$. Since $s(A, \delta, \frac{1}{8}\varepsilon)$ is true and $\varrho(p, q) < \delta$, there is a $\frac{1}{8}\varepsilon$-arc β from p to q in A, and β may be chosen so that $\beta - \{p, q\} \subset \operatorname{Int} A$. If D is the disk in σ bounded by the simple closed curve $\alpha \cup \beta$, standard θ-curve arguments imply that one of the arcs of J from p to q lies in D; since $\operatorname{dia} \alpha < \delta < \frac{1}{8}\varepsilon$ and $\operatorname{dia} \beta < \frac{1}{8}\varepsilon$, it follows that $\alpha \cup \beta$, and hence D, is contained in the $\frac{1}{4}\varepsilon$-neighborhood of α.

(2) Suppose $\alpha - \{p, q\} \subset \operatorname{Int} A$. Then $A = A_1 \cup A_2$, where A_1 is an annulus, A_2 is a disk, $A_1 \cap A_2 = \alpha$ and $B \subset A_1$. Since $\mathfrak{U}$ is a subset of the $\frac{1}{2}\delta$-neighborhood of C in $2_h^{S^2}$, $\varrho_h(A, B) < \delta$; since $\varrho_c(A, B) \leqslant \varrho_h(A, B)$, it follows that there is a δ-map $f\colon A \to B$. For each $x \in A_2$, $\varrho(x, f(x)) < \delta$ and hence, since $s(A, \delta, \frac{1}{4}\varepsilon)$ is true, there is a $\frac{1}{4}\varepsilon$-arc from x to $f(x)$ in A and since $x \in A_2$ and $f(x) \in B \subset A_1$, this arc must intersect α. Thus A_2 is contained in the $\frac{1}{4}\varepsilon$-neighborhood of α, and therefore so is one of the arcs of J from p to q.

For the general case, let $p_1, p_2, \ldots, p_n$ be the points of $\alpha \cap J$, ordered from p to q on α. For $i = 1, 2, \ldots, n-1$, the subarc $p_i p_{i+1}$ of α satisfies either (1) or (2) above and hence J contains an arc γ_i from p_i to p_{i+1} in the $\frac{1}{4}\varepsilon$-neighborhood of α. Thus J contains an arc γ from p to q which lies in the $\frac{1}{4}\varepsilon$-neighborhood of α, and since $\operatorname{dia} \alpha < \frac{1}{4}\varepsilon$, it follows that $\operatorname{dia} \gamma < \frac{3}{4}\varepsilon$, as required.

3.2. Theorem. *The space $2_h^{S^2}$ is locally connected.*

Proof. Let C be a connected ANR properly contained in S^2 and let $\mathfrak{U}$ be a neighborhood of C in $2_h^{S^2}$. It will be shown that there is a neighborhood $\mathfrak{V}$ of C in $2_h^{S^2}$ such that each element of $\mathfrak{V}$ can be joined to C by an arc lying in $\mathfrak{U}$, and it will follow, by Lemma 2.4, that $2_h^{S^2}$ is locally connected. Since $C \neq S^2$, it may be assumed that C is contained in the interior of a 2-simplex σ of S^2.

Let $D_1, \ldots, D_m$ be the components of $S^2 - C$ and for each i, let P_i be a nondegenerate continuum lying in D_i. Let d be a positive number less than the minimum of the diameters of the sets $C, P_1, P_2, \ldots, P_m$. There is a neighborhood $\mathfrak{U}_0$ of C in $2_h^{S^2}$ such that each element of $\mathfrak{U}_0$ is connected, lies in $\operatorname{Int} \sigma$ and has diameter greater than d, and it follows from Theorem 4.5 of [1] that $\mathfrak{U}_0$ may be chosen so that each complementary domain of each element of $\mathfrak{U}_0$ contains exactly one of the sets $P_1, \ldots, P_m$.

It follows from Lemma 4.1 of [1] that there exist a positive number $\varepsilon < \frac{1}{4}d$ and a neighborhood $\mathfrak{U}_1$ of C in $2_h^{S^2}$ such that $\mathfrak{U}_1 \subset \mathfrak{U}_0$ and such that if $X \in \mathfrak{U}_1$ and q is a 4ε-homeomorphism of X into S^2, then $q(X) \in \mathfrak{U}$.

By Lemma 3.1, there exist a positive number $\delta < \varepsilon$ and a neighborhood $\mathfrak{U}_2$ of C such that $\mathfrak{U}_2 \subset \mathfrak{U}_1$, and if A, $B \in \mathfrak{U}_2$, $B \subset A$ and A is an annulus, then every two points of $\mathrm{Bd}\,A$ which can be joined by a δ-arc in S^2-B can be joined by an ε-arc in $\mathrm{Bd}\,A$. Let $\varepsilon' = \frac{1}{24}\delta$ and apply Lemma 3.1 again to obtain a positive number $\delta' < \varepsilon'$ and a neighborhood $\mathfrak{U}_3$ of C in $2_h^{S^2}$ such that $\mathfrak{U}_3 \subset \mathfrak{U}_2$, and if $A, B \in \mathfrak{U}_3$, $B \subset A$ and A is an annulus, then every two points of $\mathrm{Bd}\,A$ which can be joined by a δ'-arc in S^2-B can be joined by an ε'-arc in $\mathrm{Bd}\,A$. Let $\mathfrak{V}'$ denote the intersection of $\mathfrak{U}_3$ with the $\frac{1}{2}\delta$-neighborhood of C in $2_h^{S^2}$.

Summarizing, we now have a neighborhood $\mathfrak{V}'$ of C in $2_h^{S^2}$, nondegenerate continua $P_1, \ldots, P_m$, and positive numbers $d, \varepsilon, \delta, \varepsilon', \delta'$ with the following properties:

(1) $\delta' < \varepsilon' = \frac{1}{24}\delta < \delta < \varepsilon < \frac{1}{4}d$.

(2) $\mathfrak{V}' \subset \mathfrak{C}$ and each element of $\mathfrak{V}'$ lies in $\mathrm{Int}\,\sigma$.

(3) Every element of $\mathfrak{V}'$ has diameter greater than d.

(4) If $X \in \mathfrak{V}'$, then each complementary domain of X contains exactly one of $P_1, \ldots, P_m$, and, in particular, each complementary domain of X has diameter greater than d.

(5) If $X \in \mathfrak{V}'$ and g is a 4ε-homeomorphism of X into S^2, then $g(X) \in \mathfrak{U}$. In particular $\mathfrak{V}' \subset \mathfrak{U}$.

(6) If $A, B \in \mathfrak{V}'$, $B \subset A$ and A is an annulus, then every two points of $\mathrm{Bd}\,A$ which can be joined by a δ-arc (resp., by a δ'-arc) in S^2-B can be joined by an ε-arc (resp., by an ε'-arc) in $\mathrm{Bd}\,A$.

(7) If $A, B \in \mathfrak{V}'$, then $\varrho_h(A, B) < \delta$.

It follows from the proof of Theorem 4.4 of [1] that there is an arc $\mathcal{A}$ in $2_h^{S^2}$ such that one endpoint of $\mathcal{A}$ is C, and each element of $\mathcal{A}-\{C\}$ is an annulus containing C in its interior. There is an element A of $\mathcal{A}$ such that $A \in \mathfrak{V}'$ and the subarc of $\mathcal{A}$ from A to C lies in $\mathfrak{U}$.

There is a neighborhood $\mathfrak{V}$ of C in $2_h^{S^2}$ such that $\mathfrak{V} \subset \mathfrak{V}'$ and such that every element of $\mathfrak{V}$ lies in $\mathrm{Int}\,A$. It will be shown that every element of $\mathfrak{V}$ can be joined to C by an arc lying in $\mathfrak{U}$; since, by the above argument, each element of $\mathfrak{V}$ can be joined to some annulus in $\mathfrak{V}$ by an arc in $\mathfrak{U}$ and since A is joined to C by an arc in $\mathfrak{U}$, it will be sufficient to show that every annulus in $\mathfrak{V}$ can be joined to A by an arc in $\mathfrak{U}$.

Let B be an annulus in $\mathfrak{V}$; then $B \subset \mathrm{Int}\,A$, so each complementary domain of A is contained in some complementary domain of B. By condition (4) above, each complementary domain of A, and each complementary domain of B, contains exactly one of the sets $P_1, \ldots, P_m$. Hence each complementary domain of B contains exactly one complementary domain of A.

Let D be a complementary domain of A and let D' be the complementary domain of B which contains $\overline{D}$. Let J and J' denote the boundary curves of D and D', respectively, and let U denote the connected open subset of S^2 bounded by $J \cup J'$. Then $\overline{U} \subset \mathrm{Int}\,\sigma$ and $U \subset D'-D$; in particular, $U \cap \mathrm{Bd}\,A = \emptyset$.

Let F be a finite subset of J which intersects every arc in J of diameter $\geqslant \varepsilon'$. For each $x \in F$, let $q(x)$ denote the point of B nearest x, $T(x)$ the straight line interval from x to $q(x)$, and $p(x)$ the last point of J on $T(x)$ in the order from x to $q(x)$. Since $T(x)-\{q(x)\} \subset D'$, $q(x) \in J'$ and, since $D' \cap \mathrm{Bd}\,A = J$, $p(x) \in J$.

Let $G = \{p(x) |\ x \in F\}$. Choose a fixed orientation for σ, and let $p_1, \ldots, p_n$ be an ordering of the points of G such that for $i = 1, \ldots, n$, p_i and p_{i+1} (where $p_{n+1} = p_1$) are distinct, and the positively oriented arc of J from p_i to p_{i+1} contains no point of G in its interior. For each i, let $q_i = q(x)$ and $T_i = T(x)$, where x is a point of F for which $p_i = p(x)$, and let T'_i be the subinterval of T_i from p_i to q_i.

By condition (7) above, $\varrho_h(A, B) < \delta'$ and hence there is a δ'-map $f\colon A \to B$. For each x in F, $q(x)$ is the nearest point of B to x, so $\varrho(x, q(x)) \leqslant \varrho(x, f(x)) < \delta'$. In particular, each T_i, and hence each T'_i, has length less than δ'.

The intervals $T'_1, \ldots, T'_n$ may not be disjoint, but no two of them can have a point in common other than an endpoint on J'. It can be shown by an elementary, but somewhat detailed, argument that $T'_1, \ldots, T'_n$ may be replaced by disjoint arcs $\alpha_1, \ldots, \alpha_n$ such that for each i, α_i is a δ'-arc from p_i to a point $p'_i \in J'$ and $\alpha_i - \{p_i, p'_i\} \subset U$.

For $i = 1, 2, \ldots, n$, let β_i denote the positive arc of J from p_i to p_{i+1} and β'_i the positive arc of J' from p'_i to p'_{i+1}. Then U is the union of disjoint open disks $U_1, \ldots, U_n$, where U_i is bounded by $\alpha_i \cup \beta_i \cup \alpha_{i+1} \cup \beta'_i$.

Suppose that for some i, $\mathrm{dia}\,\beta_i \geqslant \frac{1}{3}\delta$. Since $\varepsilon' = \frac{1}{24}\delta$, $\mathrm{dia}\,\beta_i \geqslant 8\varepsilon'$ and it follows that some component S of $\beta_i - \big(N_{\varepsilon'}(P_i) \cup N_{\varepsilon'}(P_{i+1})\big)$ has diameter $\geqslant \varepsilon'$. Since F intersects every subarc of J of diameter $\geqslant \varepsilon'$, S contains a point x of F. For some j, $p_j = p(x)$ and hence $\varrho(x, p_j) < \delta'$, and this implies, by condition (6), that there is an ε'-arc γ from x to p_j in J. Since $x \in S \subset \beta_i$ and the distance from x to either endpoint of β_i is not less than ε', it follows that $\gamma \subset \beta_i - \{p_i, p_{i+1}\}$. But this is impossible since β_i cannot contain p_j in its interior. Hence for each i, $\mathrm{dia}\,\beta_i < \frac{1}{3}\delta$. Since also each of α_i, α_{i+1} has diameter less than $\delta' < \frac{1}{3}\delta$, it follows that $\mathrm{dia}(\alpha_i \cup \beta_i \cup \alpha_{i+1}) < \delta$.

It is easily seen that there is an annulus $B' \in \mathfrak{V}$ such that $B' \subset \mathrm{Int}\,B$. Then for each i, since $\alpha_i \cup \beta_i \cup \alpha_{i+1}$ is a δ-arc joining p'_i and p'_{i+1} in S^2-B', it follows from condition (6) that there is an ε-arc γ_i from p'_i to p'_{i+1} in J'. For each i, let V_i be the disk in σ bounded by $\gamma_i \cup \alpha_i \cup \beta_i \cup \alpha_{i+1}$. Since the boundary of V_i has diameter less than 4ε and $V_i \subset \sigma$, it follows that $\mathrm{dia}\,V_i < 4\varepsilon$. If for some i, $\gamma_i \neq \beta_i$, then γ_i is the negatively oriented arc of J' from p'_i to p'_{i+1}, and it follows that V_i either contains D (in case J is not the outermost boundary curve of A in σ) or else contains B. But each of these is impossible since $d > 4\varepsilon$ and by condition (3), $\operatorname{dia} B > d$ and by (4), $\operatorname{dia} D > d$. Hence for each i, $\gamma_i = \beta_i'$ and thus $V_i = U_i$ and $\operatorname{dia} U_i < 4\varepsilon$. For each i, let U_i' be a neighborhood of $\overline{U}_i$ in S^2 such that $\operatorname{dia} U_i' < 4\varepsilon$ and $U_i' \cap (\operatorname{Bd} A \cup \operatorname{Bd} B) = J \cup J'$.

Since there is a homeomorphism of S^2 onto itself which takes J and J' onto a pair of concentric circles and each α_i onto a radial interval joining these circles, it is easily seen that for every neighborhood V of $\overline{U}$ in S^2, there is an isotopy $f\colon S^2 \times I \to S^2$ such that $f_0 = \mathrm{id}$, $f_1(J) = J'$, and for each $t \in I$, (i) f_t is the identity on $S^2 - V$ and (ii) for each i, $f_t(U_i') \subset U_i'$. Since $\operatorname{dia} U_i' < 4\varepsilon$, this last condition implies that for each t, f_t is a 4ε-homeomorphism. Since V is an arbitrary neighborhood of $\overline{U}$, V may be chosen so that $\overline{V} \cap (\operatorname{Bd} A \cup \operatorname{Bd} B) = J \cup J'$ and $V \cap (P_1 \cup P_2 \cup \ldots \cup P_m) = \emptyset$.

Repeating the entire construction for each pair of corresponding boundary curves of A and B gives an isotopy $g\colon S^2 \times I \to S^2$ such that $g_0 = \mathrm{id}$, $g_1(\operatorname{Bd} A) = \operatorname{Bd} B$, and for each $t \in I$, g_t is a 4ε-homeomorphism which is the identity on $P_1 \cup \ldots \cup P_m$. Since $g_1(\operatorname{Bd} A) = \operatorname{Bd} B$ and $g_1(A)$ does not intersect $P_1 \cup \ldots \cup P_m$, it follows that $g_1(A) = B$. Since each g_t is a 4ε-homeomorphism, it follows from condition (5) that $g_t(A) \in \mathcal{U}$ for each $t \in I$. By Lemma 4.2 of [1], there is an arc from A to B lying in $\{g_t(A) \mid t \in I\}$, and hence there is an arc from A to B in $\mathcal{U}$, as required.

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Embedding certain compactifications of a half-ray

by

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Abstract. Two problems concerning embedding compactifications of a half-ray are stated and partially answered. In our investigations of these problems, certain types of continua are completely determined.

1. Introduction. Throughout this paper, *continuum* will mean a compact connected metric space containing more than one point. The question of what spaces can be remainders in compactifications of certain kinds of spaces has been of interest (see, for example, [1], [7], [8], [13], [15], and [16]). In [1], Aarts and Van Emde Boas showed that any continuum can be the remainder in some compactification of a given locally compact non-compact separable metric space. This implies, of course, that any continuum can be the remainder in some compactification of a half-ray (a half-ray is a topological space homeomorphic to $[0, +\infty)$). Compactifications of a half-ray have been studied by D. Bellamy [2], M. E. Rudin [14], Simon [15] (where the main aspect of a result in [16] was proved for the special case of a half-ray), and others. In [11, Lemma 5.6] we proved a result, a very special case of which is

LEMMA A. *If Σ is an arcwise connected circle-like continuum* (see [10]), *then any compactification of a half-ray with Σ as the remainder is embeddable in the plane.*

This result and others mentioned above, as well as our theorem in section 2 of this paper, raise for us the following questions.

PROBLEM 1. What continua K have the property that *there is* a compactification of a half-ray, with K as the remainder, such that the compactification is embeddable in R^n (Euclidean n-space)? Clearly, such continua are embeddable in R^n and have dimension less than n [5], p. 44.

PROBLEM 2. What continua K have the property that

(α_n) *any* compactification of a half-ray, with K as the remainder, is embeddable in R^n?

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