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The ideal and new interval topologies on l-groups

by

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Abstract. Lattice-ordered groups with the ideal and new interval topologies are studied to determine when they are topological groups. It is proved that any l-group with the ideal topology is a topological group. Any commutative or Archimedean l-group with only a finite number of disjoint elements in shown to be a topological group when it is given the new interval topology. An Archimedean l-group with an infinite number of disjoint elements that is not a topological group under its new interval topology is exhibited.

If the partial order in a lattice-ordered group G is used to define a topology on G, will G then be a topological group? The answer, of course, depends upon the topology defined. The case of the interval topology has been investigated by Northam [12], Conrad [6], and Jakubík [11]. In this paper, we begin the investigation for the new interval topology of Garrett Birkhoff (an investigation suggested by him in [2] (problem 114)) and for the ideal topology of Orrin Frink.

We are able to answer the question completely for the ideal topology. We show that any l-group with the ideal topology is a topological group. Our results for the new interval topology are not as complete. We show that a commutative l-group with only a finite number of disjoint elements is a topological group in the new interval topology if and only if it is a finite direct sum of ordered groups. Moreover any Archimedean l-group with only a finite number of disjoint elements is a topological group in its new interval topology. Finally an Archimedean l-group with an infinite number of disjoint elements that is not a topological group in its new interval topology is exhibited.

1. Definitions and notation. Throughout we use $<$ and $\subset$ to denote strict inequality and strict containment.

The *interval topology*, denoted by η, on a lattice L has as a subbase for the closed sets all sets of the form $\{x|\ x \leqslant a\} = a^+$ and $\{x|\ x \geqslant a\} = a^*$. All intervals of the form $[a, b] = \{x|\ a \leqslant x \leqslant b\}$ are clearly closed in this topology.

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A *closed bounded set* is formed by arbitrary intersections of finite unions of closed intervals $[a, b]$. Closed sets in the *new interval topology*, ν, are defined to be exactly those sets whose intersection with every closed bounded set is a closed bounded set. This is equivalent to requiring that a set be closed if and only if its intersection with every closed interval $[a, b]$ is a closed bounded set.

An ideal (dual ideal) in a lattice is *completely irreducible* if it cannot be expressed as the intersection of a collection of ideals (dual ideals) distinct from it. The lattice L is clearly a completely irreducible ideal and is referred to as the *trivial* ideal. The completely irreducible ideals and completely irreducible dual ideals are the subbase for the open sets of the *ideal topology*. In [3], Birkhoff and Frink established that a non-trivial ideal (dual ideal) in a lattice L is completely irreducible if and only if it is maximal subject to not containing some element of L. We use this characterization extensively.

A *lattice-ordered group* or *l-group* G is a group that is also a lattice in which $x \leqslant y$ implies $a+x+b \leqslant a+y+b$ for all $a, b, x, y \in G$. We use addition to symbolize the group operation. An *l-subgroup* of an l-group is a subgroup that is also a sublattice. An l-group that is a chain under its order relation (i.e. any two elements are comparable) is called an *ordered group* or *o-group*.

A *topological group* G is a group that is also a topological space in which the mappings $(x, y) \to x+y$ from $G \times G$ into G and $x \to -x$ from G into G are continuous, while a *semi-topological group* G is a group that is a topological space in which the translations $x \to x+a$ and $x \to a+x$ from G into G are continuous for all a as well as the reflection $x \to -x$ from G into G.

Let L_1 and L_2 be lattices. A one-one, onto function φ: $L_1 \to L_2$ such that $x < y$ if and only if $\varphi(x) < \varphi(y)$ ($x < y$ if and only if $\varphi(x) > \varphi(y)$) is called an *order isomorphism* (*dual order isomorphism*). If $L_1 = L_2$, then φ is called an *order automorphism* (*dual order automorphism*). If G_1 and G_2 are l-groups and φ: $G_1 \to G_2$ is a map which is both a group isomorphism and an order isomorphism, then we call φ an *isomorphism* and we say G_1 and G_2 are *isomorphic*.

The following lemma is useful; its proof is straightforward and is left to the reader.

1.1. LEMMA. *Let U be an ideal in a lattice L and let $b \notin U$. Then U is maximal subject to not containing b if and only if for all $a \notin U$, there exists a $u \in U$ such that $u \vee a \geqslant b$. The dual statement is also true.*

2. Semi-topological groups. The difficult part in proving that an l-group is a topological group under the interval, new interval, or ideal topologies is in showing that the group operation is continuous at the identity. Indeed, it is easy to show that the reflection $x \to -x$ is continuous and that the neighborhoods of a point are simply translates of the neighborhoods of the origin, which we do in this section.

2.1. THEOREM. *Let L_1 and L_2 be lattices. Let φ: $L_1 \to L_2$ be an order isomorphism and θ: $L_1 \to L_2$ be a dual order isomorphism. If both L_1 and L_2 are given the interval topology, the new interval topology, or the ideal topology, then φ and θ are homeomorphisms.*

Proof. (a) The proof for the interval topology is straightforward.

(b) Suppose L_1 and L_2 have the new interval topology. Let F be a closed set in L_1 and $[c, d]$ an arbitrary closed interval in L_2.

$$\varphi(F) \cap [c, d] = \varphi(F) \cap [\varphi\varphi^{-1}(c), \varphi\varphi^{-1}(d)] = \varphi(F \cap [\varphi^{-1}(c), \varphi^{-1}(d)]) .$$

Now F is a closed set in L_1, and $[\varphi^{-1}(c), \varphi^{-1}(d)]$ is a closed bounded set in L_1. Therefore, by the definition of the new interval topology,

$$F \cap [\varphi^{-1}(c), \varphi^{-1}(d)] = \bigcap_{\alpha \in A} \Big(\bigcup_{i=1}^{n_\alpha} [c_i, d_i] \Big)$$

where A is an arbitrary index set, n_α is finite for all α, and c_i and d_i are elements in L_1. Continuing, we have

$$\varphi(F) \cap [c, d] = \varphi\Big(\bigcap_{\alpha \in A} \Big(\bigcup_{i=1}^{n_\alpha} [c_i, d_i] \Big)\Big) = \bigcap_{\alpha \in A} \Big(\bigcup_{i=1}^{n_\alpha} [\varphi(c_i), \varphi(d_i)] \Big)$$

which is a closed bounded set in L_2. Therefore $\varphi(F)$ is closed in L_2. Since the image of every closed set is closed, φ^{-1} is continuous. The proof that φ is continuous is identical. Thus φ is a homeomorphism.

The proof that θ is a homeomorphism is similar.

(c) Suppose L_1 and L_2 have the ideal topology. Let U be a non-trivial completely irreducible ideal in L_1 which is maximal subject to not containing b in L_1. Clearly $\varphi(U)$ is an ideal in L_2, and $\varphi(b) \notin \varphi(U)$. Let x be an arbitrary element of L_2 that is not in $\varphi(U)$. Then $\varphi^{-1}(x) \notin U$; and by Lemma 1.1 there is a $u \in U$ such that $\varphi^{-1}(x) \vee u \geqslant b$, which implies that $x \vee \varphi(u) \geqslant \varphi(b)$. Thus $\varphi(U)$ and $\varphi(b)$ satisfy the conditions of Lemma 1.1. Since $\varphi(U)$ is maximal subject to not containing $\varphi(b)$, it follows that $\varphi(U)$ is a completely irreducible ideal. Similarly, the image $\varphi(V)$ of any completely irreducible dual ideal V is a completely irreducible dual ideal. Thus φ maps a subbasic open set onto a subbasic open set, and so φ^{-1} is continuous. In the same manner φ can be shown to be continuous. Therefore φ is a homeomorphism.

The proof that θ is a homeomorphism is similar except that the image of a completely irreducible ideal is a completely irreducible *dual* ideal and vice versa.

2.2. THEOREM. *Let G be an l-group. If G is given the interval topology, the new interval topology or the ideal topology, then G is a semi-topological group.*

Proof. According to Birkhoff ([2], pp. 287, 290), every group translation is an order automorphism, and every reflection is a dual order automorphism. By Theorem 2.1, the translations and reflections are continuous under any of the three topologies.

Let G be an l-group with one of the three topologies. Because translations are actually homeomorphisms, for all x in G the neighborhoods of x are simply translates of the neighborhoods of 0. To show addition is continuous on G, we need only show that it is continuous at 0. Thus we have the following useful corollary.

2.3. COROLLARY. *Let G be an l-group with either the interval, new interval, or ideal topologies. Then G is a topological group if and only if $x_a \to 0$ and $y_a \to 0$ imply that $x_a + y_a \to 0$ for all nets $\{x_a\}_{a\in A}$ and $\{y_a\}_{a\in A}$ in G.*

3. The ideal topology on l-groups. In this section we show that the ideal topology on any l-group G makes G a topological group.

3.1 LEMMA. *Let L be a distributive lattice and let U be an ideal which is maximal subject to not containing $b \in L$. Then $r \in U$ if and only if $r \wedge b \in U$. The dual is also true.*

Proof. The only if is clear. Suppose $r\wedge b \in U$ but $r \notin U$. Since $r \notin U$, there exists, by Lemma 1.1, a $u \in U$ such that $u \vee r \geqslant b$. Since $r \wedge b \in U$ and $u \in U$, we have $u\vee(r\wedge b) \in U$. But $u \vee (r\wedge b) = (u\vee r)\wedge(u \vee b) \geqslant b$, and so $b \in U$. Thus we have a contradiction.

The dual is proved similarly.

3.2. LEMMA. *Let L be a distributive lattice and let U be an ideal which is maximal subject to not containing $b \in L$. If $r \notin U$, then U is maximal subject to not containing $r\wedge b$. The dual is also true.*

Proof. By Lemma 3.1, $r\wedge b \notin U$. If $a \notin U$, there exists a $u \in U$ such that $a\vee u \geqslant b \geqslant r\wedge b$. Thus, by Lemma 1.1, U is maximal subject to not containing $r\wedge b$. The dual is proved similarly.

3.3. LEMMA. *Let G be an l-group. Then for $x, y, r, s \in G$ $[(x+s)\vee(r+y)]\wedge(r+s) \geqslant (x+y)\wedge(r+s)$ and the dual inequality hold.*

Proof. According to C. Holland ([9], Theorem 2) G is isomorphic to an l-subgroup H of the group of all order automorphisms on a chain. It is sufficient to show that H satisfies the above inequalities. To describe H more fully, let C be a chain and let A be the group of order automorphisms on C. If $x, y \in A$, then $x \leqslant y$ if and only if $x(c) \leqslant y(c)$ for all $c \in C$. The group operation on A is composition. Thus if $x, y \in A$, we denote the group operation by $x\circ y$ where $x \circ y(c) = x(y(c))$. In terms of H the inequality becomes

$$[(x\circ s)\vee(r\circ y)]\wedge(r\circ s)] \geqslant (x\circ y)\wedge(r\circ s)\,.$$

We need only show that this inequality holds for each $c \in C$. Since $s(c), y(c) \in C$, either $s(c) \leqslant y(c)$ or $s(c) > y(c)$.

Case 1. Assume $s(c) \leqslant y(c)$. Then $r(s(c)) \leqslant r(y(c))$ and

$$\begin{aligned}[x(s(c))\vee r(y(c))]\wedge r(s(c)) &\geqslant r(y(c))\wedge r(s(c))\\ &= r(s(c))\\ &\geqslant x(y(c))\wedge r(s(c)).\end{aligned}$$

Case 2. Assume $s(c) > y(c)$. Thus $x(s(c)) > x(y(c))$ and

$$\begin{aligned}[x(s(c))\vee r(y(c))]\wedge r(s(c)) &\geqslant x(s(c))\wedge r(s(c))\\ &\geqslant x(y(c))\wedge r(s(c))\,.\end{aligned}$$

The dual inequality is proved similarly.

The proof of part (c) of Theorem 2.1 establishes that if φ is an order isomorphism, then the image $\varphi(U)$ of every ideal U maximal subject to not containing b is maximal subject to not containing $\varphi(b)$. In an l-group G every group translation is an order isomorphism. Thus if U is an ideal in G maximal with respect to not containing b, then $U+a$ $(=\{u+a \mid u \in U\})$ is an ideal maximal subject to not containing $b+a$, and so $U+a$ is a completely irreducible ideal. We use this fact and the knowledge that every l-group is a distributive lattice in the following theorem.

3.4. THEOREM. *Let G be an l-group and let U be a completely irreducible ideal in G which contains 0. Then there exist completely irreducible ideals V and W such that $0 \in V\cap W$ and if $x, y \in V\cap W$, then $x+y \in U$. The result also holds for completely irreducible dual ideals.*

Proof. Since the theorem is clearly true if U is trivial, we assume that U is nontrivial and thus is maximal subject to not containing $b \in G$. Suppose there exist $r, s \in U$ such that $r+s \notin U$. (If no such elements can be found, let $V = W = U$.) Let $V = U-s$ and $W = -r+U$. By Theorem 2.1, V is maximal subject to not containing $b-s$ and W is maximal subject to not containing $-r+b$. Since $r, s \in U$, we have $0 = s-s \in V$ and $0 = -r+r \in W$.

To show the final condition, let $x, y \in V\cap W$. Now $x \in V$ implies $x+s \in U$ and $y \in W$ implies $r+y \in U$. Since $x+s$ and $r+y$ are in U,

$(x+s)\vee(r+y)\in U$ and so is $[(x+s)\vee(r+y)]\wedge(r+s)\wedge b$. By Lemma 3.3,

$$[(x+s)\vee(r+y)]\wedge(r+s)\wedge b\geqslant(x+y)\wedge(r+s)\wedge b\,.$$

Therefore $(x+y)\wedge(r+s)\wedge b\in U$. By Lemma 3.2, U is maximal subject to not containing $(r+s)\wedge b$. By Lemma 3.1 $(x+y)\wedge(r+s)\wedge b\in U$ implies $x+y\in U$.

The proof for completely irreducible dual ideals is similar.

3.5. THEOREM. *Any l-group G with the ideal topology is a topological group.*

Proof. By Corollary 2.3, it is sufficient to show that if two nets $\{x_a\}_{a\in A}$ and $\{y_a\}_{a\in A}$ converge to 0, then the net $\{x_a+y_a\}$ converges to 0. We show that $\{x_a+y_a\}$ is eventually in every subbasic open set which contains 0. Let U be an arbitrary completely irreducible ideal which contains 0; completely irreducible dual ideals can be treated similarly. By Theorem 3.4 there exist completely irreducible ideals V and W such that $0\in V\cap W$ and if $x,y\in V\cap W$, then $x+y\in U$. Since $V\cap W$ is an open set containing 0 and $x_a\to 0$ and $y_a\to 0$, the nets $\{x_a\}$ and $\{y_a\}$ are eventually in $V\cap W$. Thus $\{x_a+y_a\}$ is eventually in U.

4. Direct sums, lexico-sums and the new interval topology. Let G and H be l-groups. The Cartesian product of the sets G and H can be made into an l-group by defining both the order and the group operation coordinatewise. We call this l-group the *direct sum of G and H* and denote it by $G+H$. The direct sum of n l-groups $G_1, G_2, \ldots, G_n$ is denoted by $\sum_{k=1}^{n} G_k$ or by $\sum G_k$.

4.1. THEOREM. *Let G_k be an ordered group for $k=1,2,\ldots,n$. Then $\sum_{k=1}^{n} G_k$ with the new interval topology is a topological group.*

Proof. The new interval topology on $\sum G_k$ is equal to the ideal topology on $\sum G_k$ ([1], Theorem 10). The theorem follows from Theorem 3.5.

A subset A of an l-group is called *disjoint* if all the members of A are strictly positive and if $a\wedge b=0$ for all elements a and b of A which are not equal. An l-group G is an ordered group if and only if it does not contain a pair of disjoint elements ([8], p. 76). We adopt the following conventions: If $G=\{0\}$, the trivial group, we say that G has no disjoint elements; if G is an ordered group, we say that G has one disjoint element.

A subset A of an l-group G is called *convex* if it contains the entire interval $[a,b]$ whenever it contains the endpoints a and b. An *l-ideal* is a convex, normal subgroup of G. If N is an l-ideal of an l-group G, the factor group G/N can be made into an l-group by defining the order relation between the cosets by the rule: $a+N\leqslant b+N$ if and only if $a'\leqslant b'$ for some a' in $a+N$ and b' in $b+N$.

4.2. DEFINITION. An l-group G is a *lexico-extension of an l-group N* if the following conditions hold

(i) N is an l-ideal of G,

(ii) G/N is an ordered group,

(iii) every positive element in G but not in N exceeds every element in N.

(Notation: $G=\langle N\rangle$).

In [7] Fuchs proved that if G is a lexico-extension of N, then G is isomorphic to $C\times N$, where C is an ordered group that is isomorphic to G/N and where the order on $C\times N$ is defined *lexicographically*: $(c,n)\geqslant(c',n')$ if and only if $c>c'$, or $c=c'$ and $n\geqslant n'$. The group operation on $C\times N$ is defined as follows:

$$(c,n)+(d,m)=\big(c+d,\psi(c,d)+\theta_d(n)+m\big)$$

where ψ is a map from $C\times C$ into N, and θ takes C into the group of automorphisms on N which preserve both the order and the group structure (we denote $\theta(d)$ by θ_d), and ψ and θ satisfy the following four conditions:

(1) $\theta_c\big(\theta_d(n)\big)=-\psi(d,c)+\theta_{d+c}(n)+\psi(d,c)$,

(2) $\theta_0(n)=n$,

(3) $\psi(c,0)=\psi(0,d)=0$,

(4) $\psi(c,d+b)+\psi(d,b)=\psi(c+d,b)+\theta_b\big(\psi(c,d)\big)$,

for all $b,c,d\in C$ and $n\in N$.

We emphasize that if $G=\langle N\rangle$, then the order is lexicographic no matter how the group operation is defined. Also, for all definitions of the group operation,

$$(0,a)+(0,b)=\big(0+0,\psi(0,0)+\theta_0(a)+b\big)=(0,a+b)$$

by the identities (2) and (3) above.

4.3. DEFINITION. Let $A_1, A_2, \ldots, A_n$ be ordered groups. Then by a finite alternating sequence of direct summations and lexico-extensions, an l-group can be constructed from the A_i in which each A_i is used exactly once to make a direct summation and the ordered groups used to make the lexico-extensions are arbitrary. Such groups are called *lexico-sums of the A_i*.

The concept of lexico-sum was introduced by Conrad in [5] where he proved the following theorem.

4.4. CONRAD'S THEOREM. *An l-group G is a lexico-sum of n ordered subgroups if and only if G contains n disjoint elements but does not contain $n+1$ such elements.*

The proof of the theorem indicates that in a lexico-sum it can be assumed that the sequence begins with a direct summation (see especially [5], p. 177; [8], p. 85–86) and that no direct summation is trivial ([5], p. 177–178; [8], p. 86). Some or all of the lexico-extensions may be trivial. From these assumptions we see that if $n = 2$, then the lexico-sum of A_1 and A_2 is $\langle A_1 + A_2\rangle$. If $n = 3$, then the lexico-sum of A_1, A_2, and A_3 in that order is $\langle\langle A_1+A_2\rangle+A_3\rangle$ or $\langle A_1+\langle A_2+A_3\rangle\rangle$ ([5], p. 171). If $n = 4$, two of the possible lexico-sums of A_1, A_2, A_3, and A_4 are $\langle\langle\langle A_1+A_2\rangle+A_3\rangle+A_4\rangle$ and $\langle\langle A_1+A_2\rangle+\langle A_3+A_4\rangle\rangle$. If G is the lexico-sum of $A_1, A_2, \ldots, A_n$, then $G = \langle A+B\rangle$, where A is a lexico-sum of $A_1, A_2, \ldots, A_s$ and B is a lexico-sum of $A_{s+1}, A_{s+2}, \ldots, A_n$ for a suitable ordering of the subscripts ([5], p. 171) and $s \neq 0$ and $s \neq n$. If all the lexico-extensions are trivial, then the lexico-sum of $A_1, A_2, \ldots, A_n$ is equal to the direct sum of the A_i.

4.5. Lemma. *The interval and new interval topologies agree on any nontrivial lexico-extension G of an l-group N.*

Proof. In this proof $A\setminus B = \{x|\ x \in A \text{ and } x \notin B\}$. According to Fuchs [7], G is isomorphic to $C\times N$, where C is a nontrivial ordered group and where the order on $C\times N$ is lexicographic.

(a) Let $a, b \in C$ be such that $a < b$, and let $p \in N$. Then

$$(a, p)^+ = (-\infty, a)\times N \cup \{a\}\times p^+$$

and

$$(b, p)^* = (b, +\infty)\times N \cup \{b\}\times p^*.$$

Thus

$$G\setminus[(a, p)^+] = (a, +\infty)\times N \cup \{a\}\times (N\setminus p^+),$$

and

$$G\setminus[(b, p)^*] = (-\infty, b)\times N \cup \{b\}\times N\setminus p^*) .$$

As a result,

$$\begin{aligned} V &= (G\setminus[(a, p)^+]) \cap (G\setminus[(b, p)^*]) \\ &= (a, b)\times N \cup \{a\}\times (N\setminus p^+) \cup \{b\}\times (N\setminus p^*) \end{aligned}$$

is open in the interval topology.

(b) Because the interval topology η is always weaker than or equal to the new interval topology ν, we need only show that every ν-closed set is η-closed. Let F be closed under ν, and let $(x, y) \notin F$. Since C is an ordered group, it is an unbounded chain. Thus there exist $a, b, r, s \in C$ such that $r < a < x < b < s$. Let p be an arbitrary element of N. Now $F\cap[(r, p), (s, p)]$ is a closed bounded set, and thus it is η-closed. Because $(x, y) \notin F\cap[(r, p), (s, p)]$, there exists an η-open set U such that $(x, y) \in U$ and $U\cap F\cap[(r, p), (s, p)] = \emptyset$. By part (a), $V = (a, b)\times\times N \cup \{a\}\times (N\setminus p^+) \cup \{b\}\times (N\setminus p^*)$ is η-open; and clearly $(x, y) \in V$.

Therefore $U\cap V$ is η-open, and $(x, y) \in U\cap V$. Moreover, $V \subseteq [(r, p), (s, p)]$ which implies that $U\cap V\cap F \subseteq U\cap F\cap [(r, p), (s, p)] = \emptyset$. Thus $G\setminus F$ is η-open, and F is η-closed.

4.6. Theorem. *Let $A_1, A_2, \ldots, A_n$, where $n \geqslant 2$, be ordered groups, and let G be a lexico-sum of the A_i in which there is at least one nontrivial lexico-extension. Suppose also that G is commutative. Then G is not a topological group in its new interval topology.*

Proof. The proof is by induction on the number n of ordered groups.

Case 1. If $n = 2$, then $G = \langle A_1+A_2\rangle$ where the lexico-extension is nontrivial. Now $(G, \nu) = (G, \eta)$ by Lemma 4.5. If (G, η) is commutative and is a topological group, then, according to Jakubík [11], it must be an ordered group. But this is impossible because, by Conrad's Theorem, G has two disjoint elements. Therefore, (G, ν) $(= (G, \eta))$ is not a topological group.

Case 2. We now assume that the theorem is true for r ordered groups where r is strictly less than n. Let G be a lexico-sum of n ordered groups in which there is at least one nontrivial lexico-extension. Thus $G = \langle A+B\rangle$, where A is a lexico-sum of r ordered groups, B is a lexico-sum of s ordered groups, $r+s = n$, and $r \neq 0 \neq s$ ([5], p. 171). If the last lexico-extension is nontrivial, then $(G, \nu) = (G, \eta)$, and an argument identical to the one given in Case 1 suffices. Suppose, however, that the last lexico-extension is trivial. Then $G = A+B$. Because G contains a nontrivial lexico-extension, either A or B must contain a nontrivial lexico-extension. Assume that B does. Since $s < n$, the induction hypothesis implies that (B, ν) is not a topological group. Suppose, however, that (G, ν) is a topological group. Then B $(= \{0\}\times B)$ is a topological group under its relative topology ([4], p. 225). It is easy to check that the relative topology on B is equal to the new interval topology on B, and thus (B, ν) $(= (\{0\}\times B, \nu))$ is a topological group. This contradicts the induction assumption.

Since a lexico-sum which has no nontrivial lexico-extension is a direct sum, Theorems 4.1, 4.4, and 4.6 imply the following corollary.

4.7. Corollary. *Let G be a commutative l-group that contains n disjoint elements and not $n+1$ such elements. Then G is a topological group in its new interval topology if and only if G is a finite direct sum of ordered groups.*

5. Archimedean l-groups. An l-group is *Archimedean* if for all elements a and b in the l-group, $na \leqslant b$ for every integer n implies that $a = 0$. An l-group is *complete* if it is *conditionally complete* as a lattice, i.e., if every nonempty bounded set has a least upper bound and a greatest lower bound.

5.1. Theorem. *Let G be an Archimedean l-group that has n disjoint elements and not $n+1$ such elements. Then G is the direct sum of n ordered* *groups, and therefore is a topological group under the new interval topology.*

Proof. We first show that a nontrivial lexico-extension H of an l-group N is not Archimedean. Now $H = C\times N$, where C is an ordered group and where the order on $C\times N$ is lexicographic. The group operation can be perfomed in a number of ways. Let $a = (0, p)$ where $p > 0$ in N, and let $b = (c, 0)$ where $c > 0$ in C. Under any definition of the group operation $n(0, p) = (0, np)$ for any integer n. Moreover, $(0, np) < (c, 0)$ for all n. Since $(0, p) \neq (0, 0)$, H is not Archimedean.

Clearly any l-subgroup of an Archimedean l-group is Archimedean. Using these two facts and Conrad's Theorem, Theorem 5.1 follows by an induction argument similar to that of Theorem 4.6.

5.2. Corollary. *Let G be a complete l-group that contains n disjoint elements but not $n+1$ such elements. Then G is the direct sum of n ordered groups and is a topological group under the new interval topology.*

Proof. According to Birkhoff ([2], p. 291), any complete l-group is Archimedean.

We have shown that any Archimedean or complete l-group with only a finite number of disjoint elements is a topological group in its new interval topology. The following is an example of an Archimedean l-group that has an infinite number of disjoint elements and that is not a topological group in its new interval topology.

5.3. Counterexample. *An Archimedean l-group with an infinite number of disjoint elements that is not a topological group in its new interval topology.*

Proof. We give as an example $C[0, 1]$, the additive group of all continuous real functions on the closed unit interval, which is an l-group under the natural ordering. Clearly it is Archimedean and has an infinite number of disjoint elements.

The interval topology, which is a T_1 topology, is clearly weaker than the new interval topology. This forces the new interval topology to be T_1. Any T_1 topological group is a Hausdorff space ([4], p. 223). Thus, if $C[0, 1]$ is a topological group under the new interval topology, then it must be a Hausdorff space. Since the property of being a Hausdorff space is hereditary, $[\mathbf{0}, \mathbf{1}]$, (where $\mathbf{0}$ ($\mathbf{1}$) is the function identically equal to 0 (1)) must be a Hausdorff space in its relative topology. By Theorem 3 of [10], the relative topology on $[\mathbf{0}, \mathbf{1}]$ equals the interval topology on $[\mathbf{0}, \mathbf{1}]$. Therefore, $[\mathbf{0}, \mathbf{1}]$ with the interval topology must be a Hausdorff space. But Northam [12] has shown that $[\mathbf{0}, \mathbf{1}]$ is not a Hausdorff space in its interval topology. Thus we have a contradiction.

References

[1] R. A. Alo and O. Frink, *Topologies of lattice products*, Canad. J. Math. 18 (1966), pp. 1004–1014.
[2] G. Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloq. Pub. 25, Third Edition (1967).
[3] — and O. Frink, *Representations of lattices by sets*, Trans. Amer. Math. Soc. 64 (1948), pp. 299–316.
[4] N. Bourbaki, *Elements of Mathematics, General Topology*, Part 1, Palo Alto 1966.
[5] P. Conrad, *The structure of a lattice-ordered group with a finite number of disjoint elements*, Michigan Math. J. 7 (1960), pp. 171–180.
[6] — *Some structure theorems for lattice-ordered groups*, Trans. Amer. Math. Soc. 99 (1961), pp. 212–240.
[7] L. Fuchs, *The extension of partially ordered groups*, Acta. Math. Acad. Sci. Hungar. 1 (1950), pp. 118–124.
[8] — *Partially Ordered Algebraic Systems*, New York 1963.
[9] C. Holland, *The lattice-ordered group of automorphisms of an ordered set*, Michigan Math. J. 10 (1963), pp. 399–408.
[10] F. Holley, *The new interval topology on lattice products*, Fund. Math. 66 (1969), pp. 329–336.
[11] J. Jakubík, *Interval topology of an l-group*, Colloq. Math. 11 (1963), pp. 65–72.
[12] E. S. Northam, *The interval topology of a lattice*, Proc. Amer. Math. Soc. 4 (1953), pp. 824–827.

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