# Minimal class generated by open compact and perfect mappings

by

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**Abstract.** We study minimal classes of spaces containing good spaces, such as metric spaces, $p$-spaces etc., generated by open compact mappings and perfect mappings. As Bennett showed, a paradoxical thing like Michael's space can be a member of such a class. If we assume that all spaces considered are regular, such minimal classes turn suddenly into somewhat harmonious objects. Main theorem: Let $f\colon X\to Y$ be a composition of open compact mappings and perfect mappings, with $X$ a $p$-space. Then $f$ is compact-covering. We deduce many corollaries from this theorem among which there are answers for some problems for MOBI and MOBOS by Arhangel'skiĭ.

**0. Introduction.** The present study has grown up from the idea of Arhangel'skiĭ [3], § 5, where he proposed the importance of the study for classes of spaces which are generated by good mappings starting from good spaces. He defined, among others, two classes of spaces, MOBI and MOBOS, and raised a group of problems which seems to be very interesting. Recall that MOBI or MOBOS is the class of all images of metric spaces under the compositions of open compact (1) mappings or of open compact ones and perfect (2) ones, respectively. Recently Bennett [4] showed that Michael space [11] $X$ is an element of MOBI and almost all problems for MOBI raised by Arhangel'skiĭ are solved in the negative. $X$ is Lindelöf regular and hence paracompact, yet it is neither metric, developable (3), perfectly normal nor a $p$-space (4). Bennett

(1) A mapping is said to be *open* if the image of every open set is open. It is to be *compact* if every point-inverse is compact.

(2) A mapping is said to be *closed* if the image of every closed set is closed. It is said to be *perfect* if it is closed and compact.

(3) A sequence $\mathcal{U}_i$, $i=1,2,\ldots$, of open coverings of a space $X$ is said to be a *development* if $\{S(x,\mathcal{U}_i)=\bigcup\{U\colon x\in U\in\mathcal{U}_i\}\colon i=1,2,\ldots\}$ forms a local base at $x$ for every point $x$ of $X$. A space is said to be *developable* if it has a development. A developable regular space is said to be a *Moore space*.

(4) A space is said to be *completely regular* if it is $T_1$ and has a base consisting of cozero sets. According to Arhangel'skiĭ [1], Definition 5, a space $X$ is said to be a *$p$-space* if it is completely regular and there exists a sequence $\mathcal{U}_i$, $i=1,2,\ldots$, of open collections of its Stone–Čech compactification $\beta X$ such that $\bigcap\{S(x,\mathcal{U}_i)\colon i=1,2,\ldots\}\subset X$ for every point $x$ of $X$ and such that each $\mathcal{U}_i$ covers $X$. A sequence $\{\mathcal{U}_i\}$ with this property is said to be a *pluming* of $X$.

actually constructed a metric space $Y$, a Hausdorff space $Z$, and open compact mappings $f\colon Y\to Z$ and $g\colon Z\to X$. In this paper we assume that all mappings are continuous onto, while transformations need not be onto. It is a meaningful fact that $Z$ is not a regular $(T_1+T_3)$ space. If we assume that a starting space is metric and each intermediate stop, which is to be the range of an open compact mapping and is to be the domain of another open compact mapping, is regular, then we can avoid monsters like $X$ as the last stop. This will be clarified in this paper. *Hereafter all spaces are regular and hence the domains and the ranges of all mappings are regular.* This convention should be kept even for the compositions of mappings. Both the domain and the range of each factor mapping must be automatically regular.

0.1. Definition. A mapping is said to be an OC, OP, or OCP, if it is respectively the composition of *open compact ones*, *open ones* and *perfect ones*, or *open compact ones* and *perfect ones*. Let $\mathcal{C}$ be a class of spaces. Let OC($\mathcal{C}$), OP($\mathcal{C}$), or OCP($\mathcal{C}$) be respectively the *class of all images of elements of* $\mathcal{C}$ *under* OC-, OP-, *or* OCP-*mappings*. In other words OC($\mathcal{C}$), OP($\mathcal{C}$), or OCP($\mathcal{C}$) is the *minimal class of spaces containing* $\mathcal{C}$ *which is closed under the operation of taking images of* OC-, OP-, *or* OCP-*mappings* respectively.

The main purpose of this paper is then to show that these three kinds of classes, starting from Moore spaces, absolute $G_\delta$ (5) spaces or $p$-spaces, turn suddenly into somewhat harmoniuous objects in spite of Bennett's counter example. When $\mathcal{C}$ is the class of all metric spaces, OC($\mathcal{C}$) is denoted merely by OC(metric). This type of abbreviation is used for other $\mathcal{C}$; OP (absolute $G_\delta$), OCP($p$-spaces), etc. The author thinks that OC(metric) or OCP(metric) is what Arhangel'skiĭ expected under the name of MOBI or MOBOS respectively. From this stand point of view yet affirmatively unsolved problems for MOBI and MOBOS raised by him can be asked again for OC(metric) and OCP(metric). Some of them will be solved affirmatively or at least their relationship will be clarified in this paper. The reader will notice that, as far as the present study concerns, starting from Moore spaces is more essential than from metric spaces.

Section 1 gives preliminary lemmas and notations. In Section 2 we will prove that OC-mappings or OP-mappings defined respectively on $p$-spaces or on absolute $G_\delta$ spaces are compact-covering (6). The technique in the proof of the theorem for OC-mappings, which is summarized as in Lemma 2.3 below, will be used in Section 4 to prove more general and main theorem: An OCP-mapping defined on a $p$-space is compact-covering. To prove this theorem we require other technique which will be exhibited in Section 3. In Section 5 some applications of this theorem will be given. The two among them are as follows. Under the continuum hypothesis, the weight (7) of each space in OCP(Moore) cannot exceed its power. Each element of OCP($p$-spaces) is of countable type (8). The first assertion contains an answer for the question of Arhangel'skiĭ [3], § 5, from our stand point of view. These two theorems reveal an aspect of the regularity of OCP-mappings. In Section 6 we prove first that a pointwise paracompact space is a Moore space if it is an element of OC(Moore). Similarly, a pointwise paracompact completely regular space is a $p$-space if it is an element of OC($p$-spaces). As a corollary we get Theorem 6.2 below: If a paracompact space is an element of OC(Moore), then it is metric. This contains an answer for the question of Arhangel'skiĭ [3], § 5, from our stand point of view. We will end this paper with Section 7 where some related questions are given. Section 8 gives a supplement.

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## 1. Preliminaries.

1.1. Notation. Let $\mathfrak{U}$ be a collection of subsets of a space $X$. Then $\mathfrak{U}^{\#}$ denotes the union of all sets in $\mathfrak{U}$.

1.2. Lemma. *Let $f\colon X\to Y$ be an open mapping and $\mathfrak{U}=\{U_\alpha\colon \alpha\in A\}$ an open collection of $X$. Then there exist an open collection $\mathfrak{V}=\{V_\beta\colon \beta\in B\}$ of $X$ and a transformation $\varphi\colon B\to A$ such that*

(i) $f(\mathfrak{U})^{\#}=f(\mathfrak{V})^{\#}$,

(ii) $\varphi(\beta)=\alpha$ *implies* $\overline{V}_\beta\subset U_\alpha$ *and* $\overline{f(V_\beta)}\subset f(U_\alpha)$ *at the same time.*

Proof. Let $\mathfrak{W}=\{W_\gamma\colon \gamma\in C\}$ be an open collection of $X$ such that $\overline{\mathfrak{W}}<$ (refines) $\mathfrak{U}$ and $\mathfrak{W}^{\#}=\mathfrak{U}^{\#}$. Let $\psi\colon C\to A$ be a transformation such that $\psi(\gamma)=\alpha$ implies $\overline{W}_\gamma\subset U_\alpha$. Let $\mathfrak{G}=\{G_\beta\colon \beta\in B\}$ be an open collection of $Y$ such that $\overline{\mathfrak{G}}<f(\mathfrak{W})$ and $\mathfrak{G}^{\#}=f(\mathfrak{W})^{\#}$. Let $\theta\colon B\to C$ be such that $\theta(\beta)=\gamma$ implies $\overline{G}_\beta\subset f(W_\gamma)$. Set $V_\beta=f^{-1}(G_\beta)\cap W_{\theta(\beta)}$. Then we have an open collection $\mathfrak{V}=\{V_\beta\colon \beta\in B\}$ of $X$. To show $f(\mathfrak{U})^{\#}=f(\mathfrak{V})^{\#}$ let $y$ be an arbitrary point of $f(\mathfrak{U})^{\#}$. Since $f(\mathfrak{U})^{\#}=f(\mathfrak{W})^{\#}$ and $f(\mathfrak{W})^{\#}=\mathfrak{G}^{\#}$, there exists $G_\beta$ with $y\in G_\beta$. Then $y\in f(W_{\theta(\beta)})$ and hence $y\in f\big(f^{-1}(G_\beta)\cap W_{\theta(\beta)}\big)=f(V_\beta)$. Set $\varphi=\psi\theta$. Then we have $\varphi\colon B\to A$. Let $\beta$ be an arbitrary index

(5) A space is said to be *absolute* $G_\delta$ if it is completely regular and a $G_\delta$ set of its Stone–Čech compactification. This concept was introduced by Čech [7] and developed by Frolík [9].

(6) A mapping $f\colon X\to Y$ is said to be *compact-covering* if each compact set of $Y$ is the image of some compact set of $X$ under $f$.

(7) The *weight* of a space $X$, denoted by $w(X)$, is the minimum of cardinals of bases of $X$.

(8) A space $X$ is said by Arhangel'skiĭ [1], Definition 3, to be of *countable type* if each compact set of $X$ is contained in a compact set $K$ of *countable character*; i.e. there exists a sequence $U_1, U_2, \ldots$ of open sets of $X$ such that for each pair $K\subset U$ with $U$ open there exists an $i$ with $K\subset U_i\subset U$.

in $B$. Set $\theta(\beta)=\gamma$ and $\psi(\gamma)=\alpha$. Then $\overline{V}_\beta\subset\overline{W}_\gamma\subset U_\alpha$ and $\overline{f(V_\beta)}\subset\bar{G}_\beta\subset f(W_\gamma)\subset f(U_\alpha)$. The proof is finished.

1.3. Lemma. *Let $f_i\colon X_i\to X_{i+1}$, $i=1,\dots,n$, be open mappings and $\mathfrak{U}=\{U_\alpha\colon \alpha\in A\}$ an open collection of $X_1$. Then there exist an open collection $\mathfrak{V}=\{V_\beta\colon \beta\in B\}$ of $X_1$ and a transformation $\varphi\colon B\to A$ satisfying the following conditions*:

(i) $f_n\dots f_1(\mathfrak{U})^{\#}=f_n\dots f_1(\mathfrak{V})^{\#}$.

(ii) $\varphi(\beta)=\alpha$ *implies* $\overline{V}_\beta\subset U_\alpha$ *and* $\overline{f_i\dots f_1(V_\beta)}\subset f_i\dots f_1(U_\alpha)$, $i=1,\dots,n$, *at the same time.*

Proof. Applying Lemma 1.2 to $\mathfrak{U}$ and $f_1\colon X_1\to X_2$, we have an open collection $\mathfrak{U}_1=\{U_{1\alpha}\colon \alpha\in A_1\}$ of $X_1$ and a transformation $\varphi_1\colon A_1\to A$ such that

(1) $$f_1(\mathfrak{U})^{\#}=f_1(\mathfrak{U}_1)^{\#},$$

(2) $$\varphi_1(\beta)=\alpha \quad\text{implies}\quad \overline{U}_{1\beta}\subset U_\alpha \text{ and } \overline{f_1(U_{1\beta})}\subset f_1(U_\alpha).$$

Applying Lemma 1.2 again to $f_1(\mathfrak{U}_1)$ and $f_2\colon X_2\to X_3$, we have an open collection $\mathfrak{U}_2=\{U_{2\alpha}\colon \alpha\in A_2\}$ of $X_2$ and a transformation $\varphi_2\colon A_2\to A_1$ such that $f_2f_1(\mathfrak{U}_1)^{\#}=f_2(\mathfrak{U}_2)^{\#}$ and such that $\varphi_2(\beta)=\alpha$ implies $\overline{U}_{2\beta}\subset f_1(U_{1\alpha})$ and $\overline{f_2(U_{2\beta})}\subset f_2f_1(U_{1\alpha})$. Continuing in this manner, we finally have an open collection $\mathfrak{U}_i=\{U_{i\alpha}\colon \alpha\in A_i\}$ of $X_i$ for $i=1,\dots,n$, and a transformation $\varphi_i\colon A_i\to A_{i-1}$ for $i=1,\dots,n$, satisfying the following two conditions:

(3) $$f_{i+1}f_i(\mathfrak{U}_i)^{\#}=f_{i+1}(\mathfrak{U}_{i+1})^{\#},\quad i=1,\dots,n-1.$$

(4) $$\varphi_i(\beta)=\alpha \quad\text{implies}\quad \overline{U}_{i\beta}\subset f_{i-1}(U_{i-1,\alpha}) \text{ and } \overline{f_i(U_{i\beta})}\subset f_if_{i-1}(U_{i-1,\alpha}),\quad i=2,\dots,n.$$

Set $B=A_n$. Define $\varphi\colon B\to A$ by setting $\varphi=\varphi_1\dots\varphi_n$. Let $\beta$ be an arbitrary index in $B$. Set $\varphi_i\dots\varphi_n(\beta)=\alpha_{i-1}$ for $i=2,\dots,n$, and $\varphi(\beta)=\alpha$. Set

$$V_{n\beta}=U_{n\beta},\ V_{n-1,\beta}=U_{n-1,\alpha_{n-1}}\cap f_{n-1}^{-1}(V_{n\beta}),\ \dots,\ V_{1\beta}=U_{1\alpha_1}\cap f_1^{-1}(V_{2\beta}).$$

Then we have formulae:

(5) $$V_{i\beta}=U_{i\alpha_i}\cap f_i^{-1}(V_{i+1,\beta}),\quad i=1,\dots,n-1.$$

Since $V_{i\beta}\subset U_{i\alpha_i}$ by (5) and $U_{i\alpha_i}\subset f_{i-1}(U_{i-1,\alpha_{i-1}})$ by (4), then $V_{i\beta}\subset f_{i-1}(U_{i-1,\alpha_{i-1}})$. Hence

$$f_{i-1}(V_{i-1,\beta})=f_{i-1}\big(U_{i-1,\alpha_{i-1}}\cap f_{i-1}^{-1}(V_{i\beta})\big)=V_{i\beta}.$$

Thus we have:

(6) $$f_{i-1}\dots f_1(V_{1\beta})=V_{i\beta},\quad i=2,\dots,n.$$



We have now an open collection $\mathfrak{V}=\{V_\beta\equiv V_{1\beta}\colon \beta\in B\}$. Let us check that this collection fulfils the second requirement. Since $V_\beta\subset U_{1\alpha_1}$ and $\varphi_1(\alpha_1)=\alpha$, then $\overline{V}_\beta\subset\overline{U}_{1\alpha_1}\subset U_\alpha$ by (2). Let $i$ be an arbitrary integer with $1\leqslant i\leqslant n$. Since $f_i\dots f_1(V_{1\beta})=V_{i+1,\beta}$ by (6), $V_{i+1,\beta}\subset U_{i+1,\alpha_{i+1}}$ by (5) and $\overline{U}_{i+1,\alpha_{i+1}}\subset f_i(U_{i\alpha_i})$ by (4), then

(7) $$\overline{f_i\dots f_1(V_{1\beta})}\subset f_i(U_{i\alpha_i}).$$

Repeated applications of (4) shows:

$$f_i(U_{i\alpha_i})\subset f_if_{i-1}(U_{i-1,\alpha_{i-1}})\subset f_if_{i-1}f_{i-2}(U_{i-2,\alpha_{i-2}})\subset\dots$$
$$\dots\subset f_i\dots f_1(U_{1\alpha_1})\subset f_i\dots f_1(U_\alpha),$$

i.e.

(8) $$f_i(U_{i\alpha_i})\subset f_i\dots f_1(U_\alpha).$$

By (7) and (8) we have the desired inequalities:

$$\overline{f_i\dots f_1(V_{1\beta})}\subset f_i\dots f_1(U_\alpha),\quad i=1,\dots,n.$$

Let us check the first requirement. From (3) we have:

(9) $$f_n\dots f_i(\mathfrak{U}_i)^{\#}=f_n\dots f_{i+1}(\mathfrak{U}_{i+1})^{\#},\quad i=1,\dots,n-1.$$

By repeated applications of (9) we have: $f_n(\mathfrak{U}_n)^{\#}=f_nf_{n-1}(\mathfrak{U}_{n-1})^{\#}=\dots=f_n\dots f_1(\mathfrak{U}_1)^{\#}$. By (1) we have: $f_n\dots f_1(\mathfrak{U}_1)^{\#}=f_n\dots f_1(\mathfrak{U})^{\#}$. Hence

(10) $$f_n(\mathfrak{U}_n)^{\#}=f_n\dots f_1(\mathfrak{U})^{\#}.$$

By (6) and the equality $V_{n\beta}=U_{n\beta}$ we have:

(11) $$f_n\dots f_1(\mathfrak{V})^{\#}=f_n(\mathfrak{U}_n)^{\#}.$$

By (10) and (11) we have the desired equality:

$$f_n\dots f_1(\mathfrak{U})^{\#}=f_n\dots f_1(\mathfrak{V})^{\#}.$$

The proof is finished.

1.4. Definition. Let us say that this type of transformation $\varphi$ is of *type* II *with respect to* $f_n\dots f_1$, $\mathfrak{V}$ *and* $\mathfrak{U}$, while the simpler type of $\varphi$ stated in Lemma 1.2 is of *type* I.

1.5. Lemma. *A completely regular space $X$ is a $p$-space if and only if there exists a sequence $\{\mathcal{H}_i\}$ of open collections of $\beta X$ with $\mathcal{H}_i^{\#}\supset X$ for each $i$ satisfying the condition:*

*If $x\in X$ and $x\in H_i\in\mathcal{H}_i$ for each $i$, then $\bigcap H_i\subset X$.*

Proof. If $X$ is a $p$-space, then its pluming $\{\mathcal{H}_i\}$ evidently satisfies the condition. Conversely to show that $\{\mathcal{H}_i\}$ with the condition is a pluming, let $y$ be an arbitrary point of $\bigcap S(x,\mathcal{H}_i)$ with $x\in X$. Then we can find for each $i$ an element $H_i$ of $\mathcal{H}_i$ with $x\in H_i$ and $y\in H_i$. Since $y\in\bigcap H_i$, $y\in X$. Thus $\bigcap S(x,\mathcal{H}_i)\subset X$. The proof is finished.

Hereafter a pluming is the one with the condition in this lemma.

1.6. Notation. Hereafter the tilde for sets in $\beta X$ denotes always the closure in $\beta X$, while the bar for sets in $X$ denotes the closure in $X$.

**2. OC($p$-spaces) and OP(absolute $G_\delta$).**

2.1. Theorem. *Let $f\colon X\to Y$ be an OC-mapping of a $p$-space $X$ ($\neq \emptyset$). Then $f$ is compact-covering.*

Proof. Let $f_i\colon X_i\to X_{i+1}$, $i=1,\dots,n$, be open compact mappings, with $X_1=X$, $X_{n+1}=Y$ and $f=f_n\dots f_1$. Let $\mathcal{G}_i$, $i=1,2,\dots$, be a pluming of $X$. Let $\mathcal{H}_i$ be an open collection of $\beta X$ such that $\mathcal{H}_i^{\#}=\mathcal{G}_i^{\#}$ and $\tilde{\mathcal{H}}_i<\mathcal{G}_i$. Since a closed subset of a $p$-space is a $p$-space, it is sufficient to prove the theorem for the case when $Y$ is compact. Assume so. Let $\mathcal{U}_1=\{U_{1\alpha}\colon \alpha\in A_1\}$ be the restriction $\mathcal{H}_1|X$. Let $\mathcal{U}_1'=\{U_{1\alpha}\neq\emptyset\colon \alpha\in B_1\}$ be a finite subcollection of $\mathcal{U}_1$ such that $f(\mathcal{U}_1')^{\#}=Y$. Let $\mathcal{U}_2=\{U_{2\alpha}\colon \alpha\in A_2\}$ be an open collection of $X$ and $\varphi_1^2\colon A_2\to B_1$ be a transformation of type II with respect to $f=f_n\dots f_1$, $\mathcal{U}_2$ and $\mathcal{U}_1'$ such that $f(\mathcal{U}_2)^{\#}=f(\mathcal{U}_1')^{\#}=Y$. It can easily be seen that we can assume $\mathcal{U}_2<\mathcal{H}_2|X$ without loss of generality. Then we have a finite subcollection of $\mathcal{U}_2$, say $\mathcal{U}_2'=\{U_{2\alpha}\neq\emptyset\colon \alpha\in B_2\}$, such that $f(\mathcal{U}_2')^{\#}=Y$. Repeating this process yields a sequence $\mathcal{U}_i=\{U_{i\alpha}\colon \alpha\in A_i\}$, $i=1,2,\dots$, of open collections of $X$, a sequence $\mathcal{U}_i'=\{U_{i\alpha}\colon \alpha\in B_i\}$, $i=1,2,\dots$, of finite subcollections of $\mathcal{U}_i$, and a sequence $\varphi_i^{i+1}\colon A_{i+1}\to B_i$, $i=1,2,\dots$, of transformations satisfying the conditions:

(i) Each $\varphi_i^{i+1}$ is of type II with respect to $f_n\dots f_1$, $\mathcal{U}_{i+1}$ and $\mathcal{U}_i'$.

(ii) $f(\mathcal{U}_i')^{\#}=Y$, $i=1,2,\dots$

(iii) $\mathcal{U}_i<\mathcal{H}_i|X$.

(iv) $U_{i\alpha}\neq\emptyset$ for $\alpha\in B_i$, $i=1,2,\dots$

It is to be noted that $\{B_i;\ \varphi_i^{i+1}|B_{i+1}\}$ forms of course an inverse system. Let $x_{n+1}$ be an arbitrary point of $Y$. Set

$$C_i=\{\alpha\in B_i\colon x_{n+1}\in f(U_{i\alpha})\}.$$

Then each $C_i$ is not empty and $\{C_i\}$ forms an inverse subsystem [9] of $\{B_i\}$. Pick an arbitrary element $\langle\alpha_i\rangle$ from $\operatorname{inv\,lim} C_i$. Since $f_n^{-1}(x_{n+1})$ is compact, $f_{n-1}\dots f_1(U_{i\alpha_i})\cap f_n^{-1}(x_{n+1})\neq\emptyset$ and $\overline{f_{n-1}\dots f_1(U_{i+1,\alpha_{i+1}})}\subset f_{n-1}\dots f_1(U_{i\alpha_i})$ for $i=1,2,\dots$, then

$$\Big(\bigcap_i f_{n-1}\dots f_1(U_{i\alpha_i})\Big)\cap f_n^{-1}(x_{n+1})\neq\emptyset.$$

Pick a point $x_n$ from the left side. Then $f_n(x_n)=x_{n+1}$. In this mann we can get a point sequence $x_j$, $j=1,\dots,n$, such that

$$f_j(x_j)=x_{j+1}\quad\text{and}\quad x_{j+1}\in\bigcap_i f_j\dots f_1(U_{i\alpha_i}).$$

[9] Let $\{A_i;\varphi_i^{i+1}\}$ be an inverse system. If for each $i$ $B_i\subset A_i$ and $\varphi_i^{i+1}(B_{i+1})\subset B_i$, then $\{B_i;\ \varphi_i^{i+1}|B_{i+1}\}$ is said to be an *inverse subsystem*.



Especially $x_1\in\bigcap_i \mathcal{U}_{i\alpha_i}$. This shows that

$$Y=\bigcup\{f(\bigcap_i U_{i\alpha_i})\colon \langle\alpha_i\rangle\in\operatorname{inv\,lim} B_i\}. \tag{1}$$

Moreover this argument shows that

$$\langle\alpha_i\rangle\in\operatorname{inv\,lim} B_i\quad\text{implies}\quad f(\bigcap_i U_{i\alpha_i})\neq\emptyset, \tag{2}$$

since $\overline{f(U_{i\alpha_i})}\neq\emptyset$ and $\overline{f(U_{i\alpha_i})}\subset f(U_{i-1,\alpha_{i-1}})$. For each $U_{i\alpha}$ with $\alpha\in B_i$, choose an element $H_{i\alpha}$ of $\mathcal{H}_i$ and an element $G_{i\alpha}$ of $\mathcal{G}_i$ with

$$U_{i\alpha}\subset H_{i\alpha}\subset\tilde{H}_{i\alpha}\subset G_{i\alpha}. \tag{3}$$

Set $P_{1\alpha}=H_{1\alpha}$, where $\alpha\in B_1$, $P_{2\alpha}=H_{2\alpha}\cap H_{1\beta}$, where $\alpha\in B_2$ and $\varphi_1^2(\alpha)=\beta$, and so on. In general set

$$P_{i\alpha}=H_{i\alpha}\cap P_{i-1,\beta},\qquad \alpha\in B_i,\qquad \varphi_{i-1}^i(\alpha)=\beta,\qquad i=2,3,\dots$$

Then

$$U_{i\alpha}\subset P_{i\alpha}\subset H_{i\alpha}. \tag{4}$$

This can easily be seen by the inequalities: $U_{i\alpha}\subset U_{i-1,\beta}\subset H_{i-1,\beta}$, where $\varphi_{i-1}^i(\alpha)=\beta$, and by induction. By this construction, for each $i$ and each $\alpha\in B_{i+1}$, $P_{i+1,\alpha}\subset P_{i\beta}$, where $\beta=\varphi_i^{i+1}(\alpha)$. Thus if we set

$$\mathcal{P}_i=\{P_{i\alpha}\colon \alpha\in B_i\},\qquad i=1,2,\dots,$$

then $\psi_i^{i+1}=\varphi_i^{i+1}|B_i$ gives a refine transformation of $\mathcal{P}_{i+1}$ to $\mathcal{P}_i$. Set $P_i=\mathcal{P}_i^{\#}$. Then $\tilde{P}_i=(\tilde{\mathcal{P}}_i)^{\#}$ and $\tilde{P}_1\supset\tilde{P}_2\supset\dots$ Set

$$L=\bigcap_i\tilde{P}_i. \tag{5}$$

Then $L$ is a compact set. To show that $L\subset X$ let $x$ be an arbitrary point of $L$. Set

$$B_i'=\{\alpha\in B_i\colon x\in\tilde{P}_{i\alpha}\}.$$

Then $B_i'$ is a non-empty finite set and $\{B_i'\}$ forms an inverse subsystem of $\{B_i\}$. Pick an arbitrary element $\langle\beta_i\rangle$ from $\operatorname{inv\,lim} B_i'$. Then

$$x\in\bigcap_i\tilde{P}_{i\beta_i}. \tag{6}$$

Since (2) implies $\bigcap_i U_{i\beta_i}\neq\emptyset$, we can pick a point $p$ from this intersection. Then by (3)

$$p\in\bigcap G_{i\beta_i}\quad\text{and hence}\quad\bigcap G_{i\beta_i}\subset X. \tag{7}$$

By (3) and (4), $\bigcap U_{i\beta_i}\subset\bigcap\tilde{P}_{i\beta_i}\subset\bigcap\tilde{H}_{i\beta_i}\subset\bigcap G_{i\beta_i}$. By (6) $x\in\bigcap G_{i\beta_i}$ and hence by (7) $x\in X$. Thus $L$ is a compact set of $X$.

If $\langle \gamma_i \rangle$ is an arbitrary element of $\operatorname{inv}\lim B_i$, then $\bigcap U_{i\gamma_i} \neq \emptyset$ and this intersection is contained in $L$. Hence by (1) $f(L) = Y$ and the proof is finished.

2.2. COROLLARY. *If $X$ is a member of* $\mathrm{OC}(p\text{-spaces})$ *and $f$ an* OC-*mapping on $X$, then $f$ is compact-covering.*

Proof. Let $Z$ be a $p$-space such that $X$ is the image of $Z$ under an OC-mapping $g$. Since $fg$ is an OC-mapping from $Z$ to $Y$, $fg$ is compact-covering by Theorem 2.1. Let $K$ be an arbitrary compact set of $Y$ and $L$ a compact set of $Z$ with $fg(L) = K$. Since $g(L)$ is a compact set of $X$, $f$ is compact-covering. The proof is finished.

The following lemma, which will be frequently used in the sequel, is essentially proved in the proof of Theorem 2.1.

2.3. LEMMA. *Let $f$ be an* OC-*mapping of $X$ to $Y$. Let $\mathfrak{U}_i = \{U_{i\alpha}\colon \alpha \in A_i\}$, $i = 1, 2, \ldots$, be open collections of $X$ and $\varphi_i^{i+1}\colon A_{i+1} \to A_i$ a transformation of type* II *with respect to $f$, $\mathfrak{U}_{i+1}$, $\mathfrak{U}_i$. Let $\langle a_i \rangle$ be an element of* $\operatorname{inv}\lim\{A_i; \varphi_i^{i+1}\}$. *If $y \in \bigcap f(U_{ia_i})$, then there exists a point $x$ of $X$ with $f(x) = y$ and $x \in \bigcap U_{ia_i}$.*

The following lemma is due to Frolík [9], Theorem 3.8.

2.4. LEMMA. *A completely regular space $X$ is absolute $G_\delta$ if and only if there exists a sequence $\mathcal{G}_i$, $i = 1, 2, \ldots$, of open coverings of $X$ satisfying the following condition:*

*If $\mathcal{F}$ is a non-empty collection of closed sets of $X$ with the finite intersection property such that $F < \mathcal{G}_i$ for each $i$ and for some $F \in \mathcal{F}$, then the intersection of all elements in $\mathcal{F}$ is not empty.*

2.5. NOTATION. Let $f$ be a perfect mapping of $X$ to $Y$. Let $\mathfrak{U} = \{U_\alpha\colon \alpha \in A\}$ and $\mathfrak{V} = \{V_\beta\colon \beta \in B\}$ be open collections of $X$ and $\varphi\colon B \to A$ a refine transformation; i.e. $\varphi(\beta) = \alpha$ implies $V_\beta \subset U_\alpha$. Let $A^{\nabla}$ denote the index set which consists of all finite subsets of $A$. $B^{\nabla}$ is defined similarly. For each element $\lambda$ in $A^{\nabla}$ set $U_\lambda = \bigcup\{U_\alpha\colon \alpha \in \lambda\}$. Then we have an open collection $\{U_\lambda\colon \lambda \in A^{\nabla}\}$ of $X$, denoting it by $\mathfrak{U}^{\nabla}$. $\mathfrak{V}^{\nabla} = \{V_\mu\colon \mu \in B^{\nabla}\}$ is defined similarly. Set for $\lambda \in A^{\nabla}$

$$W_\lambda = Y - f(X - U_\lambda)\,. \tag{1}$$

Then we have an open collection $\{W_\lambda\colon \lambda \in A^{\nabla}\}$ of $Y$, denoting it by $f^{\nabla}(\mathfrak{U})$. $f^{\nabla}(\mathfrak{V}) = \{G_\mu\colon \mu \in B^{\nabla}\}$ is defined similarly. Henceforth the element of $f^{\nabla}(\mathfrak{U})$ with the index $\lambda$ is always defined by the formula (1) without special noticing. Define $\varphi^{\nabla}\colon B^{\nabla} \to A^{\nabla}$ by:

$$\varphi^{\nabla}(\mu) = \{\varphi(\beta)\colon \beta \in \mu\}\,, \qquad \mu \in B^{\nabla}\,.$$

Then $\varphi^{\nabla}(\mu) = \lambda$ implies $G_\mu \subset W_\lambda$. Thus $\varphi^{\nabla}$ is a refine transformation from $f^{\nabla}(\mathfrak{V})$ to $f^{\nabla}(\mathfrak{U})$. It is evident that, for each $\lambda \in A^{\nabla}$ and each $\mu \in B^{\nabla}$, $f^{-1}(W_\lambda) \subset U_\lambda$ and $f^{-1}(G_\mu) \subset V_\mu$. If $y$ is a point of $Y$ with $f^{-1}(y) \subset \mathfrak{U}^{\#}$, then $y$ is covered by $f^{\nabla}(\mathfrak{U})$. If we have transformations $\varphi_1\colon A_1 \to A_2$, $\varphi_2\colon A_2 \to A_3$, then $(\varphi_2\varphi_1)^{\nabla} = \varphi_2^{\nabla}\varphi_1^{\nabla}$.



2.6. THEOREM. *Let $f\colon X \to Y$ be an* OP-*mapping on an absolute $G_\delta$ space $X$. Then $f$ is compact-covering* (10).

Proof. $X$ can be joined with $Y$ as in the following system:

$$X = X_1 \xrightarrow{f_1} Y_1 \xrightarrow{g_1} X_2 \xrightarrow{f_2} \ldots \xrightarrow{g_{n-1}} X_n \xrightarrow{f_n} Y_n = Y$$

where each $f_i\colon X_i \to Y_i$ is an open mapping and each $g_i\colon Y_i \to X_{i+1}$ is a perfect one. Let $\mathcal{G}_i$, $i = 1, 2, \ldots$, be a defining sequence of open coverings of $X$ as in Lemma 2.4. Since a closed set of $X$ is absolute $G_\delta$, it is sufficient to prove the theorem for the case when $Y$ is compact and non-empty. Set

$$\mathcal{G}_1 = \mathfrak{U}_{11} = \{U_{11\alpha}\colon \alpha \in A_{11}\}\,.$$

Set

$$A_{1i}^{\nabla} = A_{1,i+1}\,, \tag{1}$$

$$g_i^{\nabla}\big(f_i(\mathfrak{U}_{1i})\big) = \mathfrak{U}_{1,i+1} = \{U_{1,i+1,\alpha}\colon \alpha \in A_{1,i+1}\}\,, \tag{2}$$

where

$$U_{1,i+1,\alpha} = X_{i+1} - g_i\big(Y_i - \{f_i(U_{1i\beta})\colon \beta \in \alpha\}\big)\,. \tag{3}$$

Since $f_n(\mathfrak{U}_{1n})^{\#} = Y_n$, $\mathfrak{U}_{1n}$ has a finite subcollection $\mathfrak{U}'_{1n} = \{U_{1n\alpha}\colon \alpha \in B_{1n}\}$ such that

$$f_n(\mathfrak{U}'_{1n})^{\#} = Y \quad \text{and} \quad U_{1n\alpha} \neq \emptyset\,, \quad \alpha \in B_{1n}\,. \tag{4}$$

Set

$$B_{1n}^{\#} = B_{1,n-1}\,, \quad B_{1,n-1}^{\#} = B_{1,n-2}\,, \ \ldots \tag{5}$$

Then we reach a finite subset $B_{11}(\neq \emptyset)$ of $A_{11}$. Set

$$\mathfrak{U}'_{1i} = \{U_{1i\alpha}\colon \alpha \in B_{1i}\}\,.$$

Then the following holds:

$$f_i(\mathfrak{U}'_{1i})^{\#} \supset g_i^{-1}(\mathfrak{U}'_{1,i+1})^{\#}\,. \tag{6}$$

By (4) and (6)

$$f(\mathfrak{U}'_{11})^{\#} = Y\,. \tag{7}$$

Let $\mathfrak{U}_{21} = \{U_{21\alpha}\colon \alpha \in A_{21}\}$ be an open collection of $X$ such that

$$\mathfrak{U}_{21}^{\#} = \mathfrak{U}_{11}'^{\#}\,, \quad \overline{\mathfrak{U}}_{21} < \mathfrak{U}'_{11} \wedge \mathcal{G}_2\,. \tag{8}$$

(10) It is to be noted that Arhangel'skiĭ [2], Theorem 14, already proved a special case of this theorem as follows: An open mapping on an absolute $G_\delta$ space is compact-covering.

Set

(9) $$A_{2i}^{\nabla} = A_{2,i+1}\,, \qquad i = 1, \dots, n-1\,,$$

(10) $$g_i^{\nabla}(f_i(\mathfrak{U}_{2i})) = \mathfrak{U}_{2,i+1} = \{U_{2,i+1,a}\colon\ a \in A_{2,i+1}\}\,.$$

Then by (6) $f_n(\mathfrak{U}_{2n})^{\#}$ covers $Y$. Hence there exists a finite subcollection $\mathfrak{U}_{21}' = \{U_{21a}\colon\ a \in B_{2n}\}$ of $\mathfrak{U}_{2n}$ such that $f_n(\mathfrak{U}_{2n}')^{\#} = Y$. By a similar way to obtain $B_{11}$ we can get a finite subcollection $\mathfrak{U}_{21}' = \{U_{21a}\colon\ a \in B_{21}\}$ of $\mathfrak{U}_{21}$ such that $f(\mathfrak{U}_{21}')^{\#} = Y$.

Continuing this process infinitely we obtain a sequence $\mathfrak{U}_{i1}' = \{U_{i1a}\colon\ a \in B_{i1}\}$, $i = 1, 2, \dots$, of finite open collections of $X$ satisfying the following conditions:

(11) $$\overline{\mathfrak{U}}_{i1}' < \mathfrak{U}_{i-1,1}' \wedge \mathcal{G}_i\,,$$

(12) $$f(\mathfrak{U}_{i1}')^{\#} = Y\,.$$

Set

$$\mathfrak{U}_{i1}'^{\#} = L_i\,, \qquad \bigcap L_i = L\,.$$

To prove $L$ is compact let $\mathcal{F}$ be the maximal filtre of $L$. Since $\mathfrak{U}_{i1}'$ is a finite covering of $L_i$, there exists an $a \in B_{i1}$ such that $U_{i1a} \cap L \in \mathcal{F}$. Since $\overline{U}_{i1a}$ refines $\mathcal{G}_i$ by (11), then $\bigcap\{\overline{F}\colon\ F \in \mathcal{F}\} \neq \emptyset$, which proves $L$ is compact.

To prove $f(L) = Y$ let $y$ be an arbitrary point of $Y$. By (11) there exists $\varphi_i^{i+1}\colon\ B_{i+1,1} \to B_{i1}$ such that $\varphi_i^{i+1}(a) = \beta$ yields $\overline{U}_{i+1,1,a} \subset U_{i1\beta}$. Set

(13) $$C_i = \{a \in B_{i1}\colon\ U_{i1a} \cap f^{-1}(y) \neq \emptyset\}\,.$$

By (12) every $C_i$ is not empty and $\{C_i\}$ forms an inverse subsystem of $\{B_{i1};\ \varphi_i^{i+1}\}$. Since $\operatorname{inv\,lim} C_i$ is not empty, it contains an element $\langle a_i\rangle$. Then we have a sequence:

$$F_i = f^{-1}(y) \cap \overline{U}_{ia_i}\,, \qquad i = 1, 2, \dots$$

Since $F_i$ refines $\mathcal{G}_i$ by (11) and $\{F_i\}$ has a finite intersection property, then $\bigcap F_i \neq \emptyset$. Pick a point $x$ from $\bigcap F_i$. Since $F_i \subset L_{i-1}$, $\bigcap F_i \subset L$. Thus $x \in L$ and $f(x) = y$. The proof is finished.

2.7. Corollary. *If $X$ is a member of* OP(absolute $G_\delta$) *and $f$ is an* OP-*mapping defined on $X$, then $f$ is compact-covering.*

### 3. OCP-system.

3.1. Definition. A system

$$X_1 \xrightarrow{f_1} Y_1 \xrightarrow{g_1} X_2 \xrightarrow{f_2} \dots \xrightarrow{g_{n-1}} X_n \xrightarrow{f_n} Y_n$$

is said to be an OCP-*system* if each $f_i$ is an OC-mapping and each $g_i$ is a perfect one. A sequence:

(1) $$\{\mathfrak{U}_i = \{U_{ia}\colon\ a \in A_i\}\colon\ i = 1, \dots, n\}$$



is said to be a *canonical* one for this system if the following four conditions are satisfied.

(i) Each $\mathfrak{U}_i$ is an open collection of $X_i$.

(ii) $A_{i+1} = A_i^{\nabla}$.

(iii) $\mathfrak{U}_{i+1}$ is $g_i^{\nabla} f_i(\mathfrak{U}_i)$. Under our convention in Notation 2.5

$$U_{ia} = X_i - g_{i-1}\big(Y_{i-1} - \bigcup\{f_{i-1}(U_{i-1,\beta})\colon\ \beta \in a\}\big)\,.$$

(iv) $g_i^{-1}(\mathfrak{U}_{i+1})^{\#} = f_i(\mathfrak{U}_i)^{\#}$.

If there exists another canonical sequence

(2) $$\{\mathfrak{V}_i = \{V_{ia}\colon\ a \in B_i\}\colon\ i = 1, \dots, n\}$$

and a refine transformation $\varphi_i\colon\ B_i \to A_i$ for each $i$, then the sequence

(3) $$\{\varphi_1, \dots, \varphi_n\}$$

is said to be a *refine transformation* from the sequence (2) to the sequence (1). If the sequence (3) satisfies the condition:

(4) $$\varphi_i(a) \supset \varphi_{i-1}^{\nabla}(a)\,, \qquad a \in B_i$$

for each $i$, then (3) is said to be an *expanding sequence*. If the condition (4) is satisfied for some $i$, say $j$, then $\varphi_j$ is said to be an *expansion* of $\varphi_{j-1}$. If $\varphi_1$ is a refine transformation and (3) is expanding, then it can easily be seen that (3) is a refine transformation.

3.2. Lemma. *Consider an* OCP-*system*:

$$X_1 \xrightarrow{f_1} Y_1 \xrightarrow{g_1} X_2 \xrightarrow{f_2} \dots \xrightarrow{g_{n-1}} X_n \xrightarrow{f_n} Y_n\,.$$

*Let $K$ be a compact set of $Y_n$. Let*

$$\{\mathfrak{U}_i = \{U_{ia}\colon\ a \in A_i\}\colon\ i = 1, \dots, n\}$$

*be a canonical sequence with $A_i$ finite and with $f_n(\mathfrak{U}_n)^{\#} \supset K$. Then there exists a canonical sequence*

$$\{\mathfrak{V}_i = \{V_{ia}\colon\ a \in B_i\}\colon\ i = 1, \dots, n\}\,,$$

*with $B_i$ finite, which refines $\{\mathfrak{U}_i\colon\ i = 1, \dots, n\}$ by an expanding sequence $\{\varphi_i\colon\ i = 1, \dots, n\}$ of transformations $\varphi_i\colon\ B_i \to A_i$ such that $f_n(\mathfrak{V}_n)^{\#} \supset K$ and such that $\varphi_n$ is of type* II *with respect to $f_n$, $\mathfrak{V}_n$, $\mathfrak{U}_n$.*

Proof (by induction). Let $\mathfrak{V}_{n1} = \{V_{n1a}\colon\ a \in B_{n1}\}$ be a finite open collection of $X_n$ with $f_n(\mathfrak{V}_{n1})^{\#} \supset K$ and $\varphi_{n1}\colon\ B_{n1} \to A_n$ a transformation of type II with respect to $f_n$, $\mathfrak{V}_{n1}$, $\mathfrak{U}_n$. Such $\mathfrak{V}_{n1}$ and $\varphi_{n1}$ exist by Lemma 1.3. Set

(1) $$B_{n-1,2} = \{\langle a, \beta\rangle \in A_{n-1} \times B_{n1}\colon\ a \in \varphi_{n1}(\beta)\}\,,$$

(2) $$V_{n-1,2,\gamma} = U_{n-1,\alpha} \cap f_{n-1}^{-1} g_{n-1}^{-1}(V_{n1\beta}), \quad \gamma = \langle \alpha, \beta \rangle \in B_{n-1,2},$$

(3) $$\mathfrak{V}_{n-1,2} = \{V_{n-1,2,\gamma}\colon \gamma \in B_{n-1,2}\}.$$

Define $\varphi_{n-1,2}\colon B_{n-1,2} \to A_{n-1}$ by:

(4) $$\varphi_{n-1,2}(\langle \alpha, \beta \rangle) = \alpha.$$

Then $\varphi_{n-1,2}$ gives a refine transformation from $\mathfrak{V}_{n-1,2}$ to $\mathfrak{U}_{n-1}$. Set

(5) $$B_{n2} = B_{n-1,2}^{\nabla},$$

(6) $$\mathfrak{V}_{n2} = g_{n-1}^{\nabla} f_{n-1}(\mathfrak{V}_{n-1,2}) = \{V_{n2\alpha}\colon \alpha \in B_{n2}\}.$$

If $\beta \in B_{n1}$ and $\varphi_{n1}(\beta) = \gamma$, then $V_{n1\beta} \subset U_{n\gamma}$ and $g_{n-1}^{-1}(U_{n\gamma}) \subset \bigcup \{f_{n-1}(U_{n-1,\alpha})\colon \alpha \in \gamma\}$. If we set

$$\tilde{\beta} = \{\langle \alpha, \beta \rangle\colon \alpha \in \varphi_{n1}(\beta)\},$$

then $V_{n1\beta} = V_{n2\tilde{\beta}}$. Thus $\mathfrak{V}_{n1}^{\#} \subset \mathfrak{V}_{n2}^{\#}$. Since it is evident that $\mathfrak{V}_{n1}^{\#} \supset \mathfrak{V}_{n2}^{\#}$, we have

(7) $$\mathfrak{V}_{n1}^{\#} = \mathfrak{V}_{n2}^{\#},$$

and hence $f_n(\mathfrak{V}_{n2})^{\#} \supset K$. Since $g_{n-1}^{-1}(\mathfrak{V}_{n1})^{\#} \supset f_{n-1}(\mathfrak{V}_{n-1,2})^{\#}$ by (2) and $g_{n-1}^{-1}(\mathfrak{V}_{n1})^{\#} = g_{n-1}^{-1}(\mathfrak{V}_{n2})^{\#}$ by (7), then $g_{n-1}^{-1}(\mathfrak{V}_{n2})^{\#} \supset f_{n-1}(\mathfrak{V}_{n-1,2})^{\#}$ and hence $g_{n-1}^{-1}(\mathfrak{V}_{n2})^{\#} = f_n(\mathfrak{V}_{n-1,2})^{\#}$. Thus $\{\mathfrak{V}_{n-1,2}, \mathfrak{V}_{n2}\}$ is canonical.

Let $\delta$ be a finite subset $\{\langle \alpha_i, \beta_i \rangle\colon i = 1, \ldots, s\}$ of $B_{n-1,2}$. Define $\varphi_{n2}\colon B_{n2} \to A_n$ by:

(8) $$\varphi_{n2}(\delta) = \bigcup \{\varphi_{n1}(\beta_i)\colon i = 1, \ldots, s\}.$$

Since $\varphi_{n-1,2}^{\nabla}(\delta) = \{\alpha_1, \ldots, \alpha_s\}$ and $\{\alpha_1, \ldots, \alpha_s\} \subset \bigcup \{\varphi_{n1}(\beta_i)\colon i = 1, \ldots, s\}$, then $\varphi_{n2}$ is an expansion of $\varphi_{n-1,2}$. Set

(9) $$\langle \alpha_i, \beta_i \rangle = \delta_i, \quad \varphi_{n1}(\beta_i) = \gamma_i, \quad \bigcup \{\gamma_i\colon i = 1, \ldots, s\} = \varepsilon.$$

Then $\varphi_{n2}(\delta) = \varepsilon$ and

$$\begin{aligned} \overline{V}_{n2\delta} &\subset \bigcup \{\overline{g_{n-1} f_{n-1}(V_{n-1,2,\delta_i})}\colon i = 1, \ldots, s\} \\ &\subset \bigcup \{\overline{V}_{n1\beta_i}\colon i = 1, \ldots, s\} \\ &\subset \bigcup \{U_{n\gamma_i}\colon i = 1, \ldots, s\} \subset U_{n\varepsilon}. \end{aligned}$$

Thus the first condition for $\varphi_{n2}$ to be of type II with respect to $f_n$, $\mathfrak{V}_{n2}$, $\mathfrak{U}_n$ is satisfied. The remaining required condition can be verified analogously and hence $\varphi_{n2}$ is of type II. It is of some interest to see that $\mathfrak{V}_{n1}$ acted as a catalyzer to produce the canonical sequence $\{\mathfrak{V}_{n-1,2}, \mathfrak{V}_{n2}\}$. This step is logically contained in the following general step. To consider this special case will help greatly to understand the general case.



Put the induction assumption in the reverse way that there exist, for $i$ with $1 < i < n$, a canonical sequence

(10) $$\{\mathfrak{V}_{n-i+j,i} = \{V_{n-i+j,i,\alpha}\colon \alpha \in B_{n-i+j,i}\}\colon j = 1, \ldots, i\},$$

with $B_{n-i+1,i}$ finite, for the system:

$$X_{n-i+1} \xrightarrow{f_{n-i+1}} Y_{n-i+1} \xrightarrow{g_{n-i+1}} \ldots \xrightarrow{g_{n-1}} X_n \xrightarrow{f_n} Y_n$$

and an expanding sequence

$$\Phi = \{\varphi_{n-i+j,i}\colon j = 1, \ldots, i\}, \quad \text{where} \quad \varphi_{n-i+j,i}\colon B_{n-i+j,i} \to A_{n-i+j},$$

satisfying the following three conditions.

(11) $\Phi$ gives a refine transformation from (10) to $\{\mathfrak{U}_i, \ldots, \mathfrak{U}_n\}$.

(12) $\varphi_{ni}$ is of type II with respect to $f_n$, $\mathfrak{V}_{ni}$, $\mathfrak{U}_n$.

(13) $$f_n(\mathfrak{V}_{ni})^{\#} \supset K.$$

Set

(14) $$B_{n-i,i+1} = \{\langle \alpha, \beta \rangle \in A_{n-i} \times B_{n-i+1,i}\colon \alpha \in \varphi_{n-i+1,i}(\beta)\},$$

(15) $$V_{n-i,i+1,\gamma} = U_{n-i,\alpha} \cap f_{n-i}^{-1} g_{n-i}^{-1}(V_{n-i+1,i,\beta}), \quad \gamma = \langle \alpha, \beta \rangle \in B_{n-i,i+1},$$

(16) $$\mathfrak{V}_{n-i,i+1} = \{V_{n-i,i+1,\gamma}\colon \gamma \in B_{n-i,i+1}\},$$

(17) $$B_{n-i+j,i+1} = B_{n-i+j-1,i+1}^{\nabla}, \quad j = 1, \ldots, i,$$

(18) $$\begin{aligned} \mathfrak{V}_{n-i+j,i+1} &= g_{n-i+j-1}^{\nabla} f_{n-i+j-1}(\mathfrak{V}_{n-i+j-1,i+1}) \\ &= \{V_{n-i+j,i+1,\alpha}\colon \alpha \in B_{n-i+j,i+1}\}, \quad j = 1, \ldots, i. \end{aligned}$$

We have now the following diagram, where we are going to define transformations $\varphi_{*,i+1}$. The double arrow indicates that the range of the corresponding transformation is the top of the column.

$$\begin{tikzcd}
\mathfrak{U}_{n-i} & \mathfrak{U}_{n-i+1} & \mathfrak{U}_{n-i+2} & \cdots\ \mathfrak{U}_n \\
 & \mathfrak{V}_{n-i+1,i} \arrow[u, "\varphi_{n-i,i}"'] & \mathfrak{V}_{n-i+2,i} \arrow[u, "\varphi_{n-i+2,i}"'] & \cdots\ \mathfrak{V}_{ni} \arrow[u, "\varphi_{ni}"'] \\
\mathfrak{V}_{n-i,i+1} \arrow[uu, "\varphi_{n-i,i+1}"'] & \mathfrak{V}_{n-i+1,i+1} \arrow[u, two heads, "\varphi_{n-i+1,i+1}"'] & \mathfrak{V}_{n-i+2,i+1} \arrow[u, two heads, "\varphi_{n-i+2,i+1}"'] & \cdots\ \mathfrak{V}_{n,i+1} \arrow[u, two heads, "\varphi_{n,i+1}"']
\end{tikzcd}$$

Our final goal is to complete the following diagram:

$$\begin{tikzcd}
\mathfrak{U}_1 & \mathfrak{U}_2 & \cdots & \mathfrak{U}_{n-2} & \mathfrak{U}_{n-1} & \mathfrak{U}_n \\
& & & & & \mathfrak{V}_{n1} \arrow[u, "\varphi_{n1}"'] \\
& & & & \mathfrak{V}_{n-1,2} \arrow[uu, "\varphi_{n-1,2}"'] & \mathfrak{V}_{n2} \arrow[u, "\varphi_{n2}"'] \\
& & & \mathfrak{V}_{n-2,3} \arrow[uuu, "\varphi_{n-2,3}"'] & \mathfrak{V}_{n-1,3} \arrow[u] & \mathfrak{V}_{n3} \arrow[u, "\varphi_{n3}"'] \\
& & & \vdots & \vdots & \vdots \\
& \mathfrak{V}_{2,n-1} \arrow[uuuuu, "\varphi_{2,n-1}"'] & \cdots & \mathfrak{V}_{n-2,n-1} \arrow[u, "\varphi_{n-2,n-1}"'] & \mathfrak{V}_{n-1,n-1} \arrow[u, "\varphi_{n-1,n-1}"'] & \mathfrak{V}_{n,n-1} \arrow[u, "\varphi_{n,n-1}"'] \\
\mathfrak{V}_{1n} \arrow[uuuuuu, "\varphi_{1n}"'] & \mathfrak{V}_{2n} \arrow[u, "\varphi_{2n}"'] & \cdots & \mathfrak{V}_{n-2,n} \arrow[u, "\varphi_{n-2,n}"'] & \mathfrak{V}_{n-1,n} \arrow[u, "\varphi_{n-1,n}"'] & \mathfrak{V}_{nn} \arrow[u, "\varphi_{nn}"']
\end{tikzcd}$$

Define $\varphi_{n-i,i+1}\colon B_{n-i,i+1} \to A_{n-i}$ by:

(19) $$\varphi_{n-i,i+1}(\langle\alpha, \beta\rangle) = \alpha\,.$$

Then $\varphi_{n-i,i+1}$ is a refine transformation from $\mathfrak{V}_{n-i,i+1}$ to $\mathfrak{U}_{n-i}$. Let $\gamma$ be an element of $B_{n-i+1,i+1}$:

(20) $$\gamma = \{\langle\alpha_1, \beta_1\rangle, \ldots, \langle\alpha_s, \beta_s\rangle\}\,.$$

Define $\varphi_{n-i+1,i+1}\colon B_{n-i+1,i+1} \to A_{n-i+1}$ by:

(21) $$\varphi_{n-i+1,i+1}(\gamma) = \bigcup\{\varphi_{n-i+1,i}(\beta_k)\colon k = 1, \ldots, s\}\,.$$

By (19)

(22) $$\varphi^{\nabla}_{n-i,i+1}(\gamma) = \{\alpha_1, \ldots, \alpha_s\}\,.$$

By (14)

(23) $$\{\alpha_1, \ldots, \alpha_s\} \subset \bigcup\{\varphi_{n-i+1,i}(\beta_k)\colon k = 1, \ldots, s\}\,.$$

By (21), (22) and (23) $\varphi_{n-i+1,i+1}$ is an expansion of $\varphi_{n-i,i+1}$. Set

(24) $$\varphi_{n-i+j+1,i+1} = \varphi^{\nabla}_{n-i+j,i+1}\,, \qquad j = 1, \ldots, i-1\,.$$

Then

(25) $$\{\varphi_{n-i,i+1}, \varphi_{n-i+1,i+1}, \ldots, \varphi_{n,i+1}\}$$

is an expanding sequence and hence a refine transformation.

By a similar way to get (7) the following holds.

(26) $$\mathfrak{V}^{\#}_{n-i+1,i} = \mathfrak{V}^{\#}_{n-i+1,i+1}\,.$$



Since $\mathfrak{V}_{n-i+j,i}$ can be assumed to be a subcollection of $\mathfrak{V}_{n-i+j,i+1}$ for $j = 1, \ldots, i$, and the sequence (10) is canonical, then with the aid of (26),

(27) $$\mathfrak{V}^{\#}_{n-i+j,i+1} = \mathfrak{V}^{\#}_{n-i+j,i}\,, \qquad j = 1, \ldots, i\,,$$

and the sequence

(28) $$\{\mathfrak{V}_{n-i+j-1,i+1}\colon j = 1, \ldots, i+1\}$$

is canonical. By (27) and (13)

$$f_n(\mathfrak{V}_{n,i+1})^{\#} = f_n(\mathfrak{V}_{ni})^{\#} \supset K\,.$$

Let $\gamma$ be an arbitrary element of $B_{n-i+1,i+1}$ with the expression (20). Then the image of $\gamma$ under $\varphi_{n-i+1,i+1}$ has the expression (21). Set

(29) $$\varphi_{n-i+1,i}(\beta_k) = \delta_k\,, \qquad k = 1, \ldots, s\,.$$

Since $V_{n-i+1,i,\beta_k} \subset U_{n-i+1,\delta_k}$ for $k = 1, \ldots, s$ and $V_{n-i+1,i+1,\gamma} \subset \bigcup\{V_{n-i+1,i,\beta_k}\colon k = 1, \ldots, s\}$ by (15), then

(30) $$\begin{aligned} V_{n-i+1,i+1,\gamma} &\subset \bigcup\{V_{n-i+1,i,\beta_k}\colon k = 1, \ldots, s\} \\ &\subset \bigcup\{U_{n-i+1,\delta_k}\colon k = 1, \ldots, s\}\,. \end{aligned}$$

Set

(31) $$\bigcup\{\delta_k\colon k = 1, \ldots, s\} = \delta\,.$$

Then $\varphi_{n-i+1,i+1}(\gamma) = \delta$ by (21) and (29), and

(32) $$\bigcup\{U_{n-i+1,\delta_k}\colon k = 1, \ldots, s\} \subset U_{n-i+1,\delta}\,.$$

By (30) and (32)

(33) $$V_{n-i+1,i+1,\gamma} \subset \bigcup\{V_{n-i+1,i,\beta_k}\colon k = 1, \ldots, s\} \subset U_{n-i+1,\delta}\,.$$

The inequalities (33) tells us that we can insert a finite sum of elements of $\mathfrak{V}_{n-i+1,i}$, which are transformed to subsets, of type $U_{n-i+1,\delta_k}$, of $U_{n-i+1,\delta}$, between an element of $\mathfrak{V}_{n-i+1,i+1}$ taken arbitrarily, and its image element of $\mathfrak{U}_{n-i+1}$. This fact is as can easily be seen inherited to the next stage:

$$\mathfrak{V}_{n-i+2,i+1} \xrightarrow{\varphi_{n-i+2,i+1}} \mathfrak{V}_{n-i+2,i} \xrightarrow{\varphi_{n-i+2,i}} \mathfrak{U}_{n-i+2}$$

and so on. We finally reach the last stage:

$$\mathfrak{V}_{n,i+1} \xrightarrow{\varphi_{n,i+1}} \mathfrak{V}_{ni} \xrightarrow{\varphi_{ni}} \mathfrak{U}_n\,.$$

The inherited property for this system assures that $\varphi_{n,i+1}$ is of type II, since $\varphi_{ni}$ is of type II by induction assumption. Thus the induction is completed and we can get a nice canonical sequence

(34) $$\mathfrak{V}_{1n}, \mathfrak{V}_{2n}, \ldots, \mathfrak{V}_{nn}\,,$$

accompanied by a sequence of index sets:

$$B_{1n},\, B_{2n},\, \ldots,\, B_{nn}$$

and by an expanding sequence of transformations:

$$\varphi_{1n},\, \varphi_{2n},\, \ldots,\, \varphi_{nn}\,.$$

Set

$$B_i = B_{in}\,, \qquad i = 1, \ldots, n\,,$$

$$\varphi_i = \varphi_{in}\,, \qquad i = 1, \ldots, n\,,$$

$$\mathfrak{V}_i = \{V_{i\alpha} = V_{in\alpha}\colon\ \alpha \in B_i\}\,, \qquad i = 1, \ldots, n\,.$$

Then these satisfy all of the required conditions and the proof is finished.

In this argument we started from $\mathfrak{V}_{n1}$ accompanied by $\varphi_{n1}$ of type II. Look at the two diagrams in the preceding proof. If we start from $\mathfrak{V}_{n-i+1,i}, \mathfrak{V}_{n-i+2,i}, \ldots, \mathfrak{V}_{ni}$ accompanied by $\varphi_{n-i+1,i}, \varphi_{n-i+2,i}, \ldots, \varphi_{ni}$ all of which are of type II, then the resultant $\varphi_{n-i+1,n}, \ldots, \varphi_{nn}$ are of type II by a quite similar argument. So we get the following.

3.3. Lemma. *Let the* OCP-*system in Lemma* 3.2 *be given. Let $K$ and $\{\mathfrak{U}_1, \ldots, \mathfrak{U}_n\}$ be those in Lemma* 3.2. *Let*

$$\{\mathfrak{W}_i = \{W_{i\alpha}\colon\ \alpha \in C_i\}\colon\ i = j, \ldots, n\}\,,$$

*be a canonical sequence for the system:*

$$X_j \xrightarrow{f_j} Y_j \xrightarrow{g_j} \ldots \longrightarrow X_n \xrightarrow{f_n} Y_n$$

*with $C_i$ finite and $f_n(\mathfrak{W}_n)^{\#} \supset K$, and*

$$\{\psi_j, \ldots, \psi_n\}\,, \quad \textit{where} \quad \psi_i\colon\ C_i \to A_i\,,$$

*an expanding sequence such that each $\psi_i$ is of type* II *with respect to $f_i$, $\mathfrak{W}_i$, $\mathfrak{U}_i$. Then there exist a canonical sequence*

$$\{\mathfrak{V}_i = \{V_{i\alpha}\colon\ \alpha \in B_i\}\colon\ i = 1, \ldots, n\}\,,$$

*with $B_i$ finite, which refines $\{\mathfrak{U}_i\}$ by an expanding sequence*

$$\{\varphi_1, \ldots, \varphi_n\}\,, \quad \textit{where} \quad \varphi_i\colon\ B_i \to A_i\,,$$

*such that $f_n(\mathfrak{V}_n)^{\#} \supset K$ and such that $\varphi_i$ is of type* II *with respect to $f_i$, $\mathfrak{V}_i$, $\mathfrak{U}_i$ for $i = j, \ldots, n$.*

3.4. Lemma. *Consider the* OCP-*system in Lemma* 3.2. *Let $K$ and $\{\mathfrak{U}_i\}$ be those in Lemma* 3.2, *where $A_i$ is not necessarily finite. Let $\mathfrak{G}$ be an open covering of $X_1$. Then there exist a canonical sequence*

$$\{\mathfrak{V}_i = \{V_{i\alpha}\colon\ \alpha \in B_i\}\colon\ i = 1, \ldots, n\}\,,$$



*with $B_i$ finite, and a sequence*

$$\{\varphi_i\colon\ i = 1, \ldots, n\}$$

*of transformations $\varphi_i\colon\ B_i \to A_i$ satisfying the following conditions.*

(i) $\varphi_{i+1} = \varphi_i^{\nabla}$, $i = 1, \ldots, n-1$.

(ii) $\{\varphi_1, \ldots, \varphi_n\}$ *is a refine transformation from $\{\mathfrak{V}_1, \ldots, \mathfrak{V}_n\}$ to $\{\mathfrak{U}_1, \ldots, \mathfrak{U}_n\}$.*

(iii) $f_n(\mathfrak{V}_n)^{\#} \supset K$.

(iv) $\mathfrak{V}_1$ *refines* $\mathfrak{G}$.

Proof. Let $\mathfrak{W}_1 = \{W_{1\alpha}\colon\ \alpha \in C_1\}$ be an open collection of $X_1$, with $\mathfrak{W}_1^{\#} = \mathfrak{U}_1^{\#}$, which refines $\mathfrak{G} \wedge \mathfrak{U}_1$. Let $\psi_1\colon\ C_1 \to A_1$ be a transformation giving a refine one from $\mathfrak{W}_1$ to $\mathfrak{U}_1$. Set

$$C_{i+1} = C_i^{\nabla}, \qquad i = 1, \ldots, n-1\,, \tag{1}$$

$$\mathfrak{W}_{i+1} = \{W_{i+1,\alpha}\colon\ \alpha \in C_{i+1}\} = g_i^{\nabla} f_i(\mathfrak{W}_i)\,, \qquad i = 1, \ldots, n-1\,. \tag{2}$$

Since $\{\mathfrak{U}_i\}$ is canonical, $f_1(\mathfrak{U}_1)^{\#} = g_1^{-1}(\mathfrak{U}_2)^{\#}$ and hence $f_1(\mathfrak{W}_1)^{\#} = g_1^{-1}(\mathfrak{U}_2)^{\#}$. Thus $\mathfrak{W}_2^{\#} = \mathfrak{U}_2^{\#}$. In general we obtain

$$\mathfrak{W}_i^{\#} = \mathfrak{U}_i^{\#}, \qquad i = 1, \ldots, n\,. \tag{3}$$

Let $\mathfrak{W}_n'$ be a finite subcollection of $\mathfrak{W}_n$ with

$$f_n(\mathfrak{W}_n')^{\#} \supset K\,. \tag{4}$$

Set

$$\mathfrak{W}_n' = \{W_{n\alpha}' = W_{n\alpha}\colon\ \alpha \in C_n'\}\,,$$
$$C_{n-1}' = \{\alpha \in C_{n-1}\colon\ \alpha \in \beta \in C_n'\}\,,$$
$$C_{n-2}' = \{\alpha \in C_{n-2}\colon\ \alpha \in \beta \in C_{n-1}'\}\,,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$C_1' = \{\alpha \in C_1\colon\ \alpha \in \beta \in C_2'\}\,.$$

Set

$$\mathfrak{W}_{n-1}' = \{W_{n-1,\alpha}' = W_{n-1,\alpha} \cap f_{n-1}^{-1} g_{n-1}^{-1}(\mathfrak{W}_n')^{\#}\colon\ \alpha \in C_{n-1}'\}\,,$$
$$\mathfrak{W}_{n-2}' = \{W_{n-2,\alpha}' = W_{n-2,\alpha} \cap f_{n-2}^{-1} g_{n-2}^{-1}(\mathfrak{W}_{n-1}')^{\#}\colon\ \alpha \in C_{n-2}'\}\,,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\mathfrak{W}_1' = \{W_{1\alpha}' = W_{1\alpha} \cap f_1^{-1} g_1^{-1}(\mathfrak{W}_2')^{\#}\colon\ \alpha \in C_1'\}\,.$$

Then

$$f_i(\mathfrak{W}_i')^{\#} = g_i^{-1}(\mathfrak{W}_{i+1}')^{\#}, \qquad i = 1, \ldots, n-1\,. \tag{5}$$

Set

$$B_1 = C_1'\,, \quad B_{i+1} = B_i^{\nabla}, \quad i = 1, \ldots, n-1\,,$$
$$\mathfrak{V}_1 = \{V_{1\alpha} = W_{1\alpha}'\colon\ \alpha \in B_1\}\,,$$

$$\mathfrak{V}_{i+1} = \{V_{1\alpha}\colon\ \alpha \in B_{i+1}\} = g_i^\nabla f_i(\mathfrak{V}_i)\,, \qquad i = 1, \ldots, n-1\,,$$
$$\varphi_1 = \psi_1 | B_1\,, \qquad \varphi_{i+1} = \varphi_i^\nabla\,, \qquad i = 1, \ldots, n-1\,.$$

Then it is easy to see that $\{\mathfrak{V}_i\}$ accompanied by $\{B_i\}$ and $\{\varphi_i\}$ satisfies the required condition and the proof is finished.

3.5. Lemma. *Let $A$, $B$, $C$ be index sets, $\psi\colon C \to B$, $\varphi\colon B \to A$ transformations and $\psi_1\colon C^\nabla \to B^\nabla$, $\varphi_1\colon B^\nabla \to A^\nabla$ their expansions. Then $\varphi_1\psi_1\colon C^\nabla \to A^\nabla$ is an expansion of $\varphi\psi\colon C \to A$.*

Proof. Let $C_0$ be a finite subset of $C$. Then

$$\psi(C_0) \subset \psi_1(C_0) \tag{1}$$

as sets of $B$, since $\psi_1$ is an expansion of $\psi$ and $\psi(C_0) = \psi^\nabla(C_0)$ as an element of $B^\nabla$. Similarly

$$\varphi\psi_1(C_0) \subset \varphi_1\psi_1(C_0) \tag{2}$$

as sets of $A$. Since $\varphi\psi(C_0) \subset \varphi\psi_1(C_0)$ by (1), then by (2)

$$\varphi\psi(C_0) \subset \varphi_1\psi_1(C_0)\,. \tag{3}$$

Since $\varphi\psi(C_0) = \varphi^\nabla\psi^\nabla(C_0)$ as an element of $A^\nabla$ and $(\varphi\psi)^\nabla = \varphi^\nabla\psi^\nabla$ by Notation 2.5, we can see by (3) that $\varphi_1\psi_1$ is an expansion of $\varphi\psi$. The proof is finished.

As an immediate corollary of this lemma we obtain the following.

3.6. Lemma. *Let the following three systems of index sets be given:*

$$A_1, \ldots, A_n\,, \quad \textit{where} \quad A_{i+1} = A_i^\nabla\,,$$
$$B_1, \ldots, B_n\,, \quad \textit{where} \quad B_{i+1} = B_i^\nabla\,,$$
$$C_1, \ldots, C_n\,, \quad \textit{where} \quad C_{i+1} = C_i^\nabla\,.$$

*Let two systems of transformations:*

$$\Phi = \{\varphi_1, \ldots, \varphi_n\}\,, \qquad \varphi_i\colon B_i \to A_i\,,$$
$$\Psi = \{\psi_1, \ldots, \psi_n\}\,, \qquad \psi_i\colon C_i \to B_i\,,$$

*be given in such a way that each $\varphi_{i+1}$ is an expansion of $\varphi_i$ and each $\psi_{i+1}$ is is an expansion of $\psi_i$. Then the composition*

$$\Phi\Psi = \{\varphi_1\psi_1, \ldots, \varphi_n\psi_n\}$$

*is a transformation of $\{C_i\}$ to $\{A_i\}$ such that each $\varphi_{i+1}\psi_{i+1}$ is an expansion of $\varphi_i\psi_i$.*

3.7. Lemma. *Consider the OCP-system in Lemma 3.2. Let $K$ and $\{\mathfrak{U}_i\}$ be the same as in Lemma 3.2. Then there exist a canonical sequence*

$$\{\mathfrak{V}_i = \{V_{i\alpha}\colon\ \alpha \in B_i\}\colon\ i = 1, \ldots, n\}\,,$$



*with $B_i$ finite, and an expanding sequence*

$$\{\varphi_1, \ldots, \varphi_n\} \quad \textit{where} \quad \varphi_i\colon B_i \to A_i\,,$$

*satisfying the following conditions.*

*(i) Each $\varphi_i$ is of type* II *with respect to $f_i$, $\mathfrak{V}_i$, $\mathfrak{U}_i$.*

*(ii) $f_n(\mathfrak{V}_n)^\# \supset K$.*

Proof (by induction). When $n = 1$, the assertion is true by Lemma 1.3. So we consider the case when $n > 1$. Let $P_j$ be the assertion that there exist a canonical sequence

$$\{\mathfrak{W}_i = \{W_{i\alpha}\colon\ \alpha \in C_i\}\colon\ i = 1, \ldots, n\}\,,$$

with $C_i$ finite, and an expanding sequence

$$\{\psi_1, \ldots, \psi_n\}\,, \qquad \text{where} \qquad \psi_i\colon C_i \to A_i\,,$$

satisfying the following conditions.

(1) $\psi_i$ is of type II with respect to $f_i$, $\mathfrak{W}_i$, $\mathfrak{U}_i$ for $i = n-j, \ldots, n$.

(2) $f_n(\mathfrak{W}_n)^\# \supset K$.

Then $P_0$ is true by Lemma 3.2. Put the induction assumption that $P_j$ is true, where $0 \leqslant j < n$. Let us deduce $P_{j+1}$ from $P_j$. Consider the system:

$$X_{n-j-1} \xrightarrow{f_{n-j-1}} Y_{n-j-1} \xrightarrow{g_{n-j-1}} \ldots \longrightarrow X_n \xrightarrow{f_n} Y_n\,. \tag{3}$$

By Lemma 1.3 there exist an open collection

$$\mathfrak{L}_{n-j-1} = \{L_{n-j-1,\alpha}\colon\ \alpha \in D_{n-j-1}\}$$

of $X_{n-j-1}$ with

$$f_{n-j-1}(\mathfrak{L}_{n-j-1})^\# = f_{n-j-1}(\mathfrak{W}_{n-j-1})^\# \tag{4}$$

and a transformation $\theta_{n-j-1}\colon D_{n-j-1} \to C_{n-j-1}$ of type II with respect to $f_{n-j-1}$, $\mathfrak{L}_{n-j-1}$, $\mathfrak{W}_{n-j-1}$. Set

$$D_i = D_{i-1}^\nabla\,, \qquad i = n-j, \ldots, n\,, \tag{5}$$

$$\theta_i = \theta_{i-1}^\nabla\,, \qquad i = n-j, \ldots, n\,, \tag{6}$$

$$\mathfrak{L}_i = \{L_{i\alpha}\colon\ \alpha \in D_i\} = g_{i-1}^\nabla f_{i-1}(\mathfrak{L}_{i-1})\,, \qquad i = n-j, \ldots, n\,. \tag{7}$$

By (4) $\mathfrak{W}_i^\# = \mathfrak{L}_i^\#$ for $i = n-j, \ldots, n$. Especially

$$f_n(\mathfrak{L}_n)^\# \supset K\,. \tag{8}$$

Thus

$$\{\mathfrak{L}_{n-j-1}, \ldots, \mathfrak{L}_n\} \tag{9}$$

is canonical. $\{\theta_{n-j-1}, \dots, \theta_n\}$ gives a refine transformation of $\{\mathcal{L}_{n-j-1}, \dots, \mathcal{L}_n\}$ to $\{\mathcal{W}_{n-j-1}, \dots, \mathcal{W}_n\}$. By Lemma 3.4 there exist a canonical sequence

$$\{\mathcal{M}_i = \{M_{i\alpha}\colon \alpha \in E_i\}\colon i = n-j-1, \dots, n\}$$

for the system (3), with $E_i$ finite, and a sequence

$$\{\xi_{n-j-1}, \dots, \xi_n\}$$

of transformations $\xi_i\colon E_i \to D_i$ satisfying the following three conditions.

(10) $$\xi_i = \xi_{i-1}^{\nabla}, \qquad i = n-j, \dots, n.$$

(11) $\{\xi_{n-j-1}, \dots, \xi_n\}$ is a refine transformation from $\{\mathcal{M}_{n-j-1}, \dots, \mathcal{M}_n\}$ to $\{\mathcal{L}_{n-j-1}, \dots, \mathcal{L}_n\}$.

(12) $$f_n(\mathcal{M}_n)^{\#} \supset K.$$

We have now the following diagram:

$$\begin{array}{ccccccc}
\mathcal{U}_{n-j-1} \leftrightarrow & A_{n-j-1} & \dots & \mathcal{U}_n \leftrightarrow & A_n \\
& \uparrow{\scriptstyle \varphi_{n-j-1}} & & & \uparrow{\scriptstyle \varphi_n} \\
\mathcal{W}_{n-j-1} \leftrightarrow & C_{n-j-1} & \dots & \mathcal{W}_n \leftrightarrow & C_n \\
& \uparrow{\scriptstyle \theta_{n-j-1}} & & & \uparrow{\scriptstyle \theta_n} \\
\mathcal{L}_{n-j-1} \leftrightarrow & D_{n-j-1} & \dots & \mathcal{L}_n \leftrightarrow & D_n \\
& \uparrow{\scriptstyle \xi_{n-j-1}} & & & \uparrow{\scriptstyle \xi_n} \\
\mathcal{M}_{n-j-1} \leftrightarrow & E_{n-j-1} & \dots & \mathcal{M}_n \leftrightarrow & E_n
\end{array}$$

Set

(13) $$\varrho_i = \varphi_i \theta_i \xi_i, \qquad i = n-j-1, \dots, n.$$

Then by Lemma 3.5 the sequence

(14) $$\{\varrho_{n-j-1}, \dots, \varrho_n\}$$

is expanding. Since $\theta_{n-j-1}$ is of type II, $\varrho_{n-j-1}$ is of type II with respect to $f_{n-j-1}, \mathcal{M}_{n-j-1}, \mathcal{U}_{n-j-1}$. By (1) $\varrho_i$ is of type II with respect to $f_i, \mathcal{M}_i, \mathcal{U}_i$ for $i = n-j, \dots, n$. Apply Lemma 3.7 to the system:

$$\begin{array}{ccccc}
\mathcal{U}_1 \dots & \mathcal{U}_{n-j-1} & \dots & \mathcal{U}_n \\
& \uparrow{\scriptstyle \varrho_{n-j-1}} & & \uparrow{\scriptstyle \varrho_n} \\
& \mathcal{M}_{n-j-1} & \dots & \mathcal{M}_n
\end{array}$$

Then the catalyzer $\mathcal{M}_{n-j-1}, \dots, \mathcal{M}_n$ produces, with the aid of $\varrho_{n-j-1}, \dots, \varrho_n$ of type II, a canonical sequence

(15) $$\{\mathcal{O}_i = \{O_{i\alpha}\colon \alpha \in F_i\}\colon i = 1, \dots, n\}$$



and an expanding sequence

(16) $$\{\eta_1, \dots, \eta_n\}, \qquad \text{where} \qquad \eta_i\colon F_i \to A_i,$$

giving a refine transformation from (15) to $\{\mathcal{U}_i\}$, such that $f_n(\mathcal{O}_n)^{\#} \supset K$ and $\eta_i$ is of type II with respect to $f_i, \mathcal{O}_i, \mathcal{U}_i$ for $i = n-j-1, \dots, n$. $P_{j+1}$ is thus deduced and the induction is completed. We know now $P_n$ is true, which assures the validity of the lemma. The proof is finished.

We are now completely ready to prove the next main theorem.

## 4. OCP ($p$-spaces).

4.1. THEOREM. *Let an* OCP-*system*:

$$X_1 \xrightarrow{f_1} Y_1 \xrightarrow{g_1} \dots \xrightarrow{g_{n-1}} X_n \xrightarrow{f_n} Y_n$$

*be given, where $X_1$ is a $p$-space. Set $f = f_n g_{n-1} \dots g_1 f_1$. Then $f$ is compact-covering.*

Proof. Let $K$ be an arbitrary non-empty compact set of $Y$. Let $\mathcal{K}_i$, $i = 0, 1, 2, \dots$, be a pluming of $X_1$. Let $\mathcal{G}_i$ be an open collection of $\beta X_1$ such that $\mathcal{G}_i^{\#} = \mathcal{K}_i^{\#}$ and $\widetilde{\mathcal{G}}_i < \mathcal{K}_i$. Set

(1) $$\mathcal{U}_{01} = \mathcal{G}_0 | X_1 = \{U_{01\alpha}\colon \alpha \in A_{01}\},$$

(2) $$A_{0,i+1} = A_{0i}^{\nabla}, \qquad i = 1, \dots, n-1,$$

(3) $$\mathcal{U}_{0,i+1} = g_i^{\nabla} f_i(\mathcal{U}_{0i}) = \{U_{0,i+1,\alpha}\colon \alpha \in A_{0,i+1}\}, \qquad i = 1, \dots, n-1.$$

Then $\mathcal{U}_{0i}^{\#} = X_i$ for each $i$. Hence

(4) $$\{\mathcal{U}_{01}, \dots, \mathcal{U}_{0n}\}$$

is canonical and of course $f_n(\mathcal{U}_{0n})^{\#} \supset K$. By Lemma 3.4 there exists a canonical sequence

(5) $$\{\mathcal{V}_{1i} = \{V_{1i\alpha}\colon \alpha \in A'_{1i}\}\colon i = 1, \dots, n\},$$

with $A'_{1i}$ finite, and a sequence

(6) $$\{\varphi'_{11}, \dots, \varphi'_{1n}\}$$

of transformations $\varphi'_{1i}\colon A'_{1i} \to A_{0i}$ satisfying the following four conditions.

(7) $$\varphi'_{1,i+1} = \varphi'^{\nabla}_{1i}, \qquad i = 1, \dots, n-1.$$

(8) $\{\varphi'_{11}, \dots, \varphi'_{1n}\}$ is a refine transformation from $\{\mathcal{V}_{11}, \dots, \mathcal{V}_{1n}\}$ to $\{\mathcal{U}_{01}, \dots, \mathcal{U}_{0n}\}$.

(9) $$f_n(\mathcal{V}_{1n})^{\#} \supset K.$$

(10) $$\mathcal{V}_{11} < \mathcal{G}_1 | X_1.$$

Apply Lemma 3.7 to the sequence (5). Then there exist a canonical sequence

(11) $$\{\mathfrak{U}_{1i} = \{U_{1i\alpha}\colon \alpha \in A_{1i}\}\colon i = 1, \ldots, n\},$$

with $A_{1i}$ finite, and an expanding sequence

(12) $$\{\varphi_{11}, \ldots, \varphi_{1n}\}, \quad \varphi_{1i}\colon A_{1i} \to A'_{1i},$$

satisfying the following three conditions.

(13) Each $\varphi_{1i}$ is of type II with respect to $f_i$, $\mathfrak{U}_{1i}$, $\mathfrak{V}_{1i}$.

(14) $\{\varphi_{1i}\}$ gives a refine transformation from $\{\mathfrak{U}_{1i}\}$ to $\{\mathfrak{V}_{1i}\}$.

(15) $$f_n(\mathfrak{U}_{1n})^{\#} \supset K.$$

Continuing applications of Lemma 3.4 and Lemma 3.7 one after the other in this manner we get two canonical sequences:

$$U_i = \{\mathfrak{U}_{ij} = \{U_{ij\alpha}\colon \alpha \in A_{ij}\}\colon j = 1, \ldots, n\}, \quad i = 0, 1, 2, \ldots,$$

$$V_i = \{\mathfrak{V}_{ij} = \{V_{ij\alpha}\colon \alpha \in A'_{ij}\}\colon j = 1, \ldots, n\}, \quad i = 1, 2, \ldots,$$

and two sequence of transformations:

(16) $$\Phi'_i = \{\varphi'_{ij}\colon j = 1, \ldots, n\}, \quad i = 1, 2, \ldots,$$

(17) $$\Phi_i = \{\varphi_{ij}\colon j = 1, \ldots, n\}, \quad i = 1, 2, \ldots,$$

satisfying the following eleven conditions for $i = 1, 2, \ldots$ and $j = 1, \ldots, n$.

(18) $A_{ij}$ is finite.

(19) $A'_{ij}$ is finite.

(20) $\varphi'_{ij}\colon A'_{ij} \to A_{i-1,j}$.

(21) $\varphi_{ij}\colon A_{ij} \to A'_{ij}$.

(22) $\varphi'_{i,j+1} = \varphi'^{\nabla}_{ij}$.

(23) $\Phi'_i$ gives a refine transformation from $V_i$ to $U_{i-1}$.

(24) $\Phi_i$ is an expanding sequence giving a refine transformation from $U_i$ to $V_i$.

(25) $f_n(\mathfrak{U}_{in})^{\#} \supset K$.

(26) $f_n(\mathfrak{V}_{in})^{\#} \supset K$.

(27) $\varphi_{ij}$ is of type II with respect to $f_i$, $\mathfrak{U}_{ij}$, $\mathfrak{V}_{ij}$.

(28) $\mathfrak{V}_{i1}$ refines $\mathcal{G}_i | X_1$.

Set

(29) $$\Psi_i = \Phi'_i \Phi_i = \{\psi_{ij} = \varphi'_{ij}\varphi_{ij}\colon j = 1, \ldots, n\}.$$

Then

(30) $\Psi_i$ is an expanding sequence



by Lemma 3.6 which gives a refine transformation from $U_i$ to $U_{i-1}$ by (23) and (27). Moreover by (22) and (27)

(31) each $\psi_{ij}$ is of type II with respect to $f_j$, $\mathfrak{U}_{ij}$, $\mathfrak{U}_{i-1,j}$.

By (24) and (28)

(32) $\mathfrak{U}_{i1}$ refines $\mathcal{G}_i | X_1$.

We have now the following diagram:

$$\begin{array}{ccccccc}
\mathfrak{U}_{11} \leftrightarrow & A_{11} & \ldots & \mathfrak{U}_{1n} \leftrightarrow & A_{1n} \\
 & \uparrow{\scriptstyle \psi_{21}} & & & \uparrow{\scriptstyle \psi_{2n}} \\
\mathfrak{U}_{21} \leftrightarrow & A_{21} & \ldots & \mathfrak{U}_{2n} \leftrightarrow & A_{2n} \\
 & \uparrow{\scriptstyle \psi_{31}} & & & \uparrow{\scriptstyle \psi_{3n}} \\
\mathfrak{U}_{31} \leftrightarrow & A_{31} & \ldots & \mathfrak{U}_{3n} \leftrightarrow & A_{3n} \\
 & \uparrow & & & \uparrow \\
 & \vdots & & & \vdots
\end{array}$$

Set

(33) $$A_i = \{\langle\alpha_1, \ldots, \alpha_n\rangle \in A_{i1} \times \ldots \times A_{in}\colon \alpha_1 \in \alpha_2 \in \ldots \in \alpha_n\}, \quad i = 1, 2, \ldots$$

To show that

(34) $$\Psi_{i+1}(A_{i+1}) \subset A_i$$

let $\langle\beta_1, \ldots, \beta_n\rangle$ be an arbitrary element of $A_{i+1}$ and $\langle\alpha_1, \ldots, \alpha_n\rangle$ its image under $\Psi_{i+1}$. Since $\Psi_{i+1} = \{\psi_{i+1,1}, \ldots, \psi_{i+1,n}\}$ is expanding by (30),

$$\alpha_1 = \psi_{i+1,1}(\beta_1) \in \psi_{i+1,1}(\beta_2) = \psi^{\nabla}_{i+1,1}(\beta_2) \subset \psi_{i+1,2}(\beta_2) = \alpha_2.$$

Analogously $\alpha_2 \in \alpha_3 \in \ldots \in \alpha_n$. Thus (34) is valid. Let us show that

(35) $$A_i \neq \emptyset.$$

Since $K \neq \emptyset$, we can find by (25) an element $\alpha_n \in A_{in}$ such that $f_n(U_{in\alpha_n}) \cap K \neq \emptyset$. Hence $\alpha_n \neq \emptyset$. Since $\{\mathfrak{U}_{i1}, \ldots, \mathfrak{U}_{in}\}$ is canonical, $A^{\nabla}_{i,n-1} = A_{in}$. Hence there exists an $\alpha_{n-1} \in A_{i,n-1}$ with $\alpha_{n-1} \in \alpha_n$ and with

$$f_n g_{n-1} f_{n-1}(U_{i,n-1,\alpha_{n-1}}) \cap K \neq \emptyset.$$

Continuing in this manner we get a sequence $\alpha_1 \in \alpha_2 \in \ldots \in \alpha_n$, where $\alpha_j \in A_{ij}$, such that

(36) $$f_n g_{n-1} \ldots g_j f_j(U_{ij\alpha_j}) \cap K \neq \emptyset, \quad j = 1, \ldots, n.$$

This will be a meaningful fact later. At any rate $\langle\alpha_1, \ldots, \alpha_n\rangle$ thus obtained is an element of $A_i$ and (35) is true. Since each $A_{ij}$ is finite by (18), each $A_i$ is finite. Therefore by (34) and (35)

(37) $$\{A_i, \Psi_i\}$$

forms an inverse system consisting of non-empty finite sets.

For each $i$ and each $\langle \alpha_1, \ldots, \alpha_n \rangle \in A_i$ set

(38) $$W(\alpha_n) = U_{in\alpha_n}, \quad W(\alpha_j \ldots \alpha_n) = U_{ij\alpha_j} \cap f_j^{-1} g_j^{-1}\big(W(\alpha_{j+1} \ldots \alpha_n)\big), \quad j = 1, \ldots, n-1.$$

Then we get an open set $W(\alpha_1 \ldots \alpha_n)$ of $X_1$. Set

(39) $$B_i = \{\sigma \in A_i\colon f\big(W(\sigma)\big) \cap K \neq \emptyset\}.$$

Then by (36)

(40) $$B_i \neq \emptyset.$$

Set

(41) $$W_i = \bigcup \{W(\sigma)\colon \sigma \in B_i\},$$

(42) $$L = \bigcap \tilde{W}_i.$$

Then $L$ is a compact set of $\beta X_1$. By the definition of $W(\ )$'s in (38), by the definition of $\Psi_i$ in (29) and by the fact that each $\psi_{ij}$ is a refine transformation by (31), the equalities, $\sigma \in B_i$ and $\Psi_i(\sigma) = \tau$, implies $\tau \in B_{i-1}$, for each $i$. In other words

(43) $\{B_i;\ \Psi_i | B_i\}$ forms an inverse system.

To show $L \subset X_1$ let $\tilde{x}$ be an arbitrary point of $L$. Set

(44) $$C_i = \{\sigma_i \in B_i\colon \tilde{x} \in \widetilde{W(\sigma_i)}\}.$$

Then $C_i \neq \emptyset$ and

(45) $\{C_i;\ \Psi_i | C_i\}$ forms an inverse system.

Since $C_i \neq \emptyset$ for each $i$, $\operatorname{invlim} C_i \neq \emptyset$. Pick an element $\langle \sigma_i \rangle$ from $\operatorname{invlim} C_i$. Then

(46) $$\tilde{x} \in \bigcap \widetilde{W(\sigma_i)}.$$

Set

(47) $$\sigma_i = \langle \alpha_{i1}, \ldots, \alpha_{in} \rangle, \quad i = 1, 2, \ldots$$

Since

(48) $$f_n\big(W(\alpha_{in})\big) \supset \overline{f_n\big(W(\alpha_{i+1,n})\big)}, \quad i = 1, 2, \ldots,$$

by (31) and (38), and $f_n\big(W(\alpha_{in})\big) \cap K \neq \emptyset$ for $i = 1, 2, \ldots$ by (38) and (39), then

(49) $$\Big(\bigcap_i f_n\big(W(\alpha_{in})\big)\Big) \cap K \neq \emptyset.$$

Pick a point $y_n$ from this intersection. Then we can get a point $x_n$ from $X_n$ with

(50) $$f_n(x_n) = y_n \quad \text{and} \quad x_n \in \bigcap_i W(\alpha_{in})$$

by Lemma 2.3.



Consider next the compact set $g_{n-1}^{-1}(x_n)$. Since

$$W(\alpha_{i,n-1}\alpha_{in}) \supset \overline{W(\alpha_{i+1,n-1}\alpha_{i+1,n})}$$

for each $i$ by (31) and $\alpha_{i,n-1} \in \alpha_{in}$ for each $i$, then

(51) $$\Big(\bigcap_i f_{n-1}\big(W(\alpha_{i,n-1}\alpha_{in})\big)\Big) \cap g_{n-1}^{-1}(x_n) \neq \emptyset.$$

Here we insert the notice that, for each $j$ with $1 \leqslant j \leqslant n$,

$$W(\alpha_{ij} \ldots \alpha_{in}) \supset \overline{W(\alpha_{i+1,j} \ldots \alpha_{i+1,n})}.$$

This can easily be seen by (31) with the consideration of definition (38). Here is another note: If $p_j \in X_j$ and $p_j \in \bigcap_i W(\alpha_{ij} \ldots \alpha_{in})$, then $f_{j-1}\big(W(\alpha_{i,j-1} \ldots \alpha_{in})\big) \cap g_{j-1}^{-1}(p_j) \neq \emptyset$ for each $i$. This can be seen by (38) and the fact that $\alpha_{i1} \in \ldots \in \alpha_{in}$ for each $i$.

With the aid of these two notices we continue the argument as in the above to obtain points $x_i \in X_i$, $y_i \in Y_i$, $i = 1, \ldots, n$, satisfying the following three conditions.

(52) $$x_j \in \bigcap_i W(\alpha_{ij} \ldots \alpha_{in}), \quad j = 1, \ldots, n.$$

(53) $$y_j \in \Big(\bigcap_i f_j\big(W(\alpha_{ij} \ldots \alpha_{in})\big)\Big) \cap g_j^{-1}(x_{j+1}), \quad j = 1, \ldots, n-1.$$

(54) $$f_j(x_j) = y_j, \quad j = 1, \ldots, n.$$

By (47), (49), (52), (53) and (54),

(55) $$x_1 \in \bigcap W(\sigma_i) \quad \text{and} \quad f(x_1) \in K.$$

Since $W(\sigma_i) \subset U_{i1\alpha_{i1}}$ by (38) and (47) and the right term refines $\mathcal{G}_i$ by (32), we can find for each $i$ an element $G_i$ of $\mathcal{G}_i$ with

(56) $$W(\sigma_i) \subset G_i.$$

Since $\tilde{\mathcal{G}}_i$ refines $\mathcal{H}_i$, we can find for each $i$ an element $H_i$ of $\mathcal{H}_i$ with

(57) $$\tilde{G}_i \subset H_i.$$

By (55), (56) and (57),

(58) $$x_1 \in \bigcap H_i.$$

Since $x_1$ is a point of $X_1$ and $\{\mathcal{H}_i\}$ is a pluming, then

(59) $$\bigcap H_i \subset X_1,$$

by Lemma 1.5. By (46), (56) and (57),

(60) $$\tilde{x} \in \bigcap H_i.$$

By (59) and (60), $\tilde{x} \in X_1$. Since $\tilde{x}$ was an arbitrary point of $L$, then $L \subset X_1$.

To show $f(L)\supset K$ let us start with taking an arbitrary point $q_n$ of $K$. Set

$$D_{in}=\{a\in A_{in}\colon\ q_n\in f_n(U_{ina})\}\,,\qquad i=1,2,\ldots$$

Then $D_{in}\neq\emptyset$ by (25). Since $\psi_{in}$ gives a refine transformation from $f_n(\mathfrak{U}_{in})$ to $f_n(\mathfrak{U}_{i-1,n})$ by (31), then $\psi_{in}(D_{in})\subset\psi_{i-1,n}(D_{i-1,n})$. Thus

(61) $$\{D_{in};\ \psi_{i+1,n}|D_{i+1,n}\colon\ i=1,2,\ldots\}$$

forms an inverse system. Since $D_{in}$ is finite by (18), $\operatorname{invlim}D_{in}\neq\emptyset$. Pick an element $\langle\beta_{1n},\beta_{2n},\ldots\rangle$ from $\operatorname{invlim}D_{in}$. Since $q_n\in\bigcap_i f_n(U_{in\beta_{in}})$ and $\psi_{in}$ is of type II with respect to $f_n$, $\mathfrak{U}_{in}$, $\mathfrak{U}_{i-1,n}$ by (31), we can find a point $p_n\in X_n$ such that

(62) $$p_n\in\bigcap_i U_{in\beta_{in}}\quad\text{and}\quad f_n(p_n)=q_n$$

by Lemma 2.3. Pick an arbitrary point $q_{n-1}$ from $g_{n-1}^{-1}(p_n)$. Since $\{\mathfrak{U}_{ij}\colon\ j=1,\ldots,n\}$ is canonical for each $i$,

(63) $$g_{n-1}^{-1}(p_n)\subset\cup\{f_{n-1}(U_{i,n-1,a})\colon\ a\in\beta_{in}\}\,,\qquad i=1,2,\ldots$$

Set

(64) $$D_{i,n-1}=\{a\in\beta_{in}\colon\ q_{n-1}\in f_{n-1}(U_{i,n-1,a})\}\,,\qquad i=1,2,\ldots$$

Then $D_{i,n-1}$ is a non-empty finite set of $A_{i,n-1}$ by (63). Since $\psi_{i,n-1}$ gives a refine transformation from $f_{n-1}(\mathfrak{U}_{i,n-1})$ to $f_{n-1}(\mathfrak{U}_{i-1,n-1})$ by (31), then

$$\psi_{i+1,n-1}(D_{i+1,n-1})\subset D_{i,n-1}\quad\text{for}\quad i=1,\ldots,n-1\,.$$

Thus

(65) $$\{D_{i,n-1};\ \psi_{i+1,n-1}|D_{i+1,n-1}\colon\ i=1,2,\ldots\}$$

forms an inverse system. Pick an element $\langle\beta_{i,n-1}\rangle$ from $\operatorname{invlim}D_{i,n-1}$. The $q_{n-1}$ is now in $\bigcap_i f_{n-1}(U_{i,n-1,\beta_{i,n-1}})$. Continuing in this manner, we get two point sequences:

$$p_i\in X_i\,,\quad q_i\in Y_i\,,\quad i=1,\ldots,n\,,$$

and

$$\beta_{ij}\in A_{ij}\,,\quad j=1,\ldots,n,\ i=1,2,\ldots,$$

satisfying the following five conditions.

(66) $$q_i\in g_i^{-1}(p_{i+1})\,,\qquad i=1,\ldots,n-1\,.$$

(67) $$p_j\in\bigcap_i U_{ij\beta_{ij}}\,,\qquad j=1,\ldots,n\,.$$

(68) $$f_j(p_j)=q_j\,,\qquad j=1,\ldots,n\,.$$



(69) $$q_j\in\bigcap_i f_j(U_{ij\beta_{ij}})\,,\qquad j=1,\ldots,n\,.$$

(70) $$\beta_{i1}\in\beta_{i2}\in\ldots\in\beta_{in}\,,\qquad i=1,2,\ldots$$

(71) $$\psi_{ij}(\beta_{ij})=\beta_{i-1,j}\,,\qquad i=2,3,\ldots,\ j=1,\ldots,n\,.$$

Set

(72) $$\tau_i=\langle\beta_{i1},\ldots,\beta_{in}\rangle\,,\qquad i=1,2,\ldots$$

Then by (70) $\tau_i$ is an element of $A_i$. By (66), (67) and (69),

(73) $$p_1\in W(\tau_i)\,,\qquad i=1,2,\ldots$$

Since $q_n\in f\big(W(\tau_i)\big)$ and $q_n\in K$, then $\tau_i\in B_i$ by the definition (39) of $B_i$. Thus by (73) and (42),

(74) $$p_1\in L\,.$$

Since $f(p_1)=q_n$ by (66) and (68), then $q_n\in f(L)$. Since $q_n$ was an arbitrary point of $K$, then $f(L)\supset K$. The proof is finished.

4.2. Corollary. *If $X$ is a member of* OCP*($p$-spaces) and $f$ is an* OCP*-mapping defined on $X$, then $f$ is compact-covering.*

## 5. Application of Theorem 4.1.

5.1. Lemma. *If $f\colon X\to Y$ is an open compact-covering mapping and $X$ is of countable type, then $Y$ is of countable type.*

5.2. Lemma. *If $f\colon X\to Y$ is a perfect mapping and $X$ is of countable type, then $Y$ is of countable type.*

These two lemmas are easy exercises.

5.3. Lemma (Čoban [15], Theorem 10) [11]. *If $X$ is a $p$-space, then $X$ is of countable type.*

5.4. Theorem. *Let $Y$ be an arbitrary element of* OCP*($p$-spaces). Then $Y$ is of countable type.*

Proof (by induction on $n$). Let

$$X_1\xrightarrow{f_1}Y_1\xrightarrow{g_1}\ldots\longrightarrow Y_{n-1}\xrightarrow{g_{n-1}}X_n\xrightarrow{f_n}Y_n$$

be an OCP-system, where $X_1$ is a $p$-space and $Y_n=Y$. When $n=1$, $Y_1$ is the image of $X_1$, which is of countable type by Lemma 5.3, under $f_1$. Since $f_1$ is an open compact-covering mapping by Theorem 2.1, $Y_1$ is of countable type by Lemma 5.1. Put the induction hypothesis that $Y_{n-1}$ is of countable type, where $n-1\geqslant 1$. By Lemma 5.2 $X_n$ is of countable type, since $g_{n-1}$ is perfect. Since $f_n$ is an open compact-covering mapping by Corollary 4.2, $Y_n$ is of countable type by Lemma 5.1. The proof is finished.

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[11] The referee kindly noted that this lemma was proved by Čoban. Cf. Lemma 8.3.

5.5. LEMMA (Arhangel'skiĭ [13], Theorem 3.21). *If $X$ is of countable type and the countable sum of compact metric sets, then $X$ is metric.*

5.6. THEOREM. *If $Y \in \mathrm{OCP}(\text{Moore})$ and $Y$ is $\sigma$-compact, i.e. the countable sum of compact sets, then $Y$ is metric.*

Proof. Each compact set of $Y$ is the continuous image of some compact set of a Moore space by Theorem 4.1 (cf. Section 8). Since each compact set of a Moore space is metric, each compact set of $Y$ is metric. Thus $Y$ is the countable sum of compact metric sets. Since $Y$ is of countable type by Theorem 5.4 (cf. Section 8), $Y$ satisfies the condition of Lemma 5.5 and is metric. The proof is finished.

5.7. THEOREM. *If $Y \in \mathrm{OCP}(p\text{-spaces})$ and $Y$ is countable, then $Y$ is metric.*

Proof. Since $Y$ is of countable type by Theorem 5.4, $Y$ is metric by Lemma 5.5 and the proof is finished.

5.8. THEOREM (CH). *If $Y \in \mathrm{OCP}(\mathit{Moore})$, then $w(Y)$, the weight of $Y$, cannot exceed its power $|Y|$.*

Proof (by induction on $n$). When $Y$ is countable, $Y$ is metric by Theorem 5.6. Consider the case when $Y$ is uncountable. Let $Y$ be a Moore space having a development $\{\mathfrak{U}_i\}$. For each point $y$ of $Y$ and each $i$ pick an element $U(y,i)$ of $\mathfrak{U}_i$ with $y \in U(y,i)$. Then $\{U(y,i): y \in Y, i = 1, 2, \ldots\}$ is a base whose power is at most $|Y|$. Hence $w(Y) \leqslant |Y|$. To consider next the general case let the following be an OCP-system starting from a Moore space $X_1$ and ending at $Y_n = Y$:

$$X_1 \xrightarrow{f_1} Y_1 \xrightarrow{g_1} X_2 \xrightarrow{f_2} \ldots \xrightarrow{g_{n-1}} X_n \xrightarrow{f_n} Y_n = Y .$$

To prove the theorem by induction on $n$ let us consider the case when $Y = Y_1$. Let $f_1$ be the composition of open compact mappings:

$$h_i\colon Z_i \to Z_{i+1}, \quad i = 1, \ldots, m,$$

such that $Z_1 = X_1$, $Z_{m+1} = Y_1$ and $f_1 = h_m \ldots h_1$. By Theorem 4.1 (cf. Section 8) each point-inverse of each $h_i$ is the continuous image of a compact Moore space and hence of a compact metric space. Hence it is compact metric and has at most the continuum power $\aleph$. Since

$$|Z_{i+1}| \leqslant |Z_i| \leqslant |Z_{i+1}| \cdot \aleph = |Z_{i+1}|,$$

then $|Z_i| = |Z_{i+1}|$ for $i = 1, \ldots, m$, and hence $|X_1| = |Y_1|$. Let $\mathfrak{B}_1$ be a base of $X_1$ with $|\mathfrak{B}_1| \leqslant |X_1|$. Since $f_1(\mathfrak{B}_1)$ is a base of $Y_1$,

$$w(Y_1) \leqslant |f_1(\mathfrak{B}_1)| \leqslant |\mathfrak{B}_1| \leqslant |X_1| = |Y_1|$$

and the theorem is true for $n = 1$.

Put the induction hypothesis that the theorem is true for $Y_{n-1}$, where $n > 1$. By the same reason as was observed above it is true that $|Y_{n-1}| = |X_n| = |Y_n|$ and $w(X_n) \geqslant w(Y_n)$. Let $\mathfrak{B}$ be a base of $Y_{n-1}$ with $\infty = |\mathfrak{B}| \leqslant |Y_{n-1}|$. Since $g_{n-1}^{\nabla}(\mathfrak{B})$ is a base of $X_n$, then



$$w(X_n) \leqslant |g_{n-1}^{\nabla}(\mathfrak{B})| \leqslant |\mathfrak{B}| \leqslant |Y_{n-1}| = |Y_n| .$$

Since $f_n g_{n-1}^{\nabla}(\mathfrak{B})$ is a base of $Y_n$, then $w(Y_n) \leqslant |f_n g_{n-1}^{\nabla}(\mathfrak{B})| \leqslant |Y_n|$. Thus the induction is completed. The proof is finished.

## 6. Pointwise paracompact spaces in OC(Moore) and OC($p$-spaces).

6.1. THEOREM. *Let $Y$ be a member of $\mathrm{OC}(\mathit{Moore})$. If $Y$ is pointwise paracompact, then $Y$ is a Moore space.*

Proof. Let $X$ be a Moore space and $f$ an OC-mapping of $X$ to $Y$. Let $\{\mathfrak{U}_i\}$ be a development of $X$. Set $\mathfrak{U}_1 = \{U_{1a}\colon a \in A_1\}$. Let $\mathfrak{V}_1 = \{V_{1a}\colon a \in A_1\}$ be a point finite open covering of $Y$ with

(1) $$V_{1a} \subset f(U_{1a}), \quad a \in A_1 .$$

Set

(2) $$W_{1a} = U_{1a} \cap f^{-1}(V_{1a}), \quad a \in A_1 .$$

Then we have an open collection $\mathfrak{W}_1 = \{W_{1a}\colon a \in A_1\}$ of $X$. Since

(3) $$f(W_{1a}) = V_{1a}, \quad a \in A_1 ,$$

by (1), then $f(\mathfrak{W}_1)$ covers $Y$. By Lemma 1.3 there exist an open collection $\mathfrak{D}_2' = \{D_{2a}'\colon a \in A_2'\}$ of $X$ and a transformation $\varphi'\colon A_2' \to A_1$ such that

(4) $$f(\mathfrak{D}_2')^{\#} = Y ,$$

(5) $\varphi'$ is of type II with respect to $f$, $\mathfrak{D}_2'$, $\mathfrak{W}_1$.

Set $\mathfrak{D}_2 = \mathfrak{U}_2 \wedge \mathfrak{D}_2' = \{D_{2a}\colon a \in A_2\}$. Let $\varphi''\colon A_2 \to A_2'$ be a transformation giving a refine one from $\mathfrak{D}_2$ to $\mathfrak{D}_2'$. Set $\varphi_1^2 = \varphi'\varphi''$. Then $\varphi_1^2$ is of type II with respect to $f$, $\mathfrak{D}_2$, $\mathfrak{W}_1$. By a similar way as in getting $\mathfrak{W}_1$ from $\mathfrak{U}_1$, we can get, from $\mathfrak{D}_2$, an open collection $\mathfrak{W}_2 = \{W_{2a}\colon a \in A_2\}$ of $X$ such that

(6) $$W_{2a} \subset D_{2a}, \quad a \in A_2 ,$$

(7) $f(\mathfrak{W}_2)$ is a point finite open covering of $Y$.

By (5) and (6) $\varphi_1^2$ is of type II with respect to $f$, $\mathfrak{W}_2$, $\mathfrak{W}_1$. Continuing in this manner, we obtain a sequence

$$\mathfrak{W}_i = \{W_{ia}\colon a \in A_i\}, \quad i = 1, 2, \ldots,$$

of open collections of $X$ and a sequence

$$\varphi_i^{i+1}\colon A_{i+1} \to A_i, \quad i = 1, 2, \ldots,$$

of transformations satisfying the following three conditions for each $i$.

(8) $f(\mathcal{W}_i)$ is a point finite open covering of $Y$.

(9) $$\mathcal{W}_i < \mathcal{U}_i .$$

(10) $\varphi_i^{i+1}$ is of type II with respect to $f$, $\mathcal{W}_{i+1}$, $\mathcal{W}_i$.

To show that $f(\mathcal{W}_i)$, $i=1,2,\ldots$, constitute a development of $Y$ assume the contrary. Then there exist a point $y$ of $Y$ and an open neighborhood $U$ of $y$ such that

(11) $$S(y, f(\mathcal{W}_i)) - U \neq \emptyset , \quad i = 1, 2, \ldots$$

Set

(12) $$B_i = \{\alpha \in A_i\colon\ y \in f(W_{i\alpha}),\ f(W_{i\alpha}) - U \neq \emptyset\} , \quad i = 1, 2, \ldots$$

Then each $B_i$ is finite by (8) and non-empty by (11). Moreover it is evident that $\{B_i\}$ forms an inverse subsystem of $\{A_i;\ \varphi_i^{i+1}\}$. Thus $\operatorname{inv\,lim} B_i \neq \emptyset$. Pick an arbitrary element $\langle\beta_i\rangle$ from $\operatorname{inv\,lim} B_i$. Since $y \in \bigcap f(W_{i\beta_i})$, then there exists by Lemma 2.3 a point $x$ of $X$ with

(13) $$f(x) = y \quad \text{and} \quad x \in \bigcap W_{i\beta_i} .$$

Since $\{W_{i\beta_i}\}$ forms a neighborhood base of $x$ by (9), then $\{f(W_{i\beta_i})\}$ has to be a neighborhood base of $y$, which contradicts to the inequalities:

$$f(W_{i\beta_i}) - U \neq \emptyset , \quad i = 1, 2, \ldots$$

The proof is finished.

Since every paracompact Moore space is metric, the following is a direct consequence of this theorem.

6.2. Theorem. *Let $Y$ be a member of* OC(*Moore*). *If $Y$ is paracompact, then $Y$ is metric.*

6.3. Lemma (Burke–Stoltenberg [6], Theorem 2.2). *A completely regular space $Y$ is a strict $p$-space* ([12]) *if and only if $Y$ has a sequence $\mathcal{V}_i$, $i=1,2,\ldots$, of open coverings of $Y$ satisfying the following two conditions.*

(i) *$Q_y = \bigcap S(y, \mathcal{V}_i)$ is compact for each point $y$ of $Y$.*

(ii) *$\{S(y, \mathcal{V}_i)\}$ is a neighborhood base of $Q_y$.*

6.4. Theorem. *Let $Y$ be a member of* OC($p$-*spaces*). *If $Y$ is a pointwise paracompact, completely regular space, then $Y$ is a $p$-space* ([13]).

([12]) According to Arhangel'skiĭ [3], Definition 5.1, a $p$-space $X$ is said to be a *strict $p$-space* if it has a pluming $\{\mathcal{U}_i\}$ such that for each $x \in X$ and for each $m$ there exists an $n$ with $\overline{S(x, \mathcal{U}_n)} \subset S(x, \mathcal{U}_m)$.

([13]) From this theorem we can see at once that Michael space [11] introduced in Section 0 is not even an element of OC($p$-spaces).



Proof. Let $f\colon X \to Y$ be an OC-mapping, where $X$ is a $p$-space. Let $\mathcal{K}_i$, $i = 1, 2, \ldots$, constitute a pluming of $X$. Let $\mathcal{G}_i$ be an open collection of $\beta X$ such that for each $i$

(1) $$\mathcal{G}_i^{\#} \supset X , \quad \tilde{\mathcal{G}}_i < \mathcal{K}_i , \quad \tilde{\mathcal{G}}_{i+1} < \mathcal{G}_i .$$

Set

(2) $$\mathcal{U}_i = \mathcal{G}_i | X , \quad i = 1, 2, \ldots , \quad \mathcal{U}_1 = \{U_{1\alpha}\colon\ \alpha \in A_1\} .$$

Let $\mathcal{V}_1 = \{V_{1\alpha}\colon\ \alpha \in A_1\}$ be a point finite open covering of $Y$ with

(3) $$V_{1\alpha} \subset f(U_{1\alpha}) , \quad \alpha \in A_1 .$$

Set

(4) $$\mathcal{W}_1 = \{W_{1\alpha} = U_{1\alpha} \cap f^{-1}(V_{1\alpha})\colon\ \alpha \in A_1\} .$$

Since $f(W_{1\alpha}) = V_{1\alpha}$, $\alpha \in A_1$, by (3) and (4), then $f(\mathcal{W}_1)$, which is identical with $\mathcal{V}_1$, covers $Y$. Set

$$\mathcal{U}_2' = \{U_{2\alpha}\colon\ \alpha \in A_2'\} = \mathcal{U}_2 \wedge \mathcal{W}_1 .$$

Applying Lemma 1.3 to $\mathcal{U}_2'$ we obtain an open collection $\mathcal{E}_2 = \{E_{2\alpha}\colon\ \alpha \in A_2\}$ of $X$ and a transformation $\varphi'\colon A_2 \to A_2'$ such that $f(\mathcal{E}_2)^{\#} = Y$ and such that $\varphi'$ is of type II with respect to $f$, $\mathcal{E}_2$, $\mathcal{U}_2'$. Let $\varphi\colon A_2' \to A_1$ be a transformation giving a refine one from $\mathcal{U}_2'$ to $\mathcal{W}_1$. Let $\mathcal{V}_2 = \{V_{2\alpha}\colon\ \alpha \in A_2\}$ be a point finite open covering of $Y$ with $V_{2\alpha} \subset f(E_{2\alpha})$, $\alpha \in A_2$. Set

$$\mathcal{W}_2 = \{W_{2\alpha} = E_{2\alpha} \cap f^{-1}(V_{2\alpha})\colon\ \alpha \in A_2\} ,$$
$$\varphi_1^2 = \varphi\varphi'\colon\ A_2 \to A_1 .$$

Then $f(\mathcal{W}_2)$, which is identical with $\mathcal{V}_2$, is a point finite open covering of $Y$ and $\varphi_1^2$ is of type II with respect to $f$, $\mathcal{W}_2$, $\mathcal{W}_1$. Continuing in this manner, we obtain a sequence

$$\mathcal{W}_i = \{W_{i\alpha}\colon\ \alpha \in A_i\} , \quad i = 1, 2, \ldots ,$$

of open collections of $X$ and transformations

$$\varphi_i^{i+1}\colon\ A_{i+1} \to A_i , \quad i = 1, 2, \ldots ,$$

satisfying the following three conditions for each $i$.

(5) $f(\mathcal{W}_i)$ is a point finite open covering of $Y$.

(6) $\varphi_i^{i+1}$ is of type II with respect to $f$, $\mathcal{W}_{i+1}$, $\mathcal{W}_i$.

(7) $\mathcal{W}_i < \mathcal{U}_i$.

Set

(8) $$\mathcal{V}_i = \{V_{i\alpha} = f(W_{i\alpha})\colon\ \alpha \in A_i\} , \quad i = 1, 2, \ldots$$

To show that this sequence satisfies the two conditions in Lemma 6.3 let $y$ be an arbitrary point of $Y$ and $Q_y$ the intersection in Lemma 6.3. Set

(9) $$B_i = \{\alpha \in A_i\colon\ y \in V_{i\alpha}\}\,, \qquad i = 1, 2, \ldots$$

Then by (5) $B_i$ is finite and non-empty. Since it can easily be seen from (9) that

(10) $$\varphi_i^{i+1}(B_{i+1}) \subset B_i\,, \qquad i = 1, 2, \ldots,$$

$\{B_i\}$ forms an inverse subsystem of $\{A_i;\ \varphi_i^{i+1}\}$. Set

(11) $$W_i = \bigcup \{W_{i\alpha}\colon\ \alpha \in B_i\}\,, \qquad i = 1, 2, \ldots, \qquad L = \bigcap W_i\,.$$

Then $L$ is closed in $X$, since $\overline{W}_{i+1} \subset W_i$ by (6) and (10). Set

(12) $$K = \bigcap \widetilde{W}_i\,.$$

Then $K$ is a compact set of $\beta X$. To show that $K \subset X$ let $\tilde{x}$ be an arbitrary point of $K$. Set

(13) $$C_i = \{\alpha \in B_i\colon\ \tilde{x} \in \widetilde{W}_{i\alpha}\}\,, \qquad i = 1, 2, \ldots$$

Let $\alpha$ be an arbitrary index of $C_{i+1}$ and $\beta = \varphi_i^{i+1}(\alpha)$. Then $\tilde{x} \in \widetilde{W}_{i+1,\alpha}$ and $W_{i+1,\alpha} \subset W_{i\beta}$. Hence $\tilde{x} \in \widetilde{W}_{i\beta}$, which shows that $\beta \in C_i$ since $\varphi_i^{i+1}(C_{i+1}) \subset B_i$ by (10) and (13). Thus

(14) $$\varphi_i^{i+1}(C_{i+1}) \subset C_i\,, \qquad i = 1, 2, \ldots,$$

which shows that $\{C_i\}$ forms an inverse subsystem of $\{B_i\}$. Pick an arbitrary element $\langle \alpha_i \rangle$ from $\operatorname{invlim} C_i$. Then

(15) $$\tilde{x} \in \bigcap \widetilde{W}_{i\alpha_i}\,.$$

Since $y \in \bigcap V_{i\alpha_i}$, there exists by (6) and Lemma 2.3 a point $x$ of $X$ with

(16) $$f(x) = y \quad \text{and} \quad x \in \bigcap W_{i\alpha_i}\,.$$

Since $\widetilde{\mathfrak{W}}_i$ refines $\mathcal{H}_i$ by (7), (2), (1), then there exists for each $i$ an element $H_i$ of $\mathcal{H}_i$ with

(17) $$\widetilde{W}_{i\alpha_i} \subset H_i\,.$$

Since $x \in \bigcap H_i$ by (16), then $\bigcap H_i \subset X$. Since $\tilde{x} \in \bigcap H_i$ by (15) and (17), then $\tilde{x} \in X$. Thus we know that $K \subset X$. Since $L$ is closed in $X$ and $L \subset K$ by (11) and (12), then $L$ is compact.

Let us show that $Q_y$ is compact. Since $B_i$ is finite, then

$$\overline{S(y, \mathfrak{V}_{i+1})} \subset S(y, \mathfrak{V}_i)$$

by (9) and (10). Hence $Q_y$ is closed. To show $f(L) \supset Q_y$ let $y'$ be an arbitrary point of $Q_y$. Set

(18) $$D_i = \{\alpha \in B_i\colon\ y' \in V_{i\alpha}\}\,, \qquad i = 1, 2, \ldots$$



Then each $D_i$ is non-empty and $\{D_i\}$ forms an inverse subsystem of $\{B_i\}$. Pick an arbitrary element $\langle \beta_i \rangle$ from $\operatorname{invlim} D_i$. Since $y' \in \bigcap V_{i\beta_i}$, there exists a point $x'$ of $X$ with $x' \in \bigcap W_{i\beta_i}$ and $f(x') = y'$. Since $x' \in L$ by (11), then $y' \in f(L)$. Since $y'$ was an arbitrary point of $Q_y$, then $Q_y \subset f(L)$ and $Q_y$ is now compact.

To show $\{S(y, \mathfrak{V}_i)\}$ forms a neighborhood base of $Q_y$ assume the contrary. Then there exists an open neighborhood $V$ of $Q_y$ such that

(19) $$S(y, \mathfrak{V}_i) - V \neq \emptyset\,, \qquad i = 1, 2, \ldots$$

Set

(20) $$E_i = \{\alpha \in B_i\colon\ V_{i\alpha} - V \neq \emptyset\}\,, \qquad i = 1, 2, \ldots$$

Then each $E_i$ is non-empty and $\{E_i\}$ forms an inverse subsystem of $\{B_i\}$ by (6). Hence $\operatorname{invlim} E_i \neq \emptyset$ and we can pick an element $\langle \gamma_i \rangle$ from $\operatorname{invlim} E_i$. Since $y \in \bigcap V_{i\gamma_i}$, there exists, by Lemma 2.3, a point $x_0$ of $X$ with

(21) $$x_0 \in \bigcap W_{i\gamma_i}\,, \qquad f(x_0) = y\,.$$

Set

(22) $$P_{x_0} = \bigcap S(x_0, \mathfrak{W}_i),\ S(x_0, \mathfrak{W}_i) = U_i\,, \qquad i = 1, 2, \ldots$$

Since $\mathfrak{W}_i$ is point finite by (5),

(23) $$\overline{U}_{i+1} \subset U_i\,, \qquad i = 1, 2, \ldots$$

Since as can easily be seen $f(P_{x_0}) \subset Q_y$, then $P_{x_0} \subset f^{-1}(V)$.

Let $U$ be an open set of $\beta X$ with $U \cap X = f^{-1}(V)$. If $U_i$ is not contained in $f^{-1}(V)$ for any $i$, then $\widetilde{U}_i - U \neq \emptyset$ for any $i$. Hence $\{\widetilde{U}_i - U\colon\ i = 1, 2, \ldots\}$ is a decreasing sequence of non-empty compact sets by (23) and the intersection of them, say $F$, is not empty. Since each $U_i$ is a finite sum of elements of $\mathfrak{W}_i$ by the point finiteness of $\mathfrak{W}_i$, then by (7), (2) and (1) we obtain

(24) $$\widetilde{U}_i \subset S(x_0, \mathcal{H}_i)\,, \qquad i = 1, 2, \ldots$$

Since $\bigcap S(x_0, \mathcal{H}_i) \subset X$, then $\bigcap \widetilde{U}_i \subset X$. Thus

$$F = \bigcap \widetilde{U}_i - U = (\bigcap \widetilde{U}_i - U) \cap X$$
$$= \bigcap \overline{U}_i - f^{-1}(V) = P_{x_0} - f^{-1}(V) = \emptyset\,,$$

which is impossible. Hence $f^{-1}(V)$ contains some $U_k$; i.e.

(25) $$S(x_0, \mathfrak{W}_k) \subset f^{-1}(V)\,.$$

Since $W_{k\gamma_k} \subset S(x_0, \mathfrak{W}_k)$ by (21), then by (25)

(26) $$V_{k\gamma_k} = f(W_{k\gamma_k}) \subset V\,.$$

Since $\gamma_k \in E_k$, then $V_{k\gamma_k} - V \neq \emptyset$, which contradicts to (26). We conclude therefore that $\{S(y, \mathfrak{V}_i)\}$ is a neighborhood base of $Q_y$. Since the two conditions of Lemma 6.3 are now satisfied, then $Y$ is a $p$-space. The proof is finished.

6.5. Theorem. *Let $f: X \to Y$ be an OC-mapping, where $X$ is a $p$-space and $Y$ is a pointwise paracompact space. Then there exists a $G_\delta$ set $W$ of $X$ such that $f(W) = Y$ and $f|W$ is a compact mapping.*

Proof. Let $\mathfrak{W}_i$, $\mathfrak{V}_i$, $A_i$, $B_i$, $\varphi_i^{i+1}$, $W_i$ be those constructed in the preceding proof. Set

$$W = \bigcap \mathfrak{W}_i^{\#}.$$

Then $W$ is a $G_\delta$ set of $X$. Let $y$ be an arbitrary point of $Y$ and $L$ the set defined by (11) in the preceding proof. To prove $f^{-1}(y) \cap W \subset L$ let $x$ be an arbitrary point of $f^{-1}(y) \cap W$. Set

$$S_i = \{\alpha \in A_i \colon x \in W_{i\alpha}\}, \qquad i = 1, 2, \ldots$$

Since $x \in W_{i\alpha}$ implies $y \in V_{i\alpha}$, then $S_i \subset B_i$. Thus $x \in W_i$ for each $i$ and hence $x \in L$, which shows $f^{-1}(y) \cap W \subset L$. It is evident that $W \supset L$. Therefore

$$f^{-1}(y) \cap W = f^{-1}(y) \cap W \cap L = f^{-1}(y) \cap L.$$

Since $L$ is compact, then $f^{-1}(y) \cap L$ is compact and hence $f^{-1}(y) \cap W$ is compact. Since $f^{-1}(y) \cap L \neq \emptyset$ by the preceding proof, then $f^{-1}(y) \cap W \neq \emptyset$. Since $y$ was an arbitrary point of $Y$, the inequality $f^{-1}(y) \cap W \neq \emptyset$ implies $f(W) = Y$. The proof is finished.

## 7. Problems.

7.1. Problem. Are all members of OC(metric) pointwise paracompact?

7.2. Problem. Are all members of OC(metric) developable?

7.3. Problem. Are all members of OC(metric) or OCP(metric) $p$-spaces?

These three problems are corresponding ones, in the sense stated in the introduction, to those for MOBI and MOBOS raised by Arhangel'skiĭ [3], § 5. An affirmative answer for Problem 7.1 would imply an affirmative one for Problem 7.2 by our Theorem 6.1. The first half of Problem 7.7 and Problem 7.12 below are also essentially due to Arhangel'skiĭ [3], § 5.

7.4. Notation. Let $\mathfrak{C}$ be a class of spaces. Let oc($\mathfrak{C}$) be the class of all images of elements of $\mathfrak{C}$ under open compact mappings.

7.5. Problem. Is oc(oc(metric)) strictly larger than oc(metric)?



Bing [5], Example B, shows that oc(metric) contains an element which is not metric. A space $X$ is a member of oc(metric) if and only if $X$ is pointwise paracompact and developable, by Hanai [10], Theorem 5.

7.6. Problem. If a completely regular space $X$ is a member of oc($p$-spaces), then is $X$ a $p$-space?

7.7. Problem. OC(metric) = OCP(metric)?

7.8. Problem. OC($p$-spaces) = OCP($p$-spaces)?

Recently Worrell [12] constructed a completely regular space which is not a $p$-space but the image of a $p$-space under a perfect mapping. This space shows that OCP($p$-spaces) is strictly larger than the class of $p$-spaces.

7.9. Problem. Are OC( ), OP( ), OCP( ) treated in this paper countably productive?

All of them are finitely productive by their definition.

7.10. Problem. Let $Y \in$ OCP($p$-spaces). Then $w(Y) \leqslant |Y|$?

7.11. Problem. Let $X$ be a pointwise paracompact $p$-space. Then is $X$ the image of a paracompact $p$-space under an open compact mapping?

Čoban [8], Theorem 3, solved this problem in the affirmative for the case when $X$ is, moreover, normal or hereditarily pointwise paracompact.

7.12. Problem. Let $X$ be a member of OC(metric) or OCP(metric). Is every closed set of $X$ a $G_\delta$ set? Does the family of Baire sets of $X$ coincide with the family of Borel sets?

7.13. Problem. Let $X \in$ OCP(Moore). If $X$ has the property $L(\aleph_\alpha)$, $\alpha \geqslant 0$, i.e. if every open covering of $X$ has a subcovering consisting of $\aleph_\alpha$ elements, then $w(X) \leqslant \aleph_\alpha$?

7.14. Problem. Let $X$ be a pointwise paracompact completely regular space. If $X$ is a member of OP(absolute $G_\delta$), then is $X$ absolute $G_\delta$?

7.15. Problem. Let $X$ be a pointwise paracompact completely regular space. If $X$ is a member of OCP($p$-spaces), then is $X$ a $p$-spaces?

7.16. Problem. Is each element of OP(absolute $G_\delta$) the image of a paracompact absolute $G_\delta$ space under an open compact-covering mapping? If it is true, find an equivalent intrinsic definition.

7.17. Problem. Let $f: X \to Y$ be an OC-mapping, where $X$ is a metric space and $Y$ is a pointwise paracompact space. Then is there a $G_\delta$ cross-section of $f$? In other words, is there a $G_\delta$ set $X'$ of $X$ with $f(X') = Y$ and with $f|X'$ one-one?

Let $f: X \to Y$ be an open compact mapping, where $X$ is of countable type. Then it is natural from Theorem 4.1 to quote whether $f$ is compact-covering. The referee kindly informed that Proizwolow [17] solved this problem (in the negative).

**8. Supplement.** In this paper $p$-spaces are assumed to be completely regular. We can weaken this condition to be merely regular as follows:

8.1. Definition. A space $X$ is said to be a *weak $p$-space* if it is regular and has a sequence $\mathfrak{U}_i$, $i = 1, 2, \ldots$, of open coverings of $X$ satisfying: If $x \in U_i \in \mathfrak{U}_i$, $i = 1, 2, \ldots$, then (i) $\bigcap_{i=1}^{\infty} \overline{U}_i$ is compact and (ii) $\bigcap_{i=1}^{\infty} \overline{U}_i \subset U$ with $U$ open implies $\bigcap_{i=1}^{n} \overline{U}_i \subset U$ for some $n$. This sequence is said to be a *defining* one.

This condition was introduced by Burke [14], Theorem 1.3, to characterize completely regular spaces to be $p$-spaces. It can easily be seen that the space $T$ in Engelking [16], Example 4 in p. 85, is an example of a weak $p$-space which is not a $p$-space. Since every Moore space (which is assumed to be merely regular) is evidently a weak $p$-space, the class of weak $p$-spaces offers a class containing all $p$-spaces and all Moore spaces. The author does not know whether each Moore space is completely regular. The property to be a weak $p$-space is inherited under the operations taking $G_\delta$ sets, closed sets, perfect preimages and countable products. Replacing $p$-spaces in the preceding sections with weak $p$-spaces, each proposition for $p$-spaces is true for weak $p$-spaces with trivial miner change in the proof. Especially corresponding to Theorems 4.1 and 5.4 the following are true: (i) An OCP-mapping on a weak $p$-space is compact-covering. (ii) Each element of OCP (weak $p$-spaces) is of countable type. The latter needs Lemma 8.3 below instead of Lemma 5.3.

Lemma 8.2. *Let $X$ be a weak $p$-space with a defining sequence $\{\mathfrak{U}_i = \{U_{i\alpha}\colon\ \alpha \in A_i\}\colon\ i = 1, 2, \ldots\}$. Let $B_i$ be a finite subset of $A_i$ and $\varphi_i^{i+1}\colon B_{i+1} \to B_i$ a transformation such that $\varphi_i^{i+1}(\alpha) = \beta$ implies $\overline{U}_{i+1,\alpha} \subset U_{i\beta}$ and such that $\langle \alpha_i \rangle \in \operatorname{invlim}\{B_i;\ \varphi_i^{i+1}\}$ implies $\bigcap_{i=1}^{\infty} U_{i\alpha_i} \neq \emptyset$. Set*

$$U_i = \bigcup \{U_{i\alpha}\colon\ \alpha \in B_i\}, \qquad K = \bigcap U_i .$$

*Then $K$ is compact and $\{U_i\}$ forms a neighborhood base of $K$ in $X$.*

Proof. It suffices to consider the case: $K \neq \emptyset$. Let $\mathcal{F}$ be a maximal filtre of subsets of $K$. Set

$$C_i = \{\alpha \in B_i\colon\ U_{i\alpha} \cap K \in \mathcal{F}\} .$$

Then $C_i \neq \emptyset$ and $\{C_i\}$ forms an inverse subsystem of $\{B_i\}$. Pick an element $\langle \alpha_i \rangle$ from $\operatorname{invlim} C_i$. Set

$$L = \bigcap_{i=1}^{\infty} U_{i\alpha_i} .$$



Then $L$ is a non-empty compact set with $L \subset K$. Let $F$ be an arbitrary element of $\mathcal{F}$. Assume $\overline{F} \cap L = \emptyset$. Then $\overline{F} \cap U_{j\alpha_j} = \emptyset$ for some $j$, a contradiction. Thus a point of $L$ adheres $\mathcal{F}$, which proves that $K$ is compact.

To prove $\{U_i\}$ forms a neighborhood base of $K$ in $X$ assume the contrary. Let $U$ be an open set of $X$ with $K \subset U$ and with $U_i - U \neq \emptyset$ for any $i$. Set

$$D_i = \{\alpha \in B_i\colon\ U_{i\alpha} - U \neq \emptyset\} .$$

Then $D_i \neq \emptyset$ and $\{D_i\}$ forms an inverse subsystem of $\{B_i\}$. Pick an element $\langle \beta_i \rangle$ from $\operatorname{invlim} D_i$. Set

$$M = \bigcap_{i=1}^{\infty} U_{i\beta_i} .$$

Since $M \subset K$, then $U_{k\beta_k} \subset U$ for some $k$, a contradiction. The proof is finished.

Lemma 8.3. *A weak $p$-space $X$ is of countable type.*

Proof. Let $Q$ be a non-empty compact set of $X$ and $\{\mathfrak{V}_i\}$ a defining sequence of open coverings of $X$. Let $\mathfrak{U}_i = \{U_{i\alpha}\colon\ \alpha \in A_i\}$ be an open covering of $X$ such that (i) $\mathfrak{U}_i$ refines $\mathfrak{V}_i$, (ii) $\overline{\mathfrak{U}}_{i+1}$ refines $\mathfrak{U}_i$, and (iii) all but a finite number of elements of $\mathfrak{U}_i$ do not meet $Q$. Then $\{\mathfrak{U}_i\}$ is also a defining one. Let $\varphi_i^{i+1}\colon A_{i+1} \to A_i$ be a transformation such that $\varphi_i^{i+1}(\alpha) = \beta$ implies $\overline{U}_{i+1,\alpha} \subset U_{i\beta}$. Set

$$B_i = \{\alpha \in A_i\colon\ U_{i\alpha} \cap Q \neq \emptyset\} .$$

Then $B_i \neq \emptyset$ and $\{B_i\}$ forms an inverse subsystem of $\{A_i;\ \varphi_i^{i+1}\}$. Since the condition of Lemma 8.2 is satisfied, $K$ defined in Lemma 8.2 is a compact set of countable character. Since $Q \subset K$, $X$ is of countable type and the proof is finished.

Added in proof. The author has just solved Problems 7.10, 7.13, 7.14, 7.15, 7.16 in the affirmative with same of them in move general form.

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# On decomposing the plane into $\aleph_0$ connected or one-to-one curves

by

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**Abstract.** The following results are proven: (1) Assuming $2^{\aleph_0} = \aleph_1$ the plane is the union of denumerably many connected curves whose set of axes consist of two directions; (2) the plane is the union of denumerably many one-to-one curves; and (3) assuming $2^{\aleph_0} \leqslant \aleph_n$ the plane is the union of denumerably many one-to-one curves whose set of axes consist of $(n+2)$-directions.

A *curve* is a planar set with the property that each line in a certain direction, called the axis of the curve, intersects the curve at most once. In other words, for a suitable rotation of the coordinate axes the curve is the graph of a real function. In 1919 Sierpiński (see [4]) showed, assuming the continuum hypothesis, that the plane is the union of denumerably many curves whose set of axes consists of two perpendicular directions. He later showed in [5] that the plane is the union of denumerably many mutually congruent curves.

In 1963 Davies [1] succeeded in proving without any cardinality assumptions that the plane is the union of denumerably many curves whose set of axes is infinite. Moreover, under the hypothesis that $2^{\aleph_0} \leqslant \aleph_n$, Davies [2] proved that the plane is the union of denumerably many curves whose set of axes consists of $n+2$ directions. It is unknown whether this conclusion can be improved to $n+1$ directions, as suggested by Sierpiński's result when $n = 1$. It is the purpose of this paper to extend the above results of Sierpiński and Davies to apply to some special types of curves namely those which are connected (as planar subsets) and those which are one-to-one (i.e., graphs of one-to-one real functions).

Specifically we will establish the following results:

THEOREM 1. *Assuming $2^{\aleph_0} = \aleph_1$, the plane is the union of denumerably many connected curves whose set of axes consists of two directions.*

We conjecture that the above result remains valid infinitely many axes when the continuum hypothesis is dropped.

THEOREM 2. *The plane is the union of denumerably many one-to-one curves whose set of axes is infinite.*