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On decomposing the plane into $\aleph_0$ connected or one-to-one curves

by

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Abstract. The following results are proven: (1) Assuming $2^{\aleph_0} = \aleph_1$ the plane is the union of denumerably many connected curves whose set of axes consist of two directions; (2) the plane is the union of denumerably many one-to-one curves; and (3) assuming $2^{\aleph_0} \leqslant \aleph_n$ the plane is the union of denumerably many one-to-one curves whose set of axes consist of $(n+2)$-directions.

A *curve* is a planar set with the property that each line in a certain direction, called the axis of the curve, intersects the curve at most once. In other words, for a suitable rotation of the coordinate axes the curve is the graph of a real function. In 1919 Sierpiński (see [4]) showed, assuming the continuum hypothesis, that the plane is the union of denumerably many curves whose set of axes consists of two perpendicular directions. He later showed in [5] that the plane is the union of denumerably many mutually congruent curves.

In 1963 Davies [1] succeeded in proving without any cardinality assumptions that the plane is the union of denumerably many curves whose set of axes is infinite. Moreover, under the hypothesis that $2^{\aleph_0} \leqslant \aleph_n$, Davies [2] proved that the plane is the union of denumerably many curves whose set of axes consists of $n+2$ directions. It is unknown whether this conclusion can be improved to $n+1$ directions, as suggested by Sierpiński's result when $n = 1$. It is the purpose of this paper to extend the above results of Sierpiński and Davies to apply to some special types of curves namely those which are connected (as planar subsets) and those which are one-to-one (i.e., graphs of one-to-one real functions).

Specifically we will establish the following results:

THEOREM 1. *Assuming $2^{\aleph_0} = \aleph_1$, the plane is the union of denumerably many connected curves whose set of axes consists of two directions.*

We conjecture that the above result remains valid infinitely many axes when the continuum hypothesis is dropped.

THEOREM 2. *The plane is the union of denumerably many one-to-one curves whose set of axes is infinite.*

THEOREM 3. *Assuming $2^{\aleph_0} \leqslant \aleph_n$, the plane is the union of denumerably many one-to-one curves whose set of axes consist of $n+2$ directions.*

It is unknown whether or not the $n+2$ of Theorem 3 can be reduced to $n+1$. It is also unknown whether the curves can be made mutually congruent in any of the above three theorems.

None of the above cited results can be improved by replacing "denumerable" by "finite" because Mazurkiewicz [3] has shown that the plane is not the union of finitely many curves. Moreover, one cannot replace "connected" in Theorem 1 by "Lebesgue measurable" or "Borel measurable", for in such a case the curves would all have null measure.

Terminology. For each $x \in R^2$ (the plane) $V(x)$ and $H(x)$ will designate the vertical and horizontal lines respectively through x. Let $\mathcal{V}$ and $\mathcal{H}$ denote the set of all vertical and all horizontal lines respectively. The cardinality of a set A will be denoted by $|A|$. We will regard real functions as identical with their graphs. We will treat cardinals as ordinals which are not equipotent with any smaller ordinal. We will also regard ordinals as equal to the set of their predecessors. We will denote $2^{\aleph_0}$ by c when convenient.

We begin with the following result which strengthens a result of Sierpiński in [6]. We will need its proof in the sequel.

LEMMA 1. *The continuum hypothesis is equivalent to the existence of a planar set X which intersects each vertical line in $\aleph_0$ points and whose complement intersects each horizontal line in $\aleph_0$ points.*

Proof ($\Rightarrow$). Let us assume that $2^{\aleph_0} = \aleph_1$. First well-order the real line R by $\{x_a\}_{a<c}$ and put $A = \{(x_a, x_\beta)\colon \omega_0 \leqslant \beta \text{ and } x_a \leqslant x_\beta\}$ and $B = \{(x_a, x_\beta)\colon \omega_0 \leqslant a \text{ and } x_\beta < x_a\}$. Then A and B are disjoint sets such that $A \cup B \cup \{(x_n, x_m)\colon n, m < \omega_0\} = R^2$. Moreover $A \cap H(x_a)$ has cardinality $\aleph_0$ when $\omega_0 \leqslant a$ and is empty when $a < \omega_0$; $B \cap V(x_a)$ has cardinality $\aleph_0$ when $\omega_0 \leqslant a$ and is empty when $a < \omega_0$.

By a simple induction argument we may choose a denumerable subset C of A such that $|C \cap V(x_n)| = \aleph_0$ for each $n < \omega_0$ and such that no two points of C have the same second coordinate. Likewise we may choose a denumerable subset D of B such that $|D \cap H(x_n)| = \aleph_0$ for each $n < \omega_0$ and such that no two points of D have the same first coordinate. Then it easily follows that $X = (B-D) \cup C \cup \{(x_n, x_m)\colon n, m < \omega_0\}$ intersects each vertical line in exactly $\aleph_0$ points and its complement $(A-C) \cup D$ intersects each horizontal line in exactly $\aleph_0$ points.

($\Leftarrow$) Suppose that $\aleph_1 < 2^{\aleph_0}$ and let X be the set of the statement. Well-order R by $\{x_a\}_{a<c}$. Then $X \cap \bigcup\{V(x_a)\colon a < \aleph_1\}$ has cardinality $\aleph_1$ and so there exists a horizontal line H having equation $y = \lambda$ which misses this set. Then $\{(x_a, \lambda)\colon a < \aleph_1\}$ is disjoint from X so that $|H - X| > \aleph_1$

which contradicts the assumed property of X, which finishes the proof of the lemma.

Note that since the set X of Lemma 1 and its complement are obviously unions of denumerably many curves, Sierpiński's theorem [4] readily follows: the plane is the union of $\aleph_0$ curves whose set of axes consist of two perpendicular directions, when $2^{\aleph_0} = \aleph_1$.

Let us now denote by $\mathcal{G}$ the family of all uncountable, nowhere dense closed sets of the plane which are neither subsets of a union of countably many vertical lines nor subsets of a union of countably many horizontal lines. Then we can strengthen Lemma 1 as follows:

LEMMA 2. *Assuming $2^{\aleph_0} = \aleph_1$, there exist complementary planar sets T_1 and T_2 such that*

$$(1) \qquad |V(x) \cap T_1| = \aleph_0 \quad \text{and} \quad |H(x) \cap T_2| = \aleph_0 \quad \text{for all } x \in R^2,$$

and

$$(2) \qquad |G \cap T_i| = c \quad \text{for} \quad i = 1 \text{ or } 2 \text{ and for all } G \in \mathcal{G}.$$

Proof. Define $\mathcal{G}^*$ to be the subfamily of $\mathcal{G}$ consisting of all G which are not subsets of a union of countably many lines, which are either vertical or horizontal.

First we will find disjoint sets Z and W such that (i) each $H(x)$ or $V(x)$ intersects Z and W in exactly one point, and (ii) each $G \in \mathcal{G}^*$ intersects Z and intersects W in c points. Let $\{S_a\}_{a<c}$ be a well-ordering of $(\mathcal{G}^* \times c) \cup \mathcal{H} \cup \mathcal{V}$. By induction on the ordinal c we will define functions z and w as follows:

Case I. If $S_a \in \mathcal{G}^* \times c$, we pick z_a and w_a to be distinct points in the set

$$1^{\text{st}}\text{coord}\, S_a - \bigcup\{V(x) \cup H(x)\colon \beta < a \text{ and } x = z_\beta \text{ or } x = w_\beta\}.$$

Case II. If $S_a \in \mathcal{H} \cup \mathcal{V}$, pick z_a in the set

$$S_a - \{w_\beta\colon \beta < a\} - \bigcup\{V(z_\beta) \cup H(z_\beta)\colon \beta < a\} \quad \text{if} \quad S_a \cap \{z_\beta\colon \beta < a\} = \emptyset$$

and otherwise pick $z_a = R^2$. Moreover, pick w_a in the set

$$S_a - \{z_\beta\colon \beta \leqslant a\} - \bigcup\{V(w_\beta) \cup H(w_\beta)\colon \beta < a\} \quad \text{if} \quad S_a \cap \{w_\beta\colon \beta < a\} = \emptyset$$

and otherwise pick $w_a = R^2$. (If w_a or $z_a = R^2$, we take $V(w_a)$ or $H(w_a)$ to be $\emptyset$.)

Having so defined the functions z and w on c, we put $Z = \{z_a\colon z_a \neq R^2\}$ and $W = \{w_a\colon w_a \neq R^2\}$. Then Z and W are disjoint and satisfy the desired properties (i) and (ii).

Obviously Z and W are disjoint and property (i) is satisfied. To show (ii), let $G \in \mathcal{G}^*$. If $\{a\colon G = 1^{\text{st}}\,\text{Coord.}\, S_a\} = F$, then $|F| = c$ so that

$\alpha \in F$ implies that z_α and w_α belong to G. However, $z_\alpha = z_\beta$ implies $\alpha = \beta$ or $z_\alpha = z_\beta = R^2$. It follows then that $|G \cap Z| = c$ and $|G \cap W| = c$.

Now consider the set $E = R^2 - Z - W$. We may apply the argument of Lemma 1 to E and $A \cap E$ and $B \cap E$ and obtain sets X_1 and $X_2 = E - X_1$ such that $|V(x) \cap X_1| = \aleph_0$ and $|H(x) \cap X_2| = \aleph_0$ for all x. Next put $T_1 = X_1 \cup Z$ and $T_2 = X_2 \cup W$.

Then T_1 and T_2 satisfy the conclusion of the statement of the lemma. Clearly $|V(x) \cap T_1| = \aleph_0$ and $|H(x) \cap T_2| = \aleph_0$ for all x. Suppose $G \in \mathcal{G}$. If $G \in \mathcal{G}^*$, then G hits both Z and W and, consequently T_1 and T_2 in c points each. If $G \in \mathcal{G} - \mathcal{G}^*$, then it must be the case that G is contained in the union of $\aleph_0$ lines, so that $|G \cap H| = c$ and $|G \cap V| = c$ for some horizontal line H and some vertical line V. However, X_2 hits H in $\aleph_0$ points and both Z and W hit H in one point each. Therefore, $|G \cap H \cap X_1| = c$ and $|G \cap T_1| = c$. Likewise, $|G \cap T_2| = c$, which completes the proof of the lemma.

LEMMA 3. *If F is a planar closed set whose vertical and horizontal projections are both uncountable, then there exists a disjoint family of c closed subsets of F each of whose vertical and horizontal projections are uncountable.*

Proof. Case I. Suppose there exist Cantor (i.e., nowhere dense perfect) sets P and Q and numbers r and s such that $P \times \{r\} \subseteq F$ and $\{s\} \times Q \subseteq F$. Then, as is well known, we can decompose P and Q into families of c disjoint Cantor sets $\{P_\alpha\}_{\alpha<c}$ and $\{Q_\alpha\}_{\alpha<c}$ respectively. Then the family $\{(P_\alpha \times \{r\}) \cup (\{s\} \times Q_\alpha)\}_{\alpha<c}$ is the desired family.

Case II. Suppose case I doesn't hold. Then, say for each Cantor set P contained in the vertical projection of F, $|\{y: (x, y) \in F \text{ and } x \in P\}| = c$. Let P be a Cantor set contained in the vertical projection, and let $\{P_\alpha\}_{\alpha<c}$ be a decomposition of P into c disjoint Cantor sets. Then the family $\{F \cap \{(x, y): x \in P_\alpha\}\}_{\alpha<c}$ is the desired family.

THEOREM 1. *Assuming $2^{\aleph_0} = \aleph_1$, the plane is the union of denumerably many disjoint connected curves whose set of axes consists of two directions.*

Proof. It suffices to show that each of the sets T_1 and T_2 guaranteed by Lemma 2 is the union of denumerably many disjoint connected curves whose set of axes consists of one direction. We will prove it for T_1; the proof for T_2 is similar.

Let $\{G_\alpha\}_{\alpha<c}$ be a well-ordering of $\mathcal{G} \cup \mathcal{K}$. Let $\{z_\alpha\}_{\alpha<c}$ be a well-ordering of T_1.

We will define by induction on the ordinal $\aleph_1 \cdot \omega_0$ functions x and y and simultaneously define a sequence of functions $\{g_k\}_{k=0}^{\infty}$.

First we define x and y on the initial segment $\aleph_1$ of $\aleph_1 \cdot \omega_0$ as follows:

Define x_0 and y_0 to be the first two points (relative to the well-ordering of T_1) of $G_0 \cap T_1$.

Now having defined x and y on the set of ordinals α we choose x_α to be the first point in

$$G_\alpha \cap T_1 - \bigcup \{V(x_\beta):\ \beta < \alpha\} - \{y_\beta:\ \beta < \alpha\}$$

and choose y_α to be the first point in

$$G_\alpha \cap T_1 - \bigcup \{V(y_\beta):\ \beta < \alpha\} - \{x_\beta:\ \beta \leqslant \alpha\}.$$

Both selections are possible since the above sets are non-empty since $|\alpha| \leqslant \aleph_0$ and $G_\alpha \in \mathcal{G}$. This, then defines x and y on $\aleph_1$. Next define $g_0 = \{x_\alpha:\ \alpha < \aleph_1\}$ and $t_0 = \{y_\alpha:\ \alpha < \aleph_1\}$.

Then g_0 is a connected function with domain R such that $G_\alpha \cap (T_1 - g_0)$ is not a subset of the union of denumerably many vertical lines for any α.

It is immediate that both g_0 and t_0 are functions. To show that $R = \operatorname{Dmn} g_0$ let us suppose that $R - \operatorname{Dmn} g_0 \neq \emptyset$. Suppose z is the first point in T_1 such that $V(z)$ misses g_0. Since z may be some y_β and $V(z)$ hits $\{y_\beta:\ \beta < c\}$ at most once, let w be the first point in $V(z) \cap T_1 - \{y_\beta:\ \beta < \aleph_1\}$. Then $w = z_\gamma$ for some $\gamma < c$. Let $\{G_{\xi(\alpha)}\}_{\alpha<c}$ be the collection of all members of $\{G_\alpha\}_{\alpha<c}$ which contain w. There exists a 1–1 function μ such that for each α, $x_{\xi(\alpha)} = z_{\mu(\alpha)}$. Since $|\operatorname{Rng}\mu| = c$ there exists an α such that $\mu(\alpha) > \gamma$. But according to construction $x_{\xi(\alpha)}$ is the first point in

$$G_{\xi(\alpha)} \cap T_1 - \bigcup \{V(x_\beta):\ \beta < \xi(\alpha)\} - \{y_\beta:\ \beta < \xi(\alpha)\}.$$

This is a contradiction, because z_γ belongs to this set and $\mu(\alpha) > \gamma$. Hence, $\operatorname{Dmn} g_0 = R$.

Let us now show that g_0 is connected. Suppose g_0 is disconnected by two disjoint non-empty open sets O_1 and O_2. Let B be the boundary of O_1. Then it is easily shown that (1) $B =$ some G_α or (2) some $H(z) \subseteq B$ or (3) some $V(z) \subseteq B$. In either case $B \cap g_0 \neq \emptyset$, a contradiction.

Finally, let us show that $G_\alpha \cap T_1 - g_0$ is not a subset of a union of denumerably many vertical lines. Similar to the proof that $\operatorname{Dmn} g_0 = R$, we can show that $\operatorname{Dmn} t_0 = R$. However, $t_0 \cap g_0 = \emptyset$ by construction and t_0 is a function contained in T_1. By Lemma 3 it follows that $|t_0 \cap G_\alpha| = c$. Hence, $G_\alpha \cap T_1 - g_0$ is not a subset of the union of denumerably many vertical lines.

Next we continue the induction construction of x and y on the interval of ordinals $\{\alpha:\ \aleph_1 \leqslant \alpha < \aleph_1 \cdot 2\}$ and obtain a function g_1. This is done by replacing T_1 by $T_1 - g_0$ in the above construction. Specifically, define

$$x_{\aleph_1} \text{ to be the first point in } G_0 \cap (T_1 - g_0)$$

and

$$y_{\aleph_1} \text{ to be the first point in } G_0 \cap (T_1 - g_0) - \{x_{\aleph_1}\}.$$

Having defined x and y for all α such that $\aleph_1 \leqslant \alpha < \gamma < \aleph_1 \cdot 2$ we define x_γ, where $\gamma = \aleph_1 + \beta$, to be the first point in

$$G_\beta \cap (T_1 - g_0) - \bigcup \{V(x_\alpha)\colon \aleph_1 \leqslant \alpha < \gamma\} - \{y_\alpha\colon \aleph_1 \leqslant \alpha < \gamma\}$$

and y_γ to be the first point in

$$G_\beta \cap (T_1 - g_0) - \bigcup \{V(y_\alpha)\colon \aleph_1 \leqslant \alpha < \gamma\} - \{x_\alpha\colon \aleph_1 \leqslant \alpha \leqslant \gamma\}.$$

Then define $g_1 = \{x_\alpha\colon \aleph_1 \leqslant \alpha < \aleph_1 \cdot 2\}$. As before we can prove that g_1 is a connected function with domain R such that for each α, $G_\alpha \cap (T_1 - \bigcup_{i=0}^{1} g_i)$ is not a subset of a union of denumerably many vertical lines.

If we continue by induction on $\aleph_1 \cdot \omega_0$ in this manner, we obtain a sequence $\{g_k\}_{k=0}^{\infty}$ of disjoint, connected functions with domain R with the property that each g_k intersects each G_α in c points (by Lemma 3).

Since we may not have $\bigcup_{k=0}^{\infty} g_k = T_1$ we proceed as follows: for n odd decompose g_n into $f_n \cup h_n$ where f_n and h_n are functions having disjoint domains and which intersect each member of $\mathcal{G} \cup \mathcal{K}$. This decomposition is found by induction as follows: pick a_0 and b_0 to be distinct points in $G_0 \cap g_n$. Having picked a_α and $b_\alpha \in G_\alpha \cap g_n$ for each $\alpha < \beta$ we may pick a_β and b_β to be distinct points in $G_\beta - \{a_\alpha\colon \alpha < \beta\} - \{b_\alpha\colon \alpha < \beta\}$. Now define $f_n = \{a_\alpha\colon \alpha < c\}$ and $h_n = g_n - f_n$.

Next put $C = T_1 - \bigcup \{g_n\colon n \text{ odd}\}$. Since $\bigcup \{g_n\colon n \text{ even}\} \subseteq C \cap T_1$ it follows that for each $V \in \mathcal{V}$ we have $|V \cap C| = \aleph_0$. Hence, there exists a sequence of disjoint functions $\{k_n\}_{n=0}^{\infty}$ having domain R such that $C = \bigcup_{n=0}^{\infty} k_n$. For n odd put

$$F_n = f_n \cup \big(k_n \urcorner (\mathrm{Dmn}\, h_n)\big)$$

and

$$F_{n-1} = h_n \cup \big(k_n \urcorner (\mathrm{Dmn} f_n)\big).$$

Then it is easily checked that each F_k is a connected function with domain R and moreover, $\bigcup_{k=0}^{\infty} F_k = T_1$, which completes the proof of the theorem.

The proof of Theorem 2 consists of a modification of Davies' proof that the plane is the union of $\aleph_0$ curves. We will only outline this modification and refer the reader to [1] for the details.

THEOREM 2. *The plane is the union of denumerably many one-to-one curves whose set of directions is infinite.*

Proof. Let $\mathcal{L}$ consist of all lines with non-zero rational slope. A subfamily $\mathcal{N}$ of $\mathcal{L}$ is called a network provided for each x, $\mathcal{L}(x) = \{L \in \mathcal{L}\colon x \in L\}$ $\subseteq \mathcal{N}$ whenever $|\{L \in \mathcal{N}\colon x \in L\}| \geqslant 2$. Then the smallest network containing an infinite subfamily $\mathcal{M}$ of $\mathcal{L}$ has the same cardinality as $\mathcal{M}$ itself.

Let $\mathcal{A}$ consist of all finite tuples of ordinals $\langle \alpha_1, \alpha_2, \ldots, \alpha_n\rangle$ such that $\aleph_0 = |\alpha_n| < |\alpha_{n-1}| < \ldots < |\alpha_2| < |\alpha_1| < c$. Let $\mathcal{B}$ consist of all finite tuples of ordinals $\langle \alpha_1, \alpha_2, \ldots, \alpha_n\rangle$ such that $\aleph_0 \leqslant |\alpha_n| < |\alpha_{n-1}| < \ldots < |\alpha_1| < c$. We will now define a function N on $\mathcal{B}$ and, in particular, $\mathcal{A}$.

Well-order $\mathcal{L}$ by $\{l_\alpha\}_{\alpha < c}$. For each α such that $\aleph_0 \leqslant \alpha < c$ define $N(\alpha)$ to be the smallest network containing $\{l_\alpha\}_{\beta \leqslant \alpha}$. Now suppose we have defined $N(\alpha_1, \ldots, \alpha_m)$ for all $\langle \alpha_1, \ldots, \alpha_m\rangle \in \mathcal{B}$ for all $m \leqslant k$ so that $|N(\alpha_1, \ldots, \alpha_m)| = |\alpha_m|$. Suppose $\langle \alpha_1, \ldots, \alpha_k, \alpha_{k+1}\rangle \in \mathcal{B}$. Well-order $N(\alpha_1, \ldots, \alpha_k)$ as $\{l_\beta\}_{\beta < |\alpha_k|}$ and choose $N(\alpha, \ldots, \alpha_k, \alpha_{k+1})$ to be the smallest network containing $\{l_\beta\}_{\beta < \alpha_{k+1}}$. Then $|N(\alpha_1, \ldots, \alpha_{k+1})| = |\alpha_{k+1}|$. This completes the definition of the function N with domain $\mathcal{B}$.

Then $\mathcal{L} = \bigcup \{N(A)\colon A \in \mathcal{A}\}$ and $|N(A)| = \aleph_0$ for each $A \in \mathcal{A}$. Let $\precsim$ be the lexicographical ordering of $\mathcal{A}$. Define $M(A) = N(A) - \bigcup \{N(B)\colon B \precsim A\}$. Then $\{M(A)\colon A \in \mathcal{A}\}$ is a disjointed family whose union is $\mathcal{L}$.

It can be proved for each x there exists an $A \in \mathcal{A}$ such that $\mathcal{L}(x) - M(A)$ is finite. It follows that there exists a $\mathcal{V}(x) \subseteq \mathcal{L}(x) \cap M(A)$ such that $|\mathcal{V}(x)| = \aleph_0$ and $l^\perp \in \mathcal{V}(x)$ whenever $l \in \mathcal{V}(x)$, where $l^\perp$ is perpendicular to l and belongs to $\mathcal{L}(x)$.

Next let $P(A)$ consist of all points x for which $\mathcal{L}(x) - M(A)$ is finite. Then each $P(A)$ is countable and $X = \bigcup \{P(A)\colon A \in \mathcal{A}\}$. Fixing A enumerate $P(A)$ as $\{p_\alpha(A)\}_{\alpha < t}$ where $t \leqslant \aleph_0$. By induction define

$$f\big(p_0(A)\big) \in M(A) \cap \mathcal{V}\big(p_0(A)\big)$$

and

$$g\big(p_0(A)\big) = f\big(p_0(A)\big)^\perp$$

and, in general, having define $f\big(p_k(A)\big)$ and $g\big(p_k(A)\big)$ for all $k < m$ we define

$$f\big(p_m(A)\big) \in M(A) \cap \mathcal{V}\big(p_0(A)\big) - \{f\big(p_i(A)\big)\colon i < m\} - \{g\big(p_i(A)\big)\colon i < m\}$$

and

$$g\big(p_m(A)\big) = f\big(p_m(A)\big)^\perp.$$

This defines f and g on the plane. It is easily shown that both f and g are one-to-one functions and $f(x) \perp g(x)$ for all x. Now define for each non-zero rational number r, $F_r = \{x\colon f(x) \text{ has slope } r\}$. Then F_r is a one-to-one function with axis a line of slope r. Moreover, R^2 is the union of the functions F_r, completing the proof.

Before proving Theorem 3 we need the following result which is a strengthening of a result of Davies [2] in that it requires that each E_i to be finite in a direction perpendicular to θ_i also. Its proof consists of a modification of Davies' proof, which we will outline.

THEOREM A. *The hypothesis $2^{\aleph_0} \leqslant \aleph_n$ is equivalent to the assertion that for any $n+2$ distinct directions $\{\theta_i\}_{i=1}^{n+2}$ no two of which are perpendicular, there exists a decomposition of the plane into $\{E_i\}_{i=1}^{n+2}$ such that each line in direction θ_i or $\theta_i+\frac{1}{2}\pi$ intersects E_i in finitely many points.*

Proof. The proof that the assertion implies $2^{\aleph_0} \leqslant \aleph_n$ is the same as found in Davies [2]. Assume now that $2^{\aleph_0} \leqslant \aleph_n$ and we will prove the assertion. Call a line special if it has direction θ_i or $\theta_i+\frac{1}{2}\pi$ for $i = 1, \ldots$ $\ldots, n+2$. A set of special lines N is a *network* if N contains all special lines through a point p whenever it contains 2 special lines through p.

Then the following lemma may be proven, almost verbatim as in Davies' proof.

LEMMA B. *Given a network N of cardinality $\aleph_m$ there exists an ordering $\prec$ of N such that for any $L \in N$ there exists only a finite number of collections of $2m+3$ special lines, $\{l_i\}_{i=1}^{2m+3}$ in N which are concurrent with L and satisfy*

$$l_{2m+3} \prec l_{2m+2} \prec \ldots \prec l_1 \prec L.$$

Now let N be the network of all special lines. Without loss of generality we may assume that $|N| = \aleph_n$, so that there exists an ordering $\prec$ of N satisfying the conditions of Lemma B. For $p \in R^2$ let $p(\theta)$ denote the line through p in direction θ. For $p \in R^2$ consider the $2n+4$ lines $p(\theta_1), \ldots, p(\theta_{n+2})$, $p(\theta_1+\frac{1}{2}\pi, \ldots, p(\theta_{n+2}+\frac{1}{2}\pi)$. Supposing either $p(\theta_i)$ or $p(\theta_i+\frac{1}{2}\pi)$ is the biggest of this collection relative to the ordering $\prec$, we assign p to the set E_i.

It is easily shown that $\{E_i\}_{i=1}^{n+2}$ satisfies the conditions of the assertion of the Theorem.

THEOREM 3. *Assuming $2^{\aleph_0} \leqslant \aleph_n$, the plane is the union of denumerably many one-to-one curves whose set of axes consist of $n+2$ directions.*

Proof. The proof will consist of showing each of the sets E_i guaranteed in Theorem A is the union of denumerably many one-to-one functions. Let E designate one of these sets and suppose its associated directions are the coordinate axes. Suppose E and $\mathfrak{V}$ are well-ordered by c.

We pick the function f_1 as follows: first choose $x_0 \in E$. Having picked x_a for each $a < \beta$, let V be the first line in $\mathfrak{V} - \{V(x_a)\colon a < \beta\}$ which intersects E and choose x_a to be the first point in $E - \bigcup \{H(x_a)\colon a < \beta\}$. If there is no such V, put $x_a = R^2$ and consider $V(R^2) = H(R^2) = \emptyset$. Now put $f_1 = \{x_a\colon x_a \neq R^2\}$.

Then f_1 is a one-to-one function contained in E and $E - f_1$ is finite in each vertical and horizontal line. Now we repeat the above construction with respect to $E - f_1$ to obtain a one-to-one function f_2 contained in $E - f_1$ such that $E - f_1 - f_2$ is finite in each vertical and horizontal line. Continuing in this manner we obtain the sequence $\{f_n\}_{n=1}^{\infty}$.

To show $\bigcup_{n=1}^{\infty} f_n = E$, let us suppose that $E - \bigcup_{n=1}^{\infty} f_n \neq \emptyset$. Let x be the first point of $E - \bigcup_{n=1}^{\infty} f_n$. The set of predecessors of x on $H(x) \cup V(x)$ is finite, say $x_{a_1}, \ldots, x_{a_k}$. Then there exists a β_i such that $x_{a_i} \in f_{\beta_i}$. Let m be the first integer greater than $\max_{1 \leqslant i \leqslant n} \beta_i$. It follows that $x \in f_m$, a contradiction. This finishes the proof of the theorem.

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