Corollary 4.3 is a consequence of the existence of good ideals (see eg. [4]).

The following fact was stated in [4].

LEMMA 4.4. *If* $\mathcal{G}$ *is* $\varkappa^+$*-good, then it is* ω*-incomplete if and only if it is* $(\omega, \varkappa)$*-regular.*

COROLLARY 4.5. *If* $\mathcal{G}$ *is* $\varkappa^+$*-good and* ω*-incomplete and* $\mathbf{2}_{\mathcal{G}}$ *is* $\varkappa^+$*-universal, then, for every class* $\boldsymbol{K}$ *whose similarity type is of power* $\leqslant \varkappa$, *the class* $\mathcal{G}(\boldsymbol{K})$ *is compact.*

Finally, let us remark that the assumption of $\varkappa^+$-goodness in Theorem 4.1 is not necessary. The proof of it is easy. Also there is a $\varkappa$-separatistic ideal $\mathcal{G}$ such that the Boolean algebra is $\varkappa^+$-universal and $\mathcal{G}$ is not $(\omega, \varkappa)$-regular.

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Concerning closed quasi-orders on hereditarily unicoherent continua

by

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Abstract. The purpose of this paper is to define and study a class of hereditarily unicoherent Hausdorff continua, called *nearly smooth*, which admit a closed quasi-order that is closely related to the weak cutpoint order. The major result is that for each point p of a nearly smooth continuum M there exists a decomposition $\mathfrak{D}$ of M such that (i) $\mathfrak{D}$ is upper semicontinuous, (ii) the elements of $\mathfrak{D}$ are continua, (iii) the decomposition space of $\mathfrak{D}$ is a generalized tree which is smooth at the element of $\mathfrak{D}$ containing p, and (iv) $\mathfrak{D}$ is the finest decomposition satisfying (i), (ii), and (iii). In addition, characterizations of nearly smooth continua, smooth continua, and generalized trees are obtained in terms of closed quasi-orders and the set-valued function T. A preliminary result of independent interest is that every semi-aposyndetic, hereditarily unicoherent continuum is a dendroid.

The notion of weak cutpoint order has been useful in studying the structure of arcwise connected, hereditarily unicoherent continua. For example, Koch and Krule [10] have shown that a hereditarily unicoherent continuum is a generalized tree [12] if and only if there exists a point p such that the weak cutpoint order with respect to p is a closed partial order. Charatonik and Eberhart [3] have applied the notion of weak cutpoint order to obtain characterizations of smooth dendroids and to study their mapping properties.

It is the purpose of this paper to study hereditarily unicoherent continua admitting a closed quasi-order which is closely related to the weak cutpoint order. One should observe that for non-arcwise connected, hereditarily unicoherent continua the weak cutpoint order is a quasi-order and not a partial order.

It is shown that a hereditarily unicoherent continuum is smooth at a point p [6] if and only if the weak cutpoint order with respect to p is closed. This result motivates the definition of a *nearly smooth* continuum as a hereditarily unicoherent continuum admitting a closed quasi-order which "approximates" the weak cutpoint order. Characterizations of nearly smooth continua, smooth continua, and generalized trees are obtained in terms of closed quasi-orders and the set-valued function T [4].

A major result of the paper is a decomposition theorem for nearly smooth continua which implies, as a special case, the known decomposition theorem for smooth continua [6]. In particular, it is shown that if the continuum M is nearly smooth at a point p, then there exists a decomposition $\mathfrak{D}$ of M such that (i) $\mathfrak{D}$ is upper semi-continuous, (ii) the elements of $\mathfrak{D}$ are continua, (iii) the decomposition space of $\mathfrak{D}$ is a generalized tree which is smooth at the element of $\mathfrak{D}$ containing p, and (iv) $\mathfrak{D}$ is the finest among all decompositions satisfying (i), (ii), and (iii). This theorem is analogous to known decomposition theorems for certain continua which are irreducible about a finite set of points (e.g., [5], [11]); and for λ-dendroids [2].

Finally, examples of smooth and nearly smooth continua are presented.

For simplicity, the results have been stated above for hereditarily unicoherent continua; however, most of the results will be proved more generally for continua which are hereditarily unicoherent at a point [6].

The author wishes to express his appreciation to Carl Eberhart for many helpful suggestions during the course of this work.

1. Definitions and preliminary remarks. A *continuum* is a compact connected Hausdorff space. An *arc* is a continuum (not necessarily metrizable) which is irreducibly connected between some pair of points.

A continuum is *hereditarily unicoherent at the point* p [6] if the intersection of any two subcontinua, each of which contains p is connected. It is easily verified that if X is a continuum, then X is hereditarily unicoherent at p if and only if given any point $x \in X$ there exists a unique subcontinuum which is irreducible between p and x. If the continuum M is hereditarily unicoherent at the point p and $q \in M-\{p\}$, then the symbol pq will denote the unique subcontinuum of M which is irreducible between p and q.

A continuum M is said to be *smooth at the point* p [6] if M is hereditarily unicoherent at p, and if for each convergent net of points $\{a_n,\ n \in E\}$ in M the condition

(i) $\lim a_n = a$

implies that

(ii) $\{pa_n,\ n \in E\}$ is convergent, and

(iii) $\operatorname{Lim} pa_n = pa$.

A *tree* [12] is a hereditarily unicoherent, locally connected continuum. A *generalized tree* [12] is a hereditarily unicoherent, arcwise connected continuum which is smooth. According to [10] the above definition of a generalized tree is equivalent to that given originally in [12]. A *dendroid* (not necessarily metrizable) is a hereditarily decomposable, hereditarily unicoherent, arcwise connected continuum.

A continuum X is *aposyndetic* [8] at a point x with respect to a point y if there exists a subcontinuum H of X such that

$$x \in H^0 \subseteq H \subseteq X-\{y\}.$$

A continuum X is *semi-aposyndetic* at a pair of points $\{x, y\}$ if X is aposyndetic at one of the points with respect to the other. If X is a continuum and $x \in X$, then

$$T(x) = \{y \in X\colon\ X \text{ is not aposyndetic at } y \text{ with respect to } x\} \quad [4],$$

and

$$K(x) = \{y \in X;\ X \text{ is not aposyndetic at } x \text{ with respect to } y\}.$$

The sets $T(x)$ and $K(x)$ are closed, and $T(x)$ is connected [8]. Observe that

$$K(x) = \bigcap \{K;\ K \text{ is a subcontinuum of } X \text{ and } x \in K^0\}.$$

A *quasi-order* on a set X is a reflexive, transitive relation. A quasi-order on a topological space X is *closed* if its graph is closed in $X \times X$. If $\leqslant$ is a quasi-order on X and $A \subseteq X$, then the set

$$L(A) = \{x \in X;\ x \leqslant a \text{ for some } a \in A\}$$

is said to be the *lower set* of A; and the set

$$D(x) = \{y \in X;\ x \leqslant y \leqslant x\}$$

is said to be the *level set* of x.

If M is a continuum which is hereditarily unicoherent at a point p, then the quasi-order $\leqslant$ on M, defined by

$$x \leqslant y \text{ if and only if } px \subseteq py,$$

is said to be the *weak cutpoint order* with respect to p.

2. Hereditary unicoherence at a point.

Convention. Throughout the remainder of the paper, the letter M will denote a continuum which is hereditarily unicoherent at a point p.

In this section we prove several useful theorems concerning hereditary unicoherence at a point and present two examples to illustrate the notion. The first theorem is an immediate consequence of the definitions.

THEOREM 2.1. *The continuum X is hereditarily unicoherent if and only if X is hereditarily unicoherent at each of its points.*

THEOREM 2.2. *If M is arcwise connected, then M is hereditarily unicoherent.*

Proof. Suppose that M is not hereditarily unicoherent. Let H and K be subcontinua of M such that $H \cap K = A \cup B$ where A and B are mutually separated sets. Assume without loss of generality that $p \notin H \cup K$, and let px be an arc in M such that

$$px \cap (A \cup B) = \{x\}.$$

If $x \in A$, then

$$(px \cup H) \cap (px \cup K) = (px \cup A) \cup B ,$$

and consequently the intersection of two subcontinua containing p is not connected. This is a contradiction.

THEOREM 2.3. *If M is semi-aposyndetic, then M is a dendroid.*

Proof. According to Theorem 2.2, it suffices to show that px is an arc for each $x \in M - \{p\}$, and that M is hereditarily decomposable.

Suppose that N is an indecomposable subcontinuum of M with non-void interior relative to px. Then $N = \mathrm{cl}(N^0)$, and we may assume that $p \notin N$ and $x \notin N$. Let P (respectively X) denote the component of $px - N^0$ which contains p (respectively x). Choose $h \in P \cap N$ and $k \in X \cap N$. Then N is clearly irreducible between h and k.

Since M is semi-aposyndetic, we may assume that there exists a subcontinuum H of M such that

$$h \in H^0 \subseteq H \subseteq M - \{k\} .$$

Since the composant of k in N must meet $H^0 \cap N$, it follows that there exists a subcontinuum A of N such that

$$A^0 = \emptyset \quad \text{(relative to } N\text{)}, \ k \in A, \ \text{and} \ A \cap H^0 \neq \emptyset .$$

Clearly, $P \cup H \cup A \cup X$ is a subcontinuum of M containing p and x, but

$$N - (P \cup H \cup A \cup X) \neq \emptyset ,$$

which is a contradiction. Thus px contains no indecomposable subcontinuum with interior relative to px.

According to [5] there exists an arc $[a, b]$ and a monotone map $f\colon px \to [a, b]$ such that $f^{-1}(c)^0 = \emptyset$ for each $c \in [a, b]$. Without loss of generality suppose that $f^{-1}(c)$ is non-degenerate for some $c \in (a, b)$ and choose distinct points z and w such that

$$z \in \mathrm{cl}\big[f^{-1}([a, c))\big] \cap \mathrm{cl}\big[f^{-1}((c, b])\big] , \quad \text{and} \quad w \in f^{-1}(c) .$$

Then there does not exist a subcontinuum of M which misses z (respectively w) and contains w (respectively z) in its interior. This contradicts the assumption that M is semi-aposyndetic.

Consequently the map f must be one-one; in particular, px is an arc.

Now let N be any non-degenerate subcontinuum of M, and let x and y be distinct points of N. Since M is semi-aposyndetic we may assume that there exists a subcontinuum F of M such that

$$x \in F^0 \subseteq F \subseteq M - \{y\} .$$

Since M is hereditarily unicoherent (Theorem 2.2), $F \cap N$ is a proper subcontinuum of N with non-void interior relative to N. Thus N is decomposable; consequently M is hereditarily decomposable.

COROLLARY 2.1. *If M is aposyndetic, then M is a tree.*

Proof. If M is aposyndetic, then M is semi-aposyndetic; hence, M is hereditarily unicoherent. It follows easily that M is locally connected.

EXAMPLE 2.1. Let X denote any Hausdorff compactification of the half-open interval $[0, 1)$. Then X is hereditarily unicoherent at each point of $[0, 1)$. Clearly X need not be hereditarily unicoherent, since X may be chosen so that $X - [0, 1)$ is homeomorphic to any given metric continuum.

EXAMPLE 2.2. Let H be a pseudo-arc (e.g., [1]) and let p, q, and r be points which lie in distinct composants of H. Given any totally disconnected compact Hausdorff space K, define H_K to be the continuum obtained by "collapsing" the sets $\{p\} \times K$ and $\{q\} \times K$ in $H \times K$. If $f\colon H \times K \to H_K$ is the natural map, then H_K is hereditarily unicoherent at each point of $f(\{r\} \times K)$. Thus H_K is a non-unicoherent continuum which is hereditarily unicoherent at each point of a dense subset.

3. Characterizations of smooth and nearly smooth continua. Recall the convention that M denotes a continuum which is hereditarily unicoherent at p. The first result is a partial generalization of the theorem in [10].

THEOREM 3.1. *The continuum M is smooth at p if and only if the weak cutpoint order with respect to p is closed.*

Proof. Let $\leqslant$ denote the weak cutpoint order with respect to p. Let $\{(x_n, y_n),\ n \in E\}$ be a net of points in $M \times M$ such that

$$x_n \leqslant y_n \quad \text{and} \quad \lim(x_n, y_n) = (x, y) .$$

Since M is smooth at p, it follows that

$$px = \mathrm{Lim}\, px_n \subseteq \mathrm{Lim}\, py_n = py .$$

Thus $x \leqslant y$, and $\leqslant$ is closed.

Conversely, suppose that M is not smooth at p. By Theorem 2.3 of [6] there exist nets $\{a_m,\ m \in E\}$ and $\{b_m,\ m \in E\}$ such that (i) $\lim a_m = a$, (ii) $\lim b_m = b$, (iii) $b_m \in pa_m$ for each $m \in E$, and (iv) $b \in M - pa$. Thus $b_m \leqslant a_m$ for each $m \in E$, but $b \not\leqslant a$. This contradicts the hypothesis that $\leqslant$ is closed.

COROLLARY 3.1. *The continuum M is a generalized tree which is smooth at p if and only if the weak cutpoint order with respect to p is a closed partial order.*

Proof. Let $\leqslant$ denote the weak cutpoint order with respect to p.

If M is a generalized tree which is smooth at p, then $\leqslant$ is a partial order since M is arcwise connected; and $\leqslant$ is closed by Theorem 3.1.

If $\leqslant$ is a closed partial order, then M is arcwise connected by Koch's arc theorem [9]. By Theorem 2.2, M is hereditarily unicoherent; and by Theorem 3.1, M is smooth at p.

We now introduce a generalization of the notion of weak cutpoint order. A quasi-order $\leqslant$ on M is said to be an *approximate weak cutpoint order* with respect to p provided that

(i) $L(x)$ is a continuum,

(ii) $p \in L(x)$, and

(iii) $L(x) \subset px \cup K(x)$

for each $x \in M$.

Clearly the weak cutpoint order with respect to p satisfies conditions (i), (ii), and (iii); in fact, the weak cutpoint order with respect to p is contained in each approximate weak cutpoint order with respect to p. If M is aposyndetic, then $K(x) = \{x\}$, and the weak cutpoint order with respect to p is the unique approximate weak cutpoint order with respect to p.

Theorem 3.1 motivates the following generalization of the notion of smoothness. A continuum M is said to be *nearly smooth* at p in case there exists a closed approximate weak cutpoint order with respect to p on M.

COROLLARY 3.2. *If the continuum M is smooth at p, then M is nearly smooth at p.*

THEOREM 3.2. *If $\leqslant$ is a closed approximate weak cutpoint order with respect to p on M, then $x \leqslant y$ for each $y \in T(x)$.*

Proof. This follows from Theorem 4 of [3].

THEOREM 3.3. *A quasi-order $\leqslant$ on M is a closed approximate weak cutpoint order with respect to p if and only if the lower sets of $\leqslant$ are of the form*

$$L(x) = px \cup K(x).$$

Proof. Let $\leqslant$ be closed and suppose that there exists a point y such that

$$y \in [px \cup K(x)] - L(x).$$

By Theorem 3.2, $D(x) \cap T(y) = \emptyset$; hence there exists a subcontinuum H such that

$$x \in H^0 \subseteq H \subseteq M - \{y\}.$$

Thus,

$$y \in px \cup K(x) \subseteq px \cup H \subseteq M - \{y\}$$

which is a contradiction. Consequently, $L(x) = px \cup K(x)$.

Conversely, suppose that $\leqslant$ has lower sets of the form $L(x) = px \cup K(x)$. If $y \not\leqslant x$, then $y \notin L(x) = px \cup K(x)$. Hence there exists a continuum H such that

$$x \in H^0 \subseteq H \subseteq M - \{y\}.$$

Let

$$U = [M - (px \cup H)] \times H^0.$$

Then U is open in $M \times M$, and for $(z, w) \in U$ it follows that

$$L(w) \subseteq px \cup H \quad \text{and} \quad z \notin px \cup H.$$

Thus $z \not\leqslant w$, and $\leqslant$ is closed.

COROLLARY 3.3. *If the continuum M is nearly smooth at p, then there exists a unique closed approximate weak cutpoint order with respect to p.*

Notation. Let M be a continuum which is nearly smooth at p. Let $\leqslant_p$ denote the unique closed approximate weak cutpoint order with respect to p; $L_p(x)$ the lower set of x; $D_p(x)$ the level set of x; and $\mathfrak{D}_p$ the decomposition of M into level sets. This notation will be used consistently throughout the remainder of the paper.

We now obtain a usable criterion for determining when a given continuum M is nearly smooth at p. Define a relation ϱ_p on M by the condition

$$(x, y) \in \varrho_p \quad \text{if and only if } x \in py \cup K(y).$$

The relation ϱ_p is clearly reflexive but not necessarily transitive (see Example 5.2). In case ϱ_p is transitive, it follows that ϱ_p is a quasi-order whose lower sets are of the form $L(x) = px \cup K(x)$. According to Theorem 3.2, ϱ_p is a closed approximate weak cutpoint order with respect to p. We have just proved the following corollary of Theorem 3.2.

COROLLARY 3.4. *The continuum M is nearly smooth at p if and only if the relation ϱ_p is transitive.*

The remainder of this section is devoted to proving generalizations of known theorems concerning smooth dendroids [3].

THEOREM 3.4. *Let $\leqslant$ be any approximate weak cutpoint order on M with respect to p. Then $\leqslant$ is closed if and only if*

$$L(x) \cap T(x) \subseteq D(x) \quad \textit{for each } x \in M.$$

Proof. If $\leqslant$ is closed, then Theorem 3.2 implies that $L(x) \cap T(x) \subseteq D(x)$.

Conversely, suppose $x \not\leqslant y$. Observe that if $z \in T(x)$, then $pz \cap D(x) \neq \emptyset$ (since $L(x) \cap T(x) \subseteq D(x)$); hence $x \leqslant z$. In particular, $y \notin T(x)$, and there exists a subcontinuum H of M such that

$$y \in H^0 \subseteq H \subseteq M - \{x\}.$$

Now $x \notin py \cup H$, and $L(z) \subseteq py \cup H$ for each $z \in H^0$. It follows that

$$[M - (py \cup H)] \times H^0$$

is an open subset of $M \times M$ which contains (x, y) and misses $\leqslant$. This implies that $\leqslant$ is closed.

COROLLARY 3.5. *Let $D(x) = \{y \in M;\ px = py\}$. Then M is smooth at p if and only if*

$$px \cap T(x) \subseteq D(x) \quad \text{for each } x \in M.$$

Proof. Let $\leqslant$ denote the weak cutpoint order with respect to p and apply Theorem 3.4.

COROLLARY 3.6. *The continuum M is a generalized tree which is smooth at p if and only if*

$$px \cap T(x) = \{x\} \quad \text{for each } x \in M.$$

Proof. If M is a generalized tree which is smooth at p, then Corollary 3.1 and Corollary 3.5 imply that

$$px \cap T(x) = \{x\} \quad \text{for each } x \in M.$$

Let $\leqslant$ be the weak cutpoint order with respect to p. By using methods similar to those applied in the proof of Theorem 2.3 it can be shown that the level sets $D(x)$ of $\leqslant$ are degenerate. In particular, $\leqslant$ is a partial order which is closed by Corollary 3.5. Corollary 3.1 implies that M is a generalized tree which is smooth at p.

LEMMA 3.1. *Let X be a dendroid. If $\{A_n,\ n \in E\}$ is a collection of subarcs of X which is linearly ordered by inclusion, then $\mathrm{cl}(\bigcup A_n)$ is an arc.*

Proof. This lemma is an immediate consequence of Lemma 3 and Theorem 1 of [13].

THEOREM 3.5. *Let X be a hereditarily unicoherent continuum. Then X is a generalized tree if and only if given x and y in X, either*

$$xy \cap T(x) = \{x\} \quad \text{or} \quad xy \cap T(y) = \{y\}.$$

Proof. If X is a generalized tree, then the proof for metric dendroids is valid (see [3], Theorem 6).

Suppose that X is a hereditarily unicoherent continuum such that

$$xy \cap T(x) = \{x\} \quad \text{or} \quad xy \cap T(y) = \{y\}.$$

In particular, X is semi-aposyndetic; and, according to Theorem 2.3, X is a dendroid. The result now follows from Lemma 3.1 and the proof of Theorem 6 in [3].

4. Monotone decompositions of nearly smooth continua. Certain types of continua are known to admit minimal monotone upper semicontinuous decompositions whose decomposition spaces belong to some class of arcwise connected continua. For example, certain continua irreducible about a finite set (e.g., [5], [11]); λ-dendroids [2]; and smooth continua [6] are known to admit such decompositions. In this section we prove an analogous decomposition theorem for nearly smooth continua.

LEMMA 4.1. *If the continuum M is nearly smooth at p, then the level set $D_p(x)$ is a continuum for each $x \in M$.*

Proof. Suppose that $D_p(x)$ is not connected (note that $D_p(x)$ is closed), and let C_1 and C_2 be distinct components of $D_p(x)$. Choose $y \in C_1$ and $z \in C_2$. It is not difficult to verify that

$$py \cap C_2 = \emptyset \quad \text{or} \quad pz \cap C_1 = \emptyset.$$

Assume that $py \cap C_2 = \emptyset$.

If $C_1 \cap T(z) = \emptyset$, then (using the compactness of C_1) it is possible to construct a continuum H such that

$$C_1 \subseteq H^0 \subseteq H \subseteq M - \{z\}.$$

In this case we have

$$z \in L(x) = L(y) \subseteq py \cup K(y) \subseteq py \cup H$$

which is a contradiction.

If $C_1 \cap T(z) \neq \emptyset$, then $pz \subseteq py \cup T(z)$. Choose

$$w \in pz \cap [T(z) - D_p(x)].$$

Then $w \leqslant_p z$; and according to Theorem 3.2, $z \leqslant_p w$. Consequently,

$$x \leqslant_p z \leqslant_p w \leqslant_p z \leqslant_p x,$$

or $w \in D_p(x)$ which contradicts the choice of w.

LEMMA 4.2. *If $\leqslant$ is a closed quasi-order on a compact Hausdorff space X, then the decomposition $\mathfrak{D}$ into level sets is upper semicontinuous; and the induced partial order $\leqslant'$ on $X/\mathfrak{D}$ is closed.*

Proof. Define an equivalence relation ϱ by the condition

$$(x, y) \in \varrho \quad \text{if and only if} \quad x \leqslant y \text{ and } y \leqslant x.$$

Then ϱ is closed, since $\leqslant$ is closed; and consequently the decomposition of X into equivalence classes (i.e., $\mathfrak{D}$) is upper semicontinuous.

If π denotes the natural continuous map from $M \times M$ onto $M/\mathfrak{D} \times \times M/\mathfrak{D}$, then $\pi(\leqslant) = \leqslant'$. Hence $\leqslant'$ is closed.

LEMMA 4.3. *Let X be a continuum which is irreducible between two points a and b. Then the set*

$$E_a = \{x \in X;\ X \text{ is irreducible from } a \text{ to } x\}$$

has empty interior.

Proof. Suppose that V is open and that $b \in V \subseteq E_a$. Choose an open set W such that $b \in W \subseteq \mathrm{cl}(W) \subseteq V$, and let C denote the component of $X - W$ which contains a. By Theorem 2–16 of [7], $C \cap V \neq \emptyset$. Thus C is a proper subcontinuum of X containing the point a and some point of E_a. This is a contradiction.

THEOREM 4.1. *If M is nearly smooth at p, then there exists a decomposition $\mathfrak{D}_p$ of M such that*

(i) *$\mathfrak{D}_p$ is upper semicontinuous,*

(ii) *the elements of $\mathfrak{D}_p$ are continua,*

(iii) *the decomposition space of $\mathfrak{D}_p$ is a generalized tree which is smooth at $D_p(p)$, and*

(iv) *if $\mathcal{E}_p$ is a decomposition satisfying* (i), (ii), *and* (iii), *then $\mathfrak{D}_p \leqslant \mathcal{E}_p$* (i.e., *$\mathfrak{D}_p$ refines $\mathcal{E}_p$*).

Proof. Let $\mathfrak{D}_p$ denote the decomposition into level sets. Then $\mathfrak{D}_p$ is upper semi-continuous by Lemma 4.2, and the elements of $\mathfrak{D}_p$ are continua by Lemma 4.1. The induced partial order on $M/\mathfrak{D}_p$ is closed, monotone (i.e., lower sets are connected) and has a unique minimal element $D_p(p)$. By Koch's arc theorem [9], $M/\mathfrak{D}_p$ is arcwise connected. Since $M/\mathfrak{D}_p$ is hereditarily unicoherent at $D_p(p)$ (see Theorem 4.1 of [6]), it follows from Theorem 2.2 that $M/\mathfrak{D}_p$ is hereditarily unicoherent. According to [10] the weak cutpoint order with respect to $D_p(p)$ is closed. In particular, $M/\mathfrak{D}_p$ is a generalized tree which is smooth at $D_p(p)$.

Suppose that $\mathcal{E}_p = \{E_p(x)\colon x \in M\}$ is decomposition satisfying (i), (ii), and (iii) such that $\mathfrak{D}_p \not\leqslant \mathcal{E}_p$. There exists an element $D_p(z)$ of $\mathfrak{D}_p$ and distinct elements $E_p(z)$ and $E_p(w)$ of $\mathcal{E}_p$ such that $E_p(w) \cap D_p(z) \neq \emptyset$. Let $g\colon M \to M/\mathcal{E}_p$ denote the natural map and let

$$z' = g\big(E_p(z)\big), \quad w' = g\big(E_p(w)\big), \quad \text{and} \quad p' = g\big(E_p(p)\big).$$

We can assume without loss of generality that $w' \notin pz'$. Since $M/\mathcal{E}_p$ is a generalized tree, it follows from Theorem 3.3 that $K(z') \subseteq p'z'$. In particular, there exists a subcontinuum H of $M/\mathcal{E}_p$ such that

$$z' \in H^0 \subseteq H \subseteq M/\mathcal{E}_p - \{w'\}.$$

Now $g^{-1}(p'z' \cup H)$ is a continuum in M (since g is monotone) which contains $pz \cup K(z)$ and misses a point of $D_p(z)$. This contradicts the fact that

$$D_p(z) \subseteq L_p(z) \subseteq pz \cup K(z).$$

As a corollary we obtain the following known decomposition theorem for smooth continua [6].

COROLLARY 4.1. *If M is a smooth continuum, then there exists a decomposition $\mathfrak{D}$ of M such that*

(i) *$\mathfrak{D}$ is upper semicontinuous,*

(ii) *the elements of $\mathfrak{D}$ are continua,*

(iii) *the decomposition space of $\mathfrak{D}$ is arcwise connected, and*

(iv) *if $\mathcal{E}$ is a decomposition satisfying* (i), (ii), *and* (iii), *then $\mathfrak{D} \leqslant \mathcal{E}$.*

Moreover, the decomposition space of $\mathfrak{D}$ is a generalized tree and each element of $\mathfrak{D}$ has void interior.

Proof. Let M be smooth at p, and let $\mathfrak{D} = \mathfrak{D}_p$. Theorem 4.1 implies that properties (i), (ii), and (iii) are satisfied and that $M/\mathfrak{D}$ is a generalized tree. If $\mathcal{E}$ is any decomposition satisfying (i), (ii), and (iii), then by Theorem 4.1 of [6] $M/\mathcal{E}$ is a generalized tree which is smooth at the element of $\mathcal{E}$ containing p. It follows from Theorem 4.1 that $\mathfrak{D} \leqslant \mathcal{E}$. According to Lemma 4.3, each element of $\mathfrak{D}$ has void interior.

The decomposition obtained for nearly smooth continua is not as well behaved as that for smooth continua. Even in simple examples (see Example 5.3) the decomposition $\mathfrak{D}_p$ of a nearly smooth continuum depends on the point p. Also, the decomposition $\mathfrak{D}_p$ of a nearly smooth continuum is not in general the minimal monotone upper semicontinuous decomposition whose decomposition space is arcwise connected. In fact, the monotone image of a nearly smooth continuum in an arcwise connected continuum need not even be a generalized tree (see Example 5.3).

For smooth continua each element of the canonical decomposition has void interior, whereas the decomposition for nearly smooth continua may be degenerate. The next theorem characterizes those nearly smooth continua which possess degenerate decompositions.

THEOREM 4.2. *Let the continuum M be nearly smooth at p. Then $\mathfrak{D}_p$ is degenerate if and only if M is indecomposable.*

Proof. It follows from Theorem 3.3 that $\mathfrak{D}_p$ is degenerate if and only if $K(x) = M$ for each $x \in M$. Consequently, $\mathfrak{D}_p$ is degenerate if and only if M is indecomposable.

5. Examples of smooth and nearly smooth continua. We begin by discussing the examples of Section 2 in more detail.

EXAMPLE 5.1. Let X denote any Hausdorff compactification of $[0,1)$, as in Example 2.1, and assume that $X-[0,1)$ is nondegenerate. Then X is smooth at each point of $[0,1)$, but it is not smooth at any point of $X-[0,1)$, since X is not locally connected there (see Corollary 3.1 of [6]). However, X is nearly smooth at p if and only if X is hereditarily unicoherent at p. In particular, the familiar "$\sin 1/x$ curve" is nearly smooth at each point.

EXAMPLE 5.2. Let H_K denote a continuum as described in Example 2.2. Let h and k be distinct points of K and define $x = f(r, h)$, and $y = f(r, k)$ (r as previously defined). We have observed that H_K is hereditarily unicoherent at the point x. We shall show that H_K is not nearly smooth at x. According to Corollary 3.4, it suffices to prove that the relation ϱ_x is not transitive. This follows from the easily verified facts that

$$(y, p) \in \varrho_x, \quad (p, x) \in \varrho_x, \quad \text{and} \quad (y, x) \notin \varrho_x,$$

where $p = f(\{p\} \times K)$. More generally it can be shown that H_K is not nearly smooth at any of its points.

The next example was presented in [10] as a dendroid which is not a generalized tree.

EXAMPLE 5.3. In the plane let A consist of the unit segment $[0, 1]$ on the x-axis, the unit segment $[0, 1]$ on the y-axis, and the vertical segments of length $\frac{1}{2}$ erected over the points with coordinates $\big((\frac{1}{2})^n, 0\big)$, where n is a positive integer. Let B denote the reflection of A through the line $y = 1$. The continuum $X = A \cup B$ is a dendroid which is not smooth. However, X is nearly smooth at each point. Let $p = (0, 0)$, $q = (0, 2)$. Then the decomposition $\mathfrak{D}_p$ (respectively $\mathfrak{D}_q$) has as its only non-degenerate element the set $[\frac{3}{2}, 2]$ (respectively $[0, \frac{1}{2}]$) on the y-axis. Consequently, $\mathfrak{D}_p \neq \mathfrak{D}_q$; although, in this example, $M/\mathfrak{D}_p$ is homeomorphic to $M/\mathfrak{D}_q$.

EXAMPLE 5.4. Let X denote a pseudo-arc (e.g., [1]). Then X is indecomposable and hereditarily unicoherent. Consequently $K(x) = X$ and $\varrho_x = X \times X$ for each $x \in X$. It follows that X is nearly smooth at each point, and that $\mathfrak{D}_x$ is degenerate for each $x \in X$.

EXAMPLE 5.5. Let Y denote a circle of pseudo-arcs [1], and let $f\colon Y \to C$ be the open decomposition map from Y onto a circle C. Let A denote a subarc of C and define $X = f^{-1}(A)$. It can be shown that X is hereditarily unicoherent (since the pseudo-arc is hereditarily unicoherent) and that $K(x) = f^{-1}\big(f(x)\big)$ for each $x \in X$. It follows that ϱ_x is transitive for each $x \in X$. Consequently X is nearly smooth at each of its points. Since X is not locally connected at any point, X is not smooth at any point. However, a slight modification of X produces a continuum which is smooth. Let $H = f^{-1}(a)$ for some $a \in A$. Then X/H is smooth at the "point" H.

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