able union of closed subsets of local weight $<p$. However even if X is σ-discrete, we need not have $K_p(X)=\emptyset$, as Example 3.7 will show.

In the light of Theorem 2.1, and the remarks following Theorem 2.3, one might ask whether, with the given assumptions on X, k, and p, the preceding theorem may be extended to say that $K_p(X)\neq\emptyset$ if and only if every absolute Borel set Y in which X is densely embedded has cardinality $k^{\aleph_0}$. Example 3.7 also shows that this is false.

EXAMPLE 3.7. This space was constructed in [7] for different, but related, purposes. With the usual assumptions on k and p, let T_n be a discrete space of cardinality k, and fix $a_n\in T_n$. In $B(k)=\prod_{n=1}^{\infty} T_n$, with the "first difference metric", let $D_m=\{x\in B(k)\colon x_i=a_i \text{ if } i>m\}$. Two distinct points of D_m are at distance at least $1/m$, so D_m is a closed discrete subspace of $B(k)$. It follows that $D=\bigcup_{n=1}^{\infty} D_m$ is an absolute Borel (in fact, absolute F_σ) set of weight and cardinal k. But every point of D is a k-limit point of D, so $K_p(X)\supseteq K_k(X)\neq\emptyset$.

Added in proof. Part of Theorem 2.3 occurs in Bel'nov, *On metric extensions*, Soviet Math. Dokl. 13 (1972), pp. 220–224.

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Inducing approximations homotopic to maps between inverse limits

by

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Abstract. Fort, McCord, J. T. Rogers, and Tollefson have shown that maps between the limit spaces of certain types of inverse systems are ε-homotopic to maps which are induced by maps between the coordinate spaces of the inverse systems, for each $\varepsilon>0$. This result is extended here to a much wider, but still restricted, class of inverse systems, and an example is given to show the need of the remaining restrictions.

1. Introduction. We denote an inverse system with directed set D, coordinate spaces X_d, bonding maps $f_d^e\colon X_e\to X_d$, and projection maps $f_d\colon X_\infty\to X_d$ for all d and all $e\leqslant d$ in D, by (X,f,D). If f_d^e maps X_e onto X_d for all d and all $e\leqslant d$ in D, then we call (X,f,D) a *proper* inverse sequence. The reader is referred to [3] for definitions and basic properties of inverse limits. If (P,g,N) is an inverse system such that N denotes the set of all positive integers, and for each n, P_n is a polyhedron with (finite) triangulation K_n, and g_n^{n+1} is a simplicial map relative to (K_{n+1},K_n), then (P,g,N) is called a *uniformly simplicial inverse sequence*, and is also denoted by (P,K,g,N). Both the solenoidal sequences of [3] and the weak solenoidal sequences of [7] are very restricted special cases of uniformly simplicial inverse sequences.

If (X,f,D) and (Y,g,E) are inverse systems, and $\varrho\colon E\to D$ is order preserving, and for each e in E there is a map $\varphi_e\colon X_{\varrho(e)}\to Y_e$ such that for all $i\leqslant e$ in E, $\varphi_i f_{\varrho(i)}^{\varrho(e)}=g_i^e\varphi_e$, then the map $\varphi\colon X_\infty\to Y_\infty$ defined by the equations $g_e\varphi=\varphi_e f_{\varrho(e)}$, for all e in E, is called an *induced map*. In Theorem 4 we generalize the results of [3] and [7] by showing that if (X,f,D) is an inverse system of compact Hausdorff spaces and (P,K,g,N) is a uniformly simplicial inverse sequence, then every map $F\colon X_\infty\to P_\infty$ is ε-homotopic to an induced map for each $\varepsilon>0$ (i.e. no point is moved more than ε during the homotopy). An example in the last section shows the theorem does not hold if the assumption that (P,K,g,N) is uniformly simplicial is dropped. Other results related to these may be found in [6] and [7].

2. Preliminary theorems. For undefined terms and notation in this section, refer to [3], Chapter II. If K is a simplicial complex, a *simple subdivision* of K is a complex K' whose vertices consist of just one point

p_s from each open simplex s of K, such that the simplex determined by a set V of vertices of K' belongs to K' if and only if there is a sequence $s_0, \ldots, s_i$ of simplexes of K, each except the last a face of the next, such that $V = \{p_{s_0}, \ldots, p_{s_i}\}$. If K is of dimension n, and k is a positive integer, then K' is said to be *of order* k if the barycentric coordinate of p_s on each vertex of s is $\geqslant (n+1)^{-k}$ for each face s of K. The standard barycentric subdivision of K is a simple subdivision of order 1. We will consider a subdivision of K to be a collection of subsets of $|K|$, rather than as a simplicial complex with a homeomorphism onto $|K|$, as in [3]. Also if v is a vertex of K and $x \in |K|$, then $x[v]$ will always denote the barycentric coordinate of x relative to v.

LEMMA 1. *If K' is a simple subdivision of the n-dimensional complex K of order k, then the mesh of $K' \leqslant \big(1-(n+1)^{-k}\big)\cdot$(mesh of K).*

Proof. We consider K to be linearly embedded in E^j for some j, and we denote by $|p|$ the usual norm of a point p in E^j.

Suppose s is a simplex of K of diameter d, s' is a face of s, $x_0, \ldots, x_{m'}$ are vertices of s' and $x_0, \ldots, x_m$ are the vertices of s, where $m' < m$. We determine $|p_s - p_{s'}|$, since the greatest such number is the mesh of K'.

Let $p_s = \sum_{i=0}^{m} b_i x_i$ and $p_{s'} = \sum_{j=0}^{m'} b'_j x_j$, where $\sum_{i=0}^{m} b_i = \sum_{j=0}^{m'} b'_j = 1$, and by hypothesis $b_i \geqslant (n+1)^{-k}$ for $0 \leqslant i \leqslant m$ and $b'_j \geqslant (n+1)^{-k}$ for $0 \leqslant j \leqslant m'$. Then if $0 \leqslant j \leqslant m'$,

$$|p_s - x_j| = \Big|\sum_{i=0}^{m} b_i(x_i - x_j)\Big| \leqslant \sum_{i=0}^{m} b_i\,|x_i - x_j|\,.$$

Now, when $i = j$, $|x_i - x_j| = 0$, and otherwise $|x_i - x_j| \leqslant d$, so

$$|p_s - x_j| \leqslant \Big(\sum_{i=0}^{m} b_i d\Big) - b_j d = d - b_j d = (1-b_j)\,d\,.$$

But, if $\delta = 1-(n+1)^{-k}$, then $(1-b_j) \leqslant \delta$, so $|p_s - x_j| \leqslant \delta d$. Hence

$$|p_s - p_{s'}| = \Big|\sum_{j=0}^{m'} b'_j(p_s - x_j)\Big| \leqslant \sum_{j=0}^{m'} b'_j|p_s - x_j| \leqslant \sum_{j=0}^{m'} b'_j\delta d = \delta d\,.$$

Consequently, the mesh of $K' \leqslant \delta\cdot$(mesh of K), and the Lemma is proven.

LEMMA 2. *Suppose K_1 and K_2 are simplicial complexes, $f\colon K_2 \to K_1$ is a simplicial map, and K'_1 is a simple subdivision of K_1 of order k, for some positive integer k. Then there is a simple subdivision K'_2 of K_2 of order $k+1$ such that f is simplicial relative to (K'_2, K'_1).*

Proof. Suppose K_2 is of dimension n, s_2 is an open simplex of K_2, and s_1 is the open simplex $f(s_2)$ of K_1. Let $v_0, \ldots, v_n$ denote the vertices of s_1 and, for each i $(0 \leqslant i \leqslant m)$ let $v_i^0, \ldots, v_i^{n_i}$ denote all the vertices of s_2 which f maps onto v_i. Then if $b_i = p_{s_1}[v_i]$ for each i, define p_{s_2} such that its barycentric coordinate b_i^j relative to v_i^j $(0 \leqslant i \leqslant m,\ 0 \leqslant j \leqslant n_i)$ is $b_i/(n_i+1)$. It is easily seen that (1) $\sum_{i=0}^{m}\sum_{j=0}^{n_i} b_i^j = 1$, so that p_{s_2} lies in s_2, and (2) $\sum_{j=0}^{n_i} b_i^j = b_i$, so that $f(p_{s_2}) = p_{s_1}$, and (3) $b_i^j = b_i/(n_i+1) \geqslant (n+1)^{-k}(n+1)^{-1} = (n+1)^{-(k+1)}$.

It follows quickly that the points p_{s_2}, for all open simplexes s_2 of K_2, determine a simple subdivision K'_2 of K_2 of order $k+1$ and f maps the vertices of each simplex of K'_2 onto the vertices of some simplex of K'_1. It follows from the linearity of f relative to (K_2, K_1) that f is linear relative to (K'_2, K'_1). So f is simplicial relative to (K'_2, K'_1).

THEOREM 1. *Suppose (P, K, g, N) is a uniformly simplicial inverse sequence, $\varepsilon > 0$, and j is a positive integer. Then there is a sequence $K' = \{K'_i\}$ of subdivisions of the triangulations of $K = \{K_i\}$ such that (P, K', g, N) is a uniformly simplicial inverse sequence, and for each $k \leqslant j$, the mesh of $K'_k < \varepsilon$.*

Proof. Let K_1^1 denote the first barycentric subdivision of K_1. By successive applications of Lemma 2, there is, for each $k > 1$, a simple subdivision K_k^1 of K_k of order k such that g_k^{k+1} is simplicial relative to (K_k^1, K_{k-1}^1). By induction there exists, for each $i > 1$ a sequence $\{K_k^i\}$ such that K_1^i is the ith barycentric subdivision of K_1, and for each k, K_k^i is a simple subdivision of K_k^{i-1} of order k. If n is the dimension of K_k, then from successive application of Lemma 1 the mesh of $K_k^i \leqslant \big(1-(n+1)^{-k}\big)^i\cdot$(mesh of K_k). So for large enough i, the mesh of $K_k^i < \varepsilon$ for each $k \leqslant j$, and we take $K'_k = K_k^i$ for all k.

DEFINITION. Suppose K is a triangulation of a polyhedron P, X is a space and f and h are maps from X into P. Then f is said to *K-approximate* h if and only if, for each x in X, $f(x)$ lies in the closure of the open simplex of K that contains $h(x)$, i.e. if $h(x)[v] = 0$ then $f(x)[v] = 0$ for each vertex v of K. If $\varepsilon > 0$ and $\delta > 0$, then f is said to *(ε, K, δ)-approximate* h if, for each x in X and each vertex v of K, (a) if $f(x)[v] = 0$, then $h(x)[v] < \varepsilon$, and (b) if $h(x)[v] < \delta$, then $f(x)[v] = 0$. From condition (b) it follows that if f (ε, K, δ)-approximates h for any $\varepsilon > 0$, $\delta > 0$, then f K-approximates h.

THEOREM 2. *Suppose X is a topological space, P_1 and P_2 are polyhedra with triangulations K_1 and K_2 respectively, $f\colon P_2 \to P_1$ is simplicial relative to (K_2, K_1), and $h^0\colon X \to P_1$, $h^1\colon X \to P_1$, $g^0\colon X \to P_2$ are maps, where $h^0 = fg^0$ and h^1 K_1-approximates h^0. Then there is a map $g^1\colon X \to P_2$ which K_2-approximates g^0, such that $h^1 = fg^1$.*

Proof. Denote the vertices of K_1 by $v_0, \ldots, v_m$, and for each i $(0 \leqslant i \leqslant m)$, denote the vertices of K_2 that f maps onto v_i by $v_i^0, \ldots, v_i^{n_i}$.

If x is a point of X, $0 \leqslant i \leqslant m$, and $0 \leqslant j \leqslant n_i$, let $b_i^0(x) = h^0(x)[v_i]$, $b_i^1(x) = h^1(x)[v_i]$, and $a_i^j(x) = g^0(x)[v_i^j]$. These are all continuous real-valued functions. Define

$$c_i^j(x) = \begin{cases} [a_i^j b_i^1/b_i^0](x) & \text{if} \quad b_i^0(x) > 0\,, \\ 0 & \text{if} \quad b_i^0(x) = 0\,. \end{cases}$$

Since h^1 K_1-aproximates h^0 we have for each x in X and $0 \leqslant i \leqslant m$,

(1) $\qquad$ if $\quad b_i^0(x) = 0\,, \quad$ then $\quad b_i^1(x) = 0\,.$

Since $h^0 = fg^0$, $b_i^0 = \sum_{j=0}^{n_i} a_i^j$. Hence

(2) $$\sum_{j=0}^{n_i} c_i^j = b_i^1\,,$$

for, if $b_i^0(x) \neq 0$ for some x in X, then

$$\sum_{j=0}^{n_i} c_i^j(x) = \Big[\Big(\sum_{j=0}^{n_i} a_i^j\Big)(b_i^1/b_i^0)\Big](x) = [b_i^0(b_i^1/b_i^0)](x) = b_i^1(x)\,,$$

and, by (1), if $b_i^0(x) = 0$, then $\sum_{j=0}^{n_i} c_i^j(x) = 0 = b_i^1(x)$.

The function c_i^j, for each $0 \leqslant i \leqslant m$ and $0 \leqslant j \leqslant n_i$, is clearly continuous on the open set $[b_i^0]^{-1}\big((0,1]\big)$. If x is in X and $b_i^0(x) = 0$, then by (1) $b_i^1(x) = 0$. So if $\varepsilon > 0$, then x lies in an open set U in X such that $b_i^1(U) \subset [0, \varepsilon)$. But by (2), $c_i^j \leqslant b_i^1$. Hence $c_i^j(U) \subset [0, \varepsilon)$, and c_i^j is continuous at x, and the continuity of c_i^j is established.

Finally, by (2) the numbers $c_i^j(x)$ sum to $\sum_{i=0}^{m} b_i^1(x) = 1$, so there is a map $g^1\colon X \to P_2$ defined by $g^1(x)[v_i^j] = c_i^j$ for each $0 \leqslant i \leqslant m$ and $0 \leqslant j \leqslant n_i$, which K_2-approximates g^0, since $c_i^j(x) = 0$ whenever $a_i^j(x) = 0$; and $h^1 = fg^1$ by (2).

Definition. A subset H of a space X is a *cozero set* in X if there is a map $f\colon X \to [0,1]$ such that $H = f^{-1}\big((0,1]\big)$. In a completely regular space, there is a basis of open cozero sets for the open sets of the space (see [4]).

The following theorem is related to Theorem 11.9 of [3], p. 287.

Theorem 3. *Suppose P is a polyhedron with a triangulation K having vertices $v_0, \ldots, v_n$, (X, f, D) is a proper inverse system of compact Hausdorff spaces, $d \in D$, $h\colon X_\infty \to P$ is a map, and $0 < \varepsilon < (n+1)^{-1}$. Then if $0 < \delta \leqslant \varepsilon/2$, there is an element $e \geqslant d$ of D and a map $\varphi_e\colon X_e \to P$ such that $\varphi_e f_e$ (ε, K, δ)-approximates h.*

Proof. Let β denote a finite open cover of P of sufficiently fine mesh so that if x and y lie in an element of β, and $0 \leqslant i \leqslant n$, then $|x[v_1] - y[v_i]|$

$< \varepsilon/2$. Let $\alpha = h^{-1}(\beta)$, and e denote an element of $D \geqslant d$ such that X_e has a finite open cover ω of cozero sets such that $f_e^{-1}(\omega)$ refines α. If $0 \leqslant i \leqslant n$, let O_i denote the union of all elements w of ω such that, for some point y of $hf_e^{-1}(w)$, $y[v_i] \geqslant \varepsilon$. Then if u is in $hf_e^{-1}(O_i)$, there is a point y in P such that $y[v_i] \geqslant \varepsilon$ and $|y[v_i] - u[v_i]| < \varepsilon/2$; hence

(1) $\qquad$ if $\quad u \in hf_e^{-1}(O_i)\,, \quad$ then $\quad u[v_i] \geqslant \varepsilon/2\,.$

The sets $\{O_i\}$ $(0 \leqslant i \leqslant n)$ cover X_e since each element of ω lies in one of them, for otherwise there is an element w of ω such that for each point y of $hf_e^{-1}(w)$ and each i $(0 \leqslant i \leqslant n)$, $y[v_i] < \varepsilon$. But then

$$1 = \sum_{i=0}^{n} y[v_i] < (n+1)\varepsilon < (n+1)/(n+1) = 1\,,$$

which is impossible.

Note that by (1), $hf_e^{-1}(O_i)$ lies in the open star $\mathrm{st}_K(v_i)$. Hence if $0 \leqslant j_0 < \ldots < j_k \leqslant n$ and $O_{j_0}, \ldots, O_{j_k}$ intersect, then $hf_e^{-1}\big(\bigcap_{i=0}^{k} O_{j_i}\big) \subset \bigcap_{i=0}^{k} \mathrm{st}_K(v_i)$, and so $v_{j_0}, \ldots, v_{j_k}$ are the vertices of a face of K. Hence if K' is the subcomplex of K to which a simplex s of K with vertices $v_{j_0}, \ldots, v_{j_k}$ belongs if and only if the sets $O_{j_0}, \ldots, O_{j_k}$ intersect, then we may take K' as the nerve of the cover $\{O_i\}$ $(0 \leqslant i \leqslant n)$ of X_e. Now, if $0 \leqslant i \leqslant n$, then O_i is a finite union of cozero sets, and hence itself a cozero set; so there is a map $u_i\colon X_e \to [0,1]$ such that $u_i(x) > 0$ if and only if x lies in O_i. These maps may be used in the standard manner ([1], p. 175 or [3], p. 286) to obtain a barycentric map $\varphi_e\colon X_e \to |K'| \subset P$ such that if $0 \leqslant i \leqslant n$, then

(2) $\qquad \varphi_e(x)[v_i] > 0 \quad$ if and only if $\quad x \in O_i\,.$

We now verify that $\varphi_e f_e$ (ε, K, δ)-approximates h. Suppose $z \in X_\infty$ and $0 \leqslant i \leqslant n$. (a) If $h(z)[v_i] \geqslant \varepsilon$, then $f_e(z)$ lies in O_i, from the definition of O_i. Hence $\varphi_e f_e(z)[v_i] > 0$ by (2). So if $\varphi_e f_e(z)[v_i] = 0$, then $h(z)[v_i] < \varepsilon$. (b) If $\varphi_e f_e(z)[v_i] > 0$, then $f_e(z)$ lies in O_i by (2) and $h(z)[v_i] \geqslant \varepsilon/2$ by (1). So if $h(z)[v_i] < \delta \leqslant \varepsilon/2$, then $\varphi_e f_e(z)[v_i] = 0$.

3. The main theorem. The purpose of this section is the proof of the following.

Theorem 4. *Suppose (X, f, D) is a proper inverse system of compact Hausdorff spaces, (P, K, g, N) is a uniformly simplicial inverse sequence on polyhedra and $F\colon X_\infty \to P_\infty$ is a map. Then if $\varepsilon > 0$, F is ε-homotopic to an induced map $\varphi\colon X_\infty \to P_\infty$. Moreover, if $x \in X_\infty$ and $g_i F$ is a vertex of K_i, for each i, then $\varphi(x) = F(x)$.*

The next lemma provides the recursive step in the proof.

Lemma 3. *Suppose $d \in D$, $i \in N$, and $\varphi_d\colon X_d \to Y_i$ is a map such that $\varphi_d f_d$ $(2, K_i, \delta_i)$-approximates $g_i F$, for some $1 > \delta_i > 0$. Then there exist* *an element $e \geqslant d$ of D and a map φ_e: $X_e \to Y_{i+1}$ such that $\varphi_d f_d^e = g_i^{i+1}\varphi_e$ and $\varphi_e f_e$ $(2, K_{i+1}, \delta_{i+1})$-approximates $g_{i+1}F$, for some $\delta_{i+1} > 0$.*

Proof. Let $n+1$ denote the number of vertices of K_{i+1} and take $\varepsilon' < \delta_i/(n+1)$. Then by Theorem 3 there is an element $e \geqslant d$ of D and a map φ_e': $X_e \to Y_{i+1}$ such that if $0 < \delta \leqslant \varepsilon'/2$, then

(1) $$\varphi_e' f_e \ (\varepsilon', K_{i+1}, \delta)\text{-approximates } g_{i+1}F.$$

To see that $\varphi_d f_d^e$ K_i-approximates $g_i^{i+1}\varphi_e'$, let $y \in X_\infty$, $x = f_e(y)$, v denote a vertex of K_i, and $\{v^0, \ldots, v^k\} = (g_i^{i+1})^{-1}(v)$. We suppose that $g_i^{i+1}\varphi_e'(x)[v] = 0$ and show that $\varphi_d f_d^e(x)[v] = 0$. If $0 \leqslant j \leqslant k$, then $\varphi_e'(x)[v^j] = \varphi_e' f_e(y)[v^j] = 0$, since g_i^{i+1} is simplicial, and so $g_{i+1}F(y)[v^j] < \varepsilon'$, by (1). Also,

$$g_iF(y)[v] = g_i^{i+1}g_{i+1}F(y)[v] = \sum_{j=0}^{k} g_{i+1}F(y)[v^j] < (k+1)\varepsilon' \leqslant (n+1)\varepsilon' < \delta_i.$$

So, since $\varphi_d f_d$ $(2, K_i, \delta_i)$-approximates g_iF, $\varphi_d f_d^e(x)[v] = \varphi_d f_d(y)[v] = 0$.

So, since $\varphi_d f_d^e$ K_i-approximates $g_i^{i+1}\varphi_e'$, we have from Theorem 2 (by letting $h^0 = g_i^{i+1}\varphi_e'$) a map φ_e: $X_e \to Y_{i+1}$ such that $\varphi_d f_d^e = g_i^{i+1}\varphi_e$, and

(2) $$\varphi_e \ K_{i+1}\text{-approximates } \varphi_e'.$$

That $\varphi_e f_e$ $(2, K_{i+1}, \delta)$-approximates $g_{i+1}F$ follows immediately from (1) and (2).

Proof of Theorem 4. We use the standard metric on P_∞:

$$d(x, y) = \sum_{i=1}^{\infty} d_i(g_i(x), g_i(y))\cdot 2^{-i}$$

where d_i is a metric for P_i with respect to which P_i has diameter $\leqslant 1$. Let j denote a positive integer such that $2^{-j} < \varepsilon/4$. We assume that if $i \leqslant j$, then K_i has mesh $< \varepsilon/4$, for if not Theorem 1 yields subdivisions with this property which can be used instead.

By Theorem 3, there is an element $e(1)$ of D, a number $\delta_1 > 0$, and a map φ_1: $X_{e(1)} \to Y_1$ such that $\varphi_1 f_{e(1)}$ $(2, K_1, \delta_1)$-approximates g_1F. Continuing recursively with Lemma 3, there are sequences $e(1) \leqslant e(2) \leqslant \ldots$ of D, positive integers $\delta_1, \delta_2, \ldots$, and maps φ_i: $X_{e(i)} \to Y_i$ for each i such that $\varphi_i f_{e(i)}$ $(2, K_i, \delta_i)$-approximates g_iF, and $\varphi_i f_{e(i)}^{e(i+1)} = g_i^{i+1}\varphi_{i+1}$. Moreover, since for each x in X_∞ $\varphi_i f_{e(i)}(x)$ lies in the closure of the open simplex of K_i that contains $g_iF(x)$, there is a linear homotopy h_i^t: $\varphi_i f_{e(i)} \simeq g_iF$, and since g_i^{i+1} is simplicial $h_i^t = g_i^{i+1}h_{i+1}^t$. So the maps φ_i induce a map φ: $X_\infty \to P_\infty$, and the homotopies h_i^t induce a homotopy h^t: $\varphi \simeq F$. Note that for each $x \in X_\infty$, $h_i^t(x)$ remains in the closure of the open simplex of K_i that contains $g_iF(x)$ for all $0 \leqslant t \leqslant 1$. Hence if g_iF is a vertex of K_i for each i, $\varphi(x) = F(x)$, and in any case,

$$d(F(x), h^t(x)) < \sum_{i=1}^{j} (\varepsilon/4)\cdot 2^{-i} + \sum_{i=j+1}^{\infty} (1)\cdot 2^{-i} < \varepsilon/4 + \varepsilon/4 = \varepsilon/2,$$

so that no point is moved further than ε during the homotopy.

4. Examples. This section contains two examples. In the first there is a mapping from one inverse limit onto another satisfying the hypothesis of Theorem 4, but no induced map is onto. The second shows that the condition that (Y, K, g, N) be uniformly simplicial in Theorem 4 is necessary even for inducing ε-approximations to mappings, and so also for ε-homotopic approximations.

Example 1. For each i, let X_i denote a space of 2^i points, Y_i denote the interval $[0, 1]$, f_i: $X_{i+1} \to X_i$ denote a map such that each point-preimage has only 2 points, and g_i: $Y_{i+1} \to Y_i$ denote the identity map. Then X_∞ is a Cantor set, Y_∞ is homeomorphic to $[0, 1]$, and there is a standard at most 2-to-1 map from X_i onto Y_i. Suppose φ: $X_\infty \to Y_\infty$ is induced by maps φ_i: $X_{n_i} \to Y_i$ for all positive integers i, where $n_1 < n_2 < \ldots$ Then $\varphi_1(X_{n_1})$ consists of not more than 2^{n_1} points, and since g_i: $Y_\infty \to Y_i$ is a homeomorphism, $\varphi(X_\infty)$ is also finite. So no induced map throws X_∞ onto Y_∞. By taking cones over the spaces of this example, and extending the maps linearly, it can be seen that the induced map obtained in the manner described in this paper need not be onto, even if all the spaces are connected and F is onto.

Question. Suppose in Theorem 4 that X_∞ is connected and F throws X_∞ onto Y_∞. Then is F ε-homotopic to some induced map from X_∞ *onto* Y_∞ for each $\varepsilon > 0$?

Example 2. Let $X_1 = [0, 1]$, and $B = \{b_1, b_2, \ldots\}$ denote a countable dense subset of $X_1 - \{0, 1\}$. Supposing $X_1, f_1^2, X_2, f_2^3, \ldots, X_n$ to be defined, define X_{n+1} and f_n^{n+1} from X_{n+1} onto X_n such that there is an arc α in X_1 with b_n in its interior such that

(a) $(f_1^n)^{-1}(\alpha)$ is an arc in X_n,

(b) $(f_1^{n+1})^{-1}(\alpha)$ is the union of an arc β_{n+1} and two topological rays, each spiraling down to β_{n+1},

(c) $f_n^{n+1}(\beta_{n+1})$ is degenerate and $f_1^{n+1}(\beta_{n+1}) = b_n$,

(d) $f_n^{n+1} | (X_{n+1} - \beta_{n+1})$ is one-to-one.

Then X_∞ is a chainable continuum with only countably many non-degenerate arc-components $f_2^{-1}(\beta_2), f_3^{-1}(\beta_3), \ldots$, and for each n, f_n^{n+1} is a monotone map.

Since X_∞ is chainable, it is homeomorphic to the limit of a proper inverse sequence on arcs, $I_1, I_2, \ldots$, and by Brown's approximation theorem ([2], Theorem 3, p. 481), we may take the bonding map g_i^{i+1} to be a light (i.e. with point preimages totally disconnected) simplicial map from I_{i+1} onto I_i. Let F denote a homeomorphism from X_∞ onto Y_∞, $n(1), n(2), \ldots$ denote an increasing sequence of positive integers, $\varphi_{n(i)}\colon X_{n(i)} \to Y_i$ denote a map for each i such that $\varphi_{n(i)} f_{n(i)}^{n(i+1)} = g_i^{i+1} \varphi_{n(i+1)}$, and $\varphi\colon X_\infty \to Y_\infty$ denote the map induced by the sequence $\{\varphi_{n(i)}\}$.

We show first that there is a map ϱ from $X_{n(1)}$ onto $\varphi(X_\infty)$, by proving the following lemma.

LEMMA 4. *If i is a positive integer, $y \in Y_i$, and H is a connected subset of $\varphi_{n(i)}^{-1}(y)$, then $K = \varphi_{n(i+1)}(f_{n(i)}^{n(i+1)})^{-1}(H)$ is a degenerate subset of $(g_i^{i+1})^{-1}(y)$.*

Proof. Since $f_{n(i)}^{n(i+1)}$ is monotone, K is connected. If $z \in K$, then $z = \varphi_{n(i+1)}(x)$ for some $x \in (f_{n(i)}^{n(i+1)})^{-1}(H)$. Hence

$$y = \varphi_{n(i)} f_{n(i)}^{n(i+1)}(x) = g_i^{i+1} \varphi_{n(i+1)}(x) = g_i^{i+1}(z) .$$

So K lies in $(g_i^{i+1})^{-1}(y)$, which is totally disconnected since g_i^{i+1} is light, and K must be degenerate.

Using the Lemma, we proceed to define $\varrho\colon X_{n(1)} \to Y_\infty$. Suppose $x \in X_{n(1)}$, and $y_1 = \varphi_1(x)$. By the lemma, $\varphi_2(f_{n(1)}^{n(2)})^{-1}(x)$ is a point y_2 in $(g_1^2)^{-1}(y_1)$. Also, $H_2 = (f_{n(1)}^{n(2)})^{-1}(x)$ is a connected subset of $\varphi_2^{-1}(y_2)$, so by the lemma, $\varphi_3(f_{n(2)}^{n(3)})^{-1}(H_2)$ is a point y_3 of $(g_2^3)^{-1}(y_2)$. Continuing, we see that $\varphi(f_{n(1)})^{-1}(x)$ is the point $(y_1, y_2, \ldots)$ of Y_∞. We let $\varrho(x) = (y_1, y_2, \ldots)$, and observe that since $f_{n(1)}$ maps X_∞ onto $X_{n(1)}$, $\varrho(X_{n(1)}) = \varphi(X_\infty)$.

To see that ϱ is continuous, suppose $x \in X_{n(1)}$, $y = \varrho(x)$, O is an open set in Y_∞ containing y, and for some i, O_i is an open set in Y_i containing $y_i = g_i(y)$ such that $g_i^{-1}(O_i) \subset O$. Then $C = f_{n(1)}^{n(i)}[X_{n(i)} - \varphi_{n(i)}^{-1}(O_i)]$ is a compact subset of $X_{n(1)}$ which does not contain x since $(f_{n(1)}^{n(i)})^{-1}(x) \subset \varphi_{n(i)}^{-1}(y_i)$, and if $O' = X_{n(1)} - C$, $x \in O'$ and $g_i \varrho(O') = \varphi_{n(i)}[(f_{n(1)}^{n(i)})^{-1}(O')] \subset \varphi_{n(i)}[\varphi_{n(i)}^{-1}(O_i)] = O_i$. Hence $\varrho(O') \subset g_i^{-1}(O_i) \subset O$.

We have shown, then, that ϱ is a map from $X_{n(1)}$ onto $\varphi(X_\infty)$. But it is easily seen that $X_{n(1)}$ contains a sequence $a_1, a_2, \ldots, a_j$ of open arc-components such that if $1 \leqslant i < j$, Then $\bar{a}_i$ intersects $\bar{a}_{i+1}$. So, since every arc-component of Y_∞ is closed, $\varrho(X_{n(1)}) = \varphi(X_\infty)$ must lie in one arc-component of Y_∞.

Now, we take the usual metric on $X_1 = [0, 1]$ for d_1, and metrics d_i for X_i such that the diameter of $X_i = d_i\big((f_1^i)^{-1}(0), (f_1^i)^{-1}(1)\big) = 1$ for $i > 1$, and define a metric on Y_∞ by

$$d(x, y) = \sum_{i=0}^{\infty} d_i\big(f_i F^{-1}(x), f_i F^{-1}(y)\big) \cdot 2^{-i} .$$

(This is the metric induced by the homeomorphism F from the standard metric on X_∞.) Since each non-degenerate arc-component of Y_∞ is thrown

by $f_1 F^{-1}$ onto one of the points $b_1, b_2, \ldots$ the diameter δ of each such arc-component satisfies

$$\delta \leqslant \sum_{i=2}^{\infty} (1) \cdot 2^{-i} = 1/2 .$$

Hence $\varphi(X_\infty)$ has diameter $\leqslant 1/2$, while if $x = F f_1^{-1}(0)$ and $y = F f_1^{-1}(1)$,

$$d(x, y) = \sum_{i=1}^{\infty} d_i\big((f_1^i)^{-1}(x), (f_1^i)^{-1}(y)\big) \cdot 2^{-i} = \sum_{i=1}^{\infty} 1 \cdot 2^{-i} = 1 ,$$

and Y_∞ has diameter 1. So, with this metric on Y_∞, no induced map can ε-approximate F if $\varepsilon < 1/4$.

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