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A positional characterization of the $(n-1)$-dimensional Sierpiński curve in S^n $(n \neq 4)$

by

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Abstract. Let X be a compact metric continuum which can be embedded in the n-sphere S^n, say by a map $h\colon X \to S^n$, in such a manner that the components of $S^n - h(X)$ form a null sequence $U_1, U_2, \ldots$ satisfying the following conditions: (1) $S^n - U_i$ is an n-cell for each i, (2) $\mathrm{Cl}\,U_i \cap \mathrm{Cl}\,U_j = \emptyset$ if $i \neq j$ (Cl denotes closure), and (3) $\mathrm{Cl}(\bigcup U_i) = S^n$. Then X is called an $(n-1)$-*dimensional Sierpiński curve*. A beautiful theorem of G. T. Whyburn [11] states that, for $n = 2$, there is precisely one $(n-1)$-dimensional Sierpiński curve X up to homeomorphism and that properties (1), (2), and (3) are satisfied for each embedding $h\colon X \to S^2$. We observe in this note that recent developments in the topology of manifolds allow one to extend Whyburn's result directly to higher dimensions $(n \neq 4)$.

Conventions. In all proofs we shall assume that $n = 3$ or $n \geqslant 5$. Our manifolds will have no boundary. If X is an $(n-1)$-dimensional Sierpiński curve and $h\colon X \to S^n$ an embedding of the type ensured by that fact, then $h(X)$ will be called an $\mathcal{S}$-*curve*; i.e., an $\mathcal{S}$-curve is a nicely embedded Sierpiński curve. We assume the reader is thoroughly familiar with [11] and simply indicate the alterations necessary in higher dimensions.

The recent developments alluded to in the first paragraph are the following.

ANNULUS THEOREM [7]. *Let U be a connected open subset of a topological n-manifold M $(n \neq 4)$ and let B and B' be two locally flat n-cells in U. Then there is a homeomorphism $h\colon M \to M$, fixed outside U, such that $h(B) = B'$.*

APPROXIMATION THEOREM FOR CELLULAR MAPS [2] [10]. *Let $f\colon M \to N$ denote a proper cellular map of n-manifolds $(n \neq 4)$ and $\{U_a\}$ an f-saturated open covering of M (i.e., $f^{-1}f(U_a) = U_a$ for each index a). Then there is a homeomorphism $g\colon N \to M$ such that $g \circ f = \text{identity} \mod \{U_a\}$ (i.e., for each $p \in M$, there is an index a such that $\{p, g \circ f(p)\} \subset U_a$).*

COROLLARY. *Suppose K is a compact subset of M such that $f^{-1}f(p) = p$ for each $p \in K$. Then g may be chosen so that $g \circ f|K = \text{identity}$. Hence,*

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if K is also a sphere of some dimension, then K is locally flat in M if and only if $g(K)$ is locally flat in N.

Proof. Choose an f-saturated open cover $\{V_\beta\}$ of $M-K$ in $M-K$ which refines $\{U_\alpha\}$ and which has mesh approaching 0 uniformly near K. Then $f|M-K$: $M-K\to N-f(K)$ is a proper cellular map of n-manifolds. By the Approximation Theorem, there is a homeomorphism g: $N-f(K)\to M-K$ such that $g\circ f=$ identity $\mathrm{mod}\{V_\beta\}$. One easily sees that g can be extended continuously to take N onto M if one defines $g(x)=f^{-1}(x)$ for $x\in f(K)$.

Other older results which we shall need are the following.

DECOMPOSITION THEOREM [9]. *Let $B_1, B_2, \ldots$ denote a sequence, either null or finite, of disjoint flat n-cells in S^n and U an open subset of S^n which contains $\bigcup B_i$. Then there is a surjective map f: $S^n\to S^n$, fixed outside U, such that the nondegenerate point inverses of f are precisely the cells $B_1, B_2, \ldots$* (Mayer's proof extends to higher dimensions.)

CELLULARITY THEOREM [8]. *Let S denote an $(n-1)$-sphere in $S^n (n\neq 4)$, U and V the components of S^n-S. Suppose there is a sequence $S_1, S_2, \ldots$ of $(n-1)$-spheres in U such that, for each i, S_i lies in the $1/i$-neighborhood $N(S, 1/i)$ of S and separates V from $U-N(S, 1/i)$ in S^n. Then $\mathrm{Cl}\,V$ is cellular in S^n.*

Proof. If S_i and S_j are disjoint, then it is an easy matter to show that the closure of the region between them is simply connected. Using this fact, it is an easy exercise to show that McMillan's cellularity criterion [8] is satisfied for $\mathrm{Cl}\,V$ in S^n. But $\mathrm{Cl}\,V$ is a compact absolute retract by [4]. Thus $\mathrm{Cl}\,V$ is cellular in S^n by [8, Theorems 1 and 1′].

We also need an easy lemma, an observation, and a definition.

LEMMA 0. *If X is an $(n-1)$-dimensional $\mathcal{S}$-curve in S^n, then there is an embedding h: $X\to S^n$ such that the components of $S^n-h(X)$ have flat $(n-1)$-spheres as boundaries.* (Note that [5] implies that $h(X)$ is also an $\mathcal{S}$-curve in S^n.)

Proof. If U is a component of S^n-X, then S^n-U is an n-cell and may be pulled slightly into its interior in such a manner that the image of $\mathrm{Bd}\,U$ is bicollared, hence flat [5]. It follows from [5] that the image of X is also an $\mathcal{S}$-curve.

An infinite iteration of this process yields the lemma. The reader who is unfamiliar with the details of such an infinite iteration should consult [3].

Remark (cf. [11, Remark, p. 321]). If X is an $(n-1)$-dimensional $\mathcal{S}$-curve in S^n, S is a flat $(n-1)$-sphere in X, and for some component V of S^n-S, no component of $V-X$ has closure which intersects S, then $(S\cup V)\cap X$ is an $\mathcal{S}$-curve in S^n.

DEFINITION (cf. [11, Definition, p. 321]). A *subdivision* σ of an $(n-1)$-dimensional $\mathcal{S}$-curve X in S^n is a division of X into a finite number of $(n-1)$-dimensional $\mathcal{S}$-curves effected by taking a simplicial subdivision σ' of the closed region R obtained by adding to X all but a finite number $U_1, \ldots, U_m$ of its complementary domains in such a way that the $(n-1)$-skeleton of σ' lies entirely in X and contains the boundary of R but does not intersect the boundary of any component of S^n-X other than $U_1, \ldots, U_m$. The intersection of the n-cells of σ' with X gives a collection of $(n-1)$-dimensional $\mathcal{S}$-curves (see the remark) constituting the "n-cells" of the subdivision of X. The subdivision is said to have *mesh less than* ε or to be an ε-*subdivision* if each "n-cell" of the subdivision has diameter less than ε.

LEMMA 1 (cf. [11, Lemma 1, p. 321]). *Suppose X and Y are $(n-1)$-dimensional $\mathcal{S}$-curves in S^n $(n\neq 4)$, U and V are components of S^n-X and S^n-Y respectively, ε is any positive number and h is any homeomorphism of $\mathrm{Bd}\,U$ onto $\mathrm{Bd}\,V$. Then there exist ε-subdivisions of X and Y whose $(n-1)$-dimensional skeletons K and K' correspond under a homeomorphism which is an extension of h.*

Proof. Proceed exactly as in [11, Proof of Lemma 1, pp. 321–322]. As the bounded complementary domains $C_1, C_2, \ldots$ and $C_1', C_2', \ldots$ of X and Y one is to understand the components of S^n-X and S^n-Y respectively not equal to U and V. One forms the decomposition spaces $(W, \varphi$: $X\to W)$ and $(W', \varphi'$: $V\to W')$ of X and Y respectively, exactly as in Whyburn's paper. One concludes that W and W' are homeomorphic to n-cells minus a finite number (same number for W and W') of holes by the Decomposition Theorem. (One may assume that $\mathrm{Cl}\,U$, $\mathrm{Cl}\,V$, $\mathrm{Cl}\,C_i$, and $\mathrm{Cl}\,C_i'$ are all n-cells by Lemma 0.) The Annulus Theorem allows one to conclude, as in Whyburn's paper, that the homeomorphism $\varphi' h\varphi^{-1}$ of the boundary sphere $\Phi(\mathrm{Bd}\,U)$ of W onto the boundary sphere $\Phi'(\mathrm{Bd}\,V')$ of W' can be extended to a homeomorphism t: $W\to W'$.

Exactly as in [11], there is, for each $\delta>0$, a simplicial subdivision Σ of W' of mesh $<\delta$ whose $(n-1)$-skeleton G does not intersect the images of the nondegenerate elements of the decompositions (W, φ) and (W', φ') under the maps $t\varphi$ and φ' respectively. That the sets $K'=(\varphi')^{-1}(G)$ and $K=\varphi^{-1}t^{-1}(G)$ effect subdivisions σ' and σ of Y and X respectively which correspond in 1-1 fashion with Σ is obvious once we note that if S is the boundary of an n-simplex in Σ, then $(\varphi')^{-1}(S)$ and $\varphi^{-1}t^{-1}(S)$ are flat $(n-1)$-spheres in Y and X respectively by the corollary to the Approximation Theorem (cf., also, the remark).

The remainder of the proof of Lemma 1 requires no changes from [11].

THEOREM 1. *Any two $(n-1)$-dimensional Sierpiński curves $(n\neq 4)$ are homeomorphic.*

Proof. It suffices to assume that each is an S-curve in S^n. Then Theorem 1 follows directly from Lemma 1 as does [11, Theorem 3] from [11, Lemma 1]

THEOREM 2. *If X is an $(n-1)$-dimensional Sierpiński curve and $h\colon X\to S^n$ is an embedding $(n\neq 4)$, then $h(X)$ is an S-curve in S^n.*

Proof. We may assume that X is an S-curve in S^n and, by Lemma 0, that the closure of each complementary domain of X in S^n is an n-cell. Let $U_1, U_2, \ldots$ be the components of S^n-X and $V_1, V_2, \ldots$ the components of $S^n-h(X)$. It suffices to show that S^n-V_1 is an n-cell. We assume indices chosen such that $\mathrm{Bd}\,V_i = h\,\mathrm{Bd}\,U_i$.

Choose $i\neq 1$. By the Decomposition Theorem, there is a map $f\colon S^n\to S^n$, fixed on $\mathrm{Cl}\,U_i$, such that the nondegenerate point inverses of f are precisely the cells $\mathrm{Cl}\,U_j$ $(j\neq i)$. Since there are only countably many nondegenerate point inverses of f and $\mathrm{Bd}\,U_i$ is collared from $S^n-\mathrm{Cl}\,U_i$, there is a sequence $S_1, S_2, \ldots$ of $(n-1)$-spheres in $S^n-f(\bigcup_{j=1}^{\infty}\mathrm{Cl}\,U_j)$ which converges to $\mathrm{Bd}\,U_i$ homeomorphically. Then $hf^{-1}(S_1)$, $hf^{-1}(S_2), \ldots$ is a sequence of $(n-1)$-spheres in $S^n-\mathrm{Cl}\,V_i$ which converges to $\mathrm{Bd}\,V_i$ homeomorphically. We conclude from the Cellularity Theorem that $\mathrm{Cl}\,V_i$ is cellular in S^n.

Examine the decompositions G and G' of S^n-U_1 and S^n-V_1 respectively which have as nondegenerate elements $\mathrm{Cl}\,U_2, \mathrm{Cl}\,U_3, \ldots$ and $\mathrm{Cl}\,V_2, \mathrm{Cl}\,V_3, \ldots$ respectively. It follows from the result of the previous paragraph that each is a cellular upper semicontinuous decomposition of its respectively space. Further, if $(W, \varphi\colon S^n-U_1\to W)$ and $(W', \varphi'\colon S^n-V_1\to W')$ are the associated decomposition spaces, there is a unique homeomorphism $h'\colon W\to W'$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & h(X) \\ \cap & & \cap \\ S^n-U_1 & & S^n-V_1 \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi'} \\ W & \xrightarrow{h'} & W' \end{array}$$

commutes.

By the Decomposition Theorem, S^n-U_1, W, and, therefore, W' are all n-cells. We wish to show the same to be true of S^n-V_1. We prove this by applying the Approximation Theorem. Choose a φ'-saturated open cover $\{W_a\}$ of $S^n-\mathrm{Cl}\,V_1$ in $S^n-\mathrm{Cl}\,V_1$ whose mesh approaches 0 uniformly near $\mathrm{Cl}\,V_1$. The map $\varphi'|S^n-\mathrm{Cl}\,V_1\colon S^n-\mathrm{Cl}\,V_1\to W'-\varphi'\,\mathrm{Bd}\,V_1$ is a proper cellular map of n-manifolds. Hence by the Approximation Theorem, there is a homeomorphism $f\colon W'-\varphi'\,\mathrm{Bd}\,V_1\to S^n-\mathrm{Cl}\,V_1$ such that $f\circ\varphi'$ $=$ identity $\mathrm{mod}\{W_a\}$. Since the mesh of $\{W_a\}$ approaches 0 uniformly

near $\mathrm{Cl}\,V_1$, it is easy to see that the homeomorphism extends to take W' onto S^n-V_1 if one defines $f(x) = (\varphi')^{-1}(x)$ for $x\in\varphi'(\mathrm{Bd}\,V_1)$. Thus S^n-V_1 is an n-cell as desired. This completes the proof of Theorem 2.

DEFINITION. A compact space Y is called a *closed n-cell-complement* if there is an embedding $h\colon Y\to S^n$ such that $\mathrm{Cl}(S^n-Y)$ is an n-cell.

Remark. By the Hosay–Lininger Theorem ([6] contains the easiest proof), every crumpled cube in S^3 is a closed 3-cell-complement. A disk is the only closed 2-cell-complement.

THEOREM 3. *Let X be an $(n-1)$-dimensional S-curve in S^n (possibly with $n=4$), $U_1, U_2, \ldots$ the components of S^n-X, $C_1, C_2, \ldots$ a family of closed n-cell-complements, and $h_i\colon \mathrm{Bd}\,C_i\to\mathrm{Bd}\,V_i$ $(i=1,2,\ldots)$ a family of homeomorphisms. Then the identification space $X\bigcup_{h_1} C_1\bigcup_{h_2} C_2\bigcup_{h_3}\ldots$ is homeomorphic to S^n and X is an S-curve in the identification space.*

Proof. We may assume by Lemma 0 that $\mathrm{Cl}\,U_i$ is an n-cell for each i. Since each C_i is a closed n-cell-complement, we may assume that $C_i\subset U_i$ and that $\mathrm{Cl}(U_i-C_i)$ can be written an a product $\mathrm{Bd}\,C_i\times\times[0,1]$ with $c=(c,0)$ and $h(c)=(c,1)$ for each $c\in\mathrm{Bd}\,C_i$. We consider the cellular upper semi-continuous decomposition G of S^n having as nondegenerate elements the arcs $c\times[0,1]$ where $c\in\mathrm{Bd}\,C_i$ for some i. The result follows from standard decomposition space techniques (cf. [1, pp. 10–11]) after we prove the following lemma.

LEMMA. *For each integer i, each open set U containing $\mathrm{Cl}\,U_i$, and each $\varepsilon>0$, there is a homeomorphism $h\colon S^n\to S^n$, fixed on C_i and outside of U such that*

(1) $\mathrm{Diam}\,h(c\times[0,1])<\varepsilon$ *for each* $c\in\mathrm{Bd}\,C_i$

and

(2) $\mathrm{Diam}\,h(\mathrm{Cl}\,U_j)<(\mathrm{Diam}\,\mathrm{Cl}\,U_j)+\varepsilon$ *for* $j\neq i$.

Proof. Extend the collar $\mathrm{Bd}\,C_i\times[0,1]$ to a collar $\mathrm{Bd}\,C_i\times[0,2]$ in U such that each of the fibers $c\times[1,\frac{3}{2}]$ and $c\times[\frac{3}{2},2]$ $(c\in\mathrm{Bd}\,C_i)$ has diameter less than $\frac{1}{4}\varepsilon$. Consider the homeomorphism $h_1\colon S^n\to S^n$ defined by

$$h_1(x)=\begin{cases} (c,\delta t) & \text{if}\quad x=(c,t),\ c\in\mathrm{Bd}\,C_i,\ 0\leqslant t\leqslant 1, \\ (c,[3\delta-3]+[3-2\delta]t)=(c,\delta+[t-1][3-2\delta]) & \\ & \text{if}\quad x=(c,t),\ c\in\mathrm{Bd}\,C_i,\ 1\leqslant t\leqslant\frac{3}{2}, \\ x & \text{otherwise}, \end{cases}$$

where $\delta>0$ is so small that each of the fibers $c\times[0,\delta]$ has diameter less than ε.

Choose $\delta_1>0$ such that δ_1-sets have ε-images under h_1. Choose positive numbers $1=t_0<t_1<\ldots<t_k=\frac{3}{2}<t_{k+1}<t_{k+2}=2$ and an open covering $\mathcal{V}$ of $\mathrm{Bd}\,U_i$ such that any set of the form $V\times[t_r, t_{r+2}]$ $(V\in\mathcal{V}$,

$0 \leqslant r < k$) has diameter less than δ_1. Using the fact that the sequence $U_1, U_2, \ldots$ is null, choose a positive number s_{k+1}, $l < s_{k+1} < 2$, such that if $(\mathrm{Cl}\,U_j) \cap \{\mathrm{Bd}\,C_i \times (1, s_{k+1}]\} \neq \emptyset$, then $\mathrm{Cl}\,U_j \subset \mathrm{Bd}\,C_i \times (1, 2)$ and the projection π: $\mathrm{Bd}\,C_i \times (1, 2) \to \mathrm{Bd}\,C_i$ takes $\mathrm{Cl}\,U_j$ into some $V \in \mathfrak{V}$.

Choose $1 = s_0 < s_1 < \ldots < s_k < s_{k+1} < s_{k+2} = 2$ such that if $r \leqslant k$ and $\mathrm{Cl}\,U_j \cap \{\mathrm{Bd}\,C_i \times (s_0, s_r)\} \neq \emptyset$, then $\mathrm{Cl}\,C_j \subset \{\mathrm{Bd}\,C_i \times (s_0, s_{r+1})\}$. Let φ: $[1, 2] \to [1, 2]$ be the map which takes each segment $[s_r, s_{r+1}]$ linearly to $[t_r, t_{r+2}]$ $(0 \leqslant r < k+2)$ with $\varphi(s_r) = t_r$ $(0 \leqslant r \leqslant k+2)$. Define h_2: $S^n \to S^n$ by the formula

$$h_2(x) = \begin{cases} (c, \varphi(t)) & \text{if } x = (c, t),\ c \in \mathrm{Bd}\,C_1,\ 1 \leqslant t \leqslant 2, \\ x & \text{otherwise}. \end{cases}$$

Then $h = h_1 \circ h_2$: $S^n \to S^n$ satisfies the requirements of the lemma.

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On decompositions of continua

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Abstract. The aim of the paper is to prove some theorems which generalize the author's earlier results concerning decompositions of λ-dendroids [4], and also some of Thomas' results on decompositions of irreducible continua [32]. An upper semi-continuous monotone decomposition of a metric continuum is called admissible if the layers of its irreducible subcontinua are contained in the elements of the decomposition. The main results of the paper say that the decomposition space of an admissible decomposition of a continuum is hereditarily arcwise connected and that every continuum has exactly one minimal admissible decomposition (called the canonical one). The structure of elements of the canonical decomposition of a continuum is shown. Some special continua are considered in the paper, e.g. continua X such that every upper semi-continuous monotone decomposition of X with a hereditarily arcwise connected decomposition space is admissible; monostratic continua; continua every element of the canonical decomposition of which has an empty interior. Some necessary and/or sufficient conditions are stated under which continua have the properties under consideration.

1. Introduction. Upper semi-continuous monotone decompositions of continua have been studied by a large number of authors. E.g. Z. Janiszewski in [18], B. Knaster in [19] and [20], K. Kuratowski in [21] and [22] and also W. A. Wilson in [34] investigated such decompositions for continua irreducible between two points. A continuation of this topic can be found in a sequence of papers. For example, E. S. Thomas, Jr., has given in [32] a large study of monotone decompositions of irreducible continua; E. Dyer [12], M. E. Hamstrom [16], W. S. Mahavier [26], H. C. Miller [28], E. E. Moise [29] and many others have discussed interesting particular problems concerning such decompositions. Some of these results, originally made for metric continua, have been extended to Hausdorff continua — see e.g. G. R. Gordh, Jr. [14] and W. S. Mahavier [27]. Besides decompositions of continua irreducible between two points, decompositions of some other spaces have been considered. In particular, R. W. FitzGerald and P. M. Swingle have studied in [13] decompositions of Hausdorff continua, especially those which have a semi-locally connected decomposition space. Some of Miller's results of [28] concerning decompositions of continua irreducible between two points were generalized by M. J. Russell in [31] to decompositions of continua