Three questions of Borsuk concerning movability and fundamental retraction

by

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Abstract. In this paper we provide answers to three problems of Borsuk with the following results: THEOREM 1. *A movable component of a compacutm is a fundamental retract of that compactum.* THEOREM 2. *There exists a decreasing sequence of compacta* $\{X_k|\ k \geqslant 1\}$ *such that* X_{k+1} *is a retract of* X_k *for each* $k \geqslant 1$ *but* $X = \bigcap_{k=1}^{\infty} X_k$ *is not a fundamental retract of* X_1. THEOREM 3. *There exists a non-movable compactum* $Z = X \cup Y$ *where* X *and* Y *are movable,* $X \cong Y$ *and* $X \cap Y = \{point\}$.

0. Introduction (1). In [1] and [2] K. Borsuk began the development of what has come to be known as shape theory. As originated by Borsuk, the shape, or fundamental type, of a compactum (compact metric space) X is defined as follows. Let X and Y be compact subsets of Q, the Hilbert cube. A fundamental sequence, $\underline{f} = \{f_k, X, Y\}$ from X to Y, is a sequence of maps f_k: $Q \to Q$, such that for each neighborhood V of Y in Q there exists a neighborhood U of X in Q and an index k_V, such that for $k \geqslant k_V$, $f_k|_U \simeq f_{k+1}|_U$ in V. The composition of fundamental sequences $\underline{f} = \{f_k, X, Y\}$ and $\underline{g} = \{g_k, Y, Z\}$ is the fundamental sequence $\underline{gf} = \{g_k f_k, X, Z\}$. For any space X, the identity fundamental sequence is $\underline{1}_X = \{\mathrm{Id}\,Q, X, X\}$, where $\mathrm{Id}\,Q$ is the identity map on Q. Two fundamental sequences $\underline{f}$ and $\underline{g}$ are said to be *homotopic*, $\underline{f} \simeq \underline{g}$, if for every neighborhood V of Y there is a neighborhood U of X and index k_V, such that for $k \geqslant k_V$ we have $f_k|_U \simeq g_k|_U$ in V. A continuous map f: $X \to Y$ is said to *generate the fundamental sequence* $\underline{f} = \{f_k, X, Y\}$, if for every index k, $f_k = f'$: $Q \to Q$, f' being an extension of f. Fundamental sequences defined by different extensions are homotopic (Theorem 4.1 of [1]). X is said to *fundamentally dominate* Y, $X \underset{F}{\geqslant} Y$, if there exist fundamental sequences $\underline{f} = \{f_k, X, Y\}$ and

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$g = \{g_k, Y, X\}$, such that $fg \simeq 1_Y$. X is fundamentally equivalent to Y if in addition, $gf \simeq 1_X$. The relationship of fundamental equivalence is an equivalence relation; equivalence classes are called *shapes*. Shape is a topological invariant; in particular, different embeddings of the same space have the same shape. A fundamental sequence $r = \{r_k, X', X\}$ is called a *fundamental retraction* if $X \subseteq X'$ and $r_k(x) = x$ for every $x \in X$, and index k.

1. In [3] Borsuk asks the following question: Problem (6.1). Is it true that every movable component of a compactum X' is necessarily a fundamental retract of X'? In Theorem 1 and its Corollary we give an affirmative answer to this question.

Let us begin by recalling the definition of movability. Borsuk gave the original definition in terms of closed neighborhoods; however this is clearly equivalent to the following definition. A compactum X, lying in the Hilbert cube Q, is said to be *movable* if for every open neighborhood U of X (in Q) there exists an open neighborhood U_0 of X which is deformable into any neighborhood W of X in U. Movability is known to be a topological property [3]. If we have a fundamental retraction $\{r_k, X', X\}$, such that for every neighborhood V of X there is a neighborhood U of X' and an index k_V, such that for $k \geqslant k_V$ we have

$$r_k|_U \simeq r_{k+1}|_U \quad \text{in } V, \ \mathrm{rel}\, X,$$

then we say that $\{r_k, X', X\}$ is a *strong fundamental retraction*. (That is, there is a homotopy $\varphi\colon U \times I \to V$ such that $\varphi(\chi, 0) = r_k(\chi)$, $\varphi(\chi, 1) = r_{k+1}(\chi)$ and $\varphi(\chi, t) = \chi$ for each $\chi \in X$, $t \in I$.) We show that if X is a movable component of X', then X is a strong fundamental retract of X'.

Conceptually, the idea of the proof is to observe that if $X \subseteq X' \subseteq Q$, and X ($\neq X'$) is a component of X', then there is an appropriate open neighborhood system $\{W_k|\ k \geqslant 1\}$ of X' in Q and an open neighborhood system $\{U_k|\ k \geqslant 1\}$ of X, satisfying the condition that for some subsequence of indices $\{n(k)\}$, $\overline{W}_k \cap [\overline{U}_{n(k)} \setminus U_{n(k)+1}] = \emptyset$. (By a system of neighborhoods for X we mean a sequence of neighborhoods eventually inside any neighborhood of X.) It is then possible to use the movability of X to deform $\overline{W}_k \setminus U_{n(k)}$ toward X, keeping $\overline{U}_{n(k)+1}$ pointwise fixed at each level. In so doing, the set $\overline{U}_{n(k)} \setminus U_{n(k)+1}$ will probably get pushed around quite a bit, for it certainly need not be possible to deform an entire neighborhood of X toward X, rel X. Thus we must be careful, at the end of successive deformations, to avoid coming to rest on any set $\overline{U}_{n(k)} \setminus U_{n(k)+1}$. This necessitates actually working with a subsequence of the $\{U_{n(k)}\}$. We use a result from infinite dimensional topology to construct the desired deformations. The sequence of inductively defined deformations may then be composed at the $t = 1$ level to define a strong fundamental retraction. The result then holds whether or not X is connected, provided only that neighborhood systems like those described above exist, and this depends on X', not X. We preceed the proof with two easy lemmas. Considering $Q = \prod_{i=1}^{\infty} [0, 1]_i$, let $R = \prod_{i=1}^{\infty} (0, 1)_i \subseteq Q$.

LEMMA 1. *Let $X \subset R$ be a compact subset of Q. Suppose that $W \supseteq U \supseteq X$ are open neighborhoods of X in Q, A is a compact subset of W, and $\varphi\colon A \times I \to W$ is a deformation of A into U in W. Then there exists a deformation $f\colon A \times I \to W$, of A into U in W, such that $f(A, 1) \cap X = \emptyset$.*

Proof. Since every compact subset of R is a Z-set in Q [6], by Lemma 4.1 of [6] there exists a deformation $F\colon Q \times I \to Q$ such that $F(Q, t) \cap X = \emptyset$ for every $t \in (0, 1]$. This deformation also has the property that for every neighborhood V of X there exists a $t_1 \in (0, 1)$ such that $F(V, t) \subseteq V$ for $0 \leqslant t \leqslant t_1$. In particular, there exists such a t_1 for $V = U$. Define the deformation f by

$$f(x, t) = \begin{cases} \varphi(x, 2t) & \text{for } 0 \leqslant t \leqslant \frac{1}{2}, \\ F\big(\varphi(x, 1), 2(t - \frac{1}{2})t_1\big) & \text{for } \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Then $f\colon A \times I \to W$ is the required deformation.

The second lemma is easy to prove by standard arguments.

LEMMA 2. *Let (Y, d) be a metric space and X' a compact subset of Y which is not connected. Suppose X is a component of X'. Then there exists a closure contained system of open neighborhoods $\{U_k|\ k \geqslant 1\}$ of X in Y (that is, a system satisfying $\overline{U}_{k+1} \subseteq U_k$ for every $k \geqslant 1$), and a closure contained system of open neighborhoods $\{W_k|\ k \geqslant 1\}$ of X' in Y, such that $\overline{W}_k \cap [\overline{U}_{2k-1} \setminus U_{2k}] = \emptyset$ for every $k \geqslant 1$.*

We are now ready to attack the problem itself.

THEOREM 1. *Let $X \subseteq R \subseteq Q$ be a connected, movable compactum. Let X' be a compact subset of Q such that X is a component of X'. Then X is a strong fundamental retract of X'.*

Proof. We must dispense with some preliminaries before beginning the construction of the strong fundamental retraction. First, if $X = X'$ then the identity fundamental sequence, $1_X = \{\mathrm{Id}Q, X', X\}$, is a strong fundamental retraction. We therefore assume that $X' \setminus X \neq \emptyset$. Since $X \subseteq X' \subseteq Q$ is a component of X', we may apply Lemma 2 to obtain the neighborhood systems $\{U_k\}$ and $\{W_k\}$. Since X is movable, given U_k there exists an open neighborhood V'_k of X, such that V'_k is deformable into any neighborhood of X in U_k. Choose an open neighborhood V_k of X such that $\overline{V}_k \subseteq V'_k \cap U_{k+1}$. Let $\{n(k)|\ k \geqslant 1\}$ be an increasing subsequence of odd indices, such that $\overline{U}_{n(k)} \subseteq V_k$. If we now relabel $W_k = W_{\frac{n(k)+1}{2}}$, we obtain a closure contained open neighborhood system

$\{W_k \mid k \geqslant 1\}$ for X, satisfying $\overline{W}_k \cap [\overline{U}_{n(k)} \setminus U_{n(k)+1}] = \emptyset$ for every $k \geqslant 1$. As a final step in preparation let $U_0 = V_0 = Q$, so that $\overline{W}_1 \subseteq V_0$, and V_0 may certainly be deformed into any neighborhood of X in U_0, say by linear contraction to a point of X.

The proof now proceeds in three stages. First we set up a construction which will define a deformation of Q given certain sets and neighborhoods of X. This construction is then used as the basis of an inductive argument which defines a sequence of deformations, each picking up where the previous one left off. This sequence is then composed at the $t = 1$ level to define a strong fundamental retraction.

Given any index $k \geqslant 0$, let φ_k: $\overline{V}_k \times I \to U_k$ deform $\overline{V}_k$ into V_{k+1} in U_k. Suppose now that we are given an index $j(k-1) \geqslant k+1$ if $k \geqslant 1$, and $j(-1) = 1$. Let B_k be a compact subset of $\overline{V}_k$ such that $B_k \cap \overline{U}_{n(j(k-1))} = \emptyset$. Then by Lemma 1 there is a deformation ψ_k: $B_k \times I \to U_k$ of B_k into V_{k+1} in U_k, such that $\psi_k(B_k, 1) \cap X = \emptyset$. We may therefore conclude that there is an index $j(k) > j(k-1)$ (so that $j(k) \geqslant k+2$) such that $\psi_k(B_k, 1) \cap \overline{U}_{n(j(k))} = \emptyset$.

Define f_k: $[B_k \cup \overline{U}_{n(j(k-1))} \cup (\overline{U}_k \setminus U_{k+1})] \times I \to \overline{U}_k$ by

$$f_k(x, t) = \begin{cases} x & \text{if } x \in \overline{U}_{n(j(k-1))} \cup (\overline{U}_k \setminus U_{k+1}), \\ \psi_k(x, t) & \text{if } x \in B_k. \end{cases}$$

The deformation f_k is well-defined, since $B_k \subseteq \overline{V}_k \subseteq U_{k+1}$ and $B_k \cap \overline{U}_{n(j(k-1))} = \emptyset$. Consider now the restriction f'_k: $[B_k \cup \overline{U}_{n(j(k-1))} \cup (U_k \setminus \overline{U}_{k+1})] \times I \to U_k$ (since $U_{n(j(k-1))} \subseteq U_{k+2} \subseteq U_k$). Define a partial homotopy, F_k: $U_k \times \{0\} \cup [B_k \cup \overline{U}_{n(j(k-1))} \cup (U_k \setminus \overline{U}_{k+1})] \times I \to U_k$ as follows:

$$F_k(x, t) = \begin{cases} x & \text{if } t = 0, \\ f'_k(x, t) & \text{otherwise}. \end{cases}$$

Since U_k is an open subset of Q and therefore an ANR for metric spaces, the homotopy extension theorem [5] applies, and we can assert the existence of an extension $\hat{f}_k$: $U_k \times I \to U_k$ of F_k (hence of f'_k), which is also a deformation. Now define Λ_k: $Q \times I \to Q$ in the obvious way:

$$\Lambda_k(x, t) = \begin{cases} \hat{f}(x, t) & \text{if } x \in U_k, \\ x & \text{otherwise}. \end{cases}$$

Λ_k is continuous since we are essentially matching two continuous functions on the closed set $(\overline{U}_k \setminus U_{k+1}) \times I$. Note that, by construction, $\Lambda_k(x, t) = x$ for every $x \in \overline{U}_{n(j(k-1))}$, and for $j \leqslant k$, $\Lambda_k(U_j \times I) \subseteq U_j$. The deformation Λ_k can be defined for any $k \geqslant 0$, given a compact set B_k and an index $j(k-1)$, satisfying the necessary conditions. Starting with $k = 0$ and $j(-1) = 1$, we inductively define a sequence of indices $\{j(k-1)\}$ and compact sets $\{B_k\}$, and hence a sequence of deformations $\{\Lambda_k \mid k \geqslant 0\}$.

Let $B_0 = \overline{W}_1 \setminus \overline{U}_{n(1)} = \overline{W}_1 \setminus U_{n(1)}$. Then $B_0 \subseteq V_0$, B_0 is compact, and $B_0 \cap \overline{U}_{n(1)} = \emptyset$. By the previous construction there exist an index $j(0) \geqslant 2$ and a deformation Λ_0: $Q \times I \to Q$, which takes B_0 into V_1 in U_0, and satisfies $\Lambda_0(B_0, 1) \cap \overline{U}_{n(j(0))} = \emptyset$, and $\Lambda_0(x, t) = x$ for every $x \in \overline{U}_{n(1)}$, $t \in I$. Let

$$\begin{aligned} B_1 &= \Lambda_0(B_0, 1) \cup [(\overline{W}_{j(0)} \cap \overline{U}_{n(1)}) \setminus \overline{U}_{n(j(0))}] \\ &= \Lambda_0(B_0, 1) \cup [(\overline{W}_{j(0)} \cap \overline{U}_{n(1)}) \setminus U_{n(j(0))}]. \end{aligned}$$

B_1 is thus compact, $B_1 \subseteq V_1$ and $B_1 \cap \overline{U}_{n(j(0)} = \emptyset$. Therefore there exist Λ_1: $Q \times I \to Q$, deforming B_1 into V_2 in U_1, and an index $j(1) \geqslant 3$, such that $\Lambda_1(B_1, 1) \cap \overline{U}_{n(j(1))} = \emptyset$, and $\Lambda_1(x, t) = x$ for every $x \in \overline{U}_{n(j(0))}$, $t \in I$.

Suppose that for $k \geqslant 2$ compacta $B_0, B_1, \ldots, B_{k-1}$ have been defined, together with indices $1 = j(-1) < j(0) < \ldots < j(k-1)$ (so that $j(k-1) \geqslant k+1$) and deformations $\Lambda_0, \ldots, \Lambda_{k-1}$. Suppose too that Λ_{k-1}: $Q \times I \to Q$ deforms B_{k-1} into V_k in U_{k-1}, $\Lambda_{k-1}(B_{k-1}, 1) \cap \overline{U}_{n(j(k-1))} = \emptyset$, and $\Lambda_{k-1}(x, t) = x$ for every $x \in U_{n(j(k-2))}$, $t \in I$. Let

$$\begin{aligned} B_k &= \Lambda_{k-1}(B_{k-1}, 1) \cup [(\overline{W}_{j(k-1)} \cap \overline{U}_{n(j(k-2))}) \setminus U_{n(j(k-1))}] \\ &= \Lambda_{k-1}(B_{k-1}, 1) \cup [(\overline{W}_{j(k-1)} \cap \overline{U}_{n(j(k-2))}) \setminus \overline{U}_{n(j(k-2))}]. \end{aligned}$$

Then B_k is compact, $B_k \subseteq V_k$ since $j(k-2) \geqslant k$, and $B_k \cap \overline{U}_{n(j(k-1))} = \emptyset$. By the construction there exist a deformation Λ_k: $Q \times I \to Q$, taking B_k into V_{k+1} in U_k, and an index $j(k) > j(k-1)$, such that $\Lambda_k(B_k, 1) \cap \overline{U}_{n(j(k))} = \emptyset$ and $\Lambda_k(x, t) = x$ for every $x \in \overline{U}_{n(j(k-1))}$, $t \in I$. By induction the deformations $\{\Lambda_k \mid k \geqslant 0\}$ are defined. We may now let $r_0(x) = \Lambda_0(x, 1)$ for every $x \in Q$, and if r_{k-1} has been defined for $k \geqslant 1$, let $r_k(x) = \Lambda_k(r_{k-1}(x), 1)$. We claim that $\{r_k, X', X\}$ is a strong fundamental retraction; that is, given any neighborhood U of X in Q there is a neighborhood W of X' and an index k_U such that $k \geqslant k_U$ implies $r_k|_W \simeq r_{k+1}|_W$ in U, rel$X$.

In fact we show that for $i \geqslant 5$, given $U_i \subseteq U$, we have for $k \geqslant i$,

$$r_{k-1}|_{W_{j(i)}} \underset{\Lambda_k}{\simeq} r_k|_{W_{j(i)}} \quad \text{in } U_i,$$

the homotopy Λ_k being rel$X$. Recall that the deformation Λ_k was constructed so that $\Lambda_k(U_i \times I) \subseteq U_i$ for every $k \geqslant i$. Thus to establish the above assertion we need only show that $r_{k-1}(W_{j(i)}) \subseteq U_i$ for every $k \geqslant i$, and this reduces again to proving that $r_{i-1}(W_{j(i)}) \subseteq U_i$. Since for $l \leqslant i-1$, $\Lambda_l(x, t) = x$ for every $x \in U_{n(j(i-2))}$ and $t \in I$, and since $U_{n(j(i-2))} \subseteq U_i$, it remains to show that $r_{i-1}(W_{j(i)} \setminus U_{n(j(i-2))}) \subseteq U_1$. To see this let $W = W_{j(i)}$

and write

$$W_{j(i)} \setminus U_{n(j(i-2))} = W \setminus U_{n(j(i-2))}$$
$$= W \setminus U_{n(1)} \cup [(W \cap U_{n(1)}) \setminus U_{n(j(0))}] \cup [(W \cap U_{n(j(0))}) \setminus U_{n(j(1))}] \cup \ldots$$
$$\ldots \cup [(W \cap U_{n(j(i-3))}) \setminus U_{n(j(i-2))}] .$$

Note that $W \setminus U_{n(1)} \subseteq B_0$ and, in general,

$$[(W \cap U_{n(j(i-p))}) \setminus U_{n(j(i-p+1))}] \subseteq B_{i-p+2} \quad \text{for} \quad 3 \leqslant p \leqslant i+1 .$$

(Note that $i > i-p+2$ for all such p, so that $W \subseteq W_{j(i-p+1)}$.) By the way in which the sets $\{B_k\}$ were defined, $A_{k-1}(B_{k-1}, 1) \subseteq B_k$ for every $k \geqslant 1$, and we also have by construction that $A_k(x, t) = x$ for every $x \in U_{n(j(k-1))}$, $t \in I$. To define r_{i-1} we compose $A_{i-1}, \ldots, A_0$ at the $t = 1$ level. Since

$$W \setminus U_{n(1)} \subseteq B_0 ,$$

$$A_0 = A_0(W \setminus U_{n(j(i-2))}, 1)$$
$$\subseteq B_1 \cup [(W \cap U_{n(j(0))}) \setminus U_{n(j(1))}] \cup \ldots \cup [(W \cap U_{n(j(i-3))}) \setminus U_{n(j(i-2))}] .$$

Using the above information again we have

$$A_1 = A_1(A_0, 1) \subseteq B_2 \cup$$
$$\cup [(W \cap U_{n(j(1))}) \setminus U_{n(j(2))}] \cup \ldots \cup [(W \cap U_{n(j(i-3))}) \setminus U_{n(j(i-2))}] .$$

The composition continues until we arrive at a set

$$A_{i-3} = A_{i-3}(A_{i-4}, 1) \subseteq B_{i-2} \cup [(W \cap U_{n(j(i-3))}) \setminus U_{n(j(i-2))}] .$$

Then

$$A_{i-2} = A_{i-2}(A_{i-3}, 1) \subseteq B_{i-1} , \quad \text{and} \quad A_{i-1}(A_{i-2}, 1) \subseteq B_i \subseteq V_i \subseteq U_{i+1} .$$

Therefore,

$$r_{i-1}(W \setminus U_{n(j(i-2))}) \subseteq U_{i+1} \subseteq U_i ,$$

and the proof is complete.

COROLLARY. *If $X \subseteq Q$ is a connected, movable compactum, and X' is a compact subset of Q which has X' as a component, then X is a strong fundamental retract of X' in Q.*

Proof. We may certainly embed $X' \cong Y' \subseteq R$ by a homeomorphism $h\colon X' \to Q$. By Theorem 1, $h(X) = Y$ is a strong fundamental retract of Y'. By a theorem whose proof is essentially identical to Theorem 2.10 of [2], X is a strong fundamental retract of X'.

2. We turn now to a second question of Borsuk. In [2] Borsuk poses the following problem (9.7): Let $\{X_k | \ k \geqslant 1\}$ be a decreasing sequence of fundamental retracts of a compactum X_1. Is it true that the set $X = \bigcap_{k=1}^{\infty} X_k$ is a fundamental retract of X_1? We remark that already in the plane there are non-locally connected sets which are the intersection of disks, for example the "comb space", $X = \{(x, y) \subseteq E^2 | \ x = 0 \text{ or } 1/n \text{ for } n \in Z^+,\ 0 \leqslant y \leqslant 1\} \cup \{(x, y) | \ 0 \leqslant x \leqslant 1,\ y = 0\}$. Hence if X_k is a retract of X_1 for every $k \geqslant 1$, it need not be the case that $X = \bigcap_{k=1}^{\infty} X_k$ is a retract of X_1. The following result provides a negative answer to Borsuk's question.

THEOREM 2. *There exists a decreasing sequence of compacta $\{X_k | \ k \geqslant 1\}$ such that X_{k+1} is a retract (hence fundamental retract) of X_k for every $k \geqslant 1$, but $X = \bigcap_{k=1}^{\infty} X_k$ is not a fundamental retract of (or even fundamentally dominated by) X_1.*

Proof. For $i \geqslant 1$ let S^1_i denote a copy of S^1, which we regard as the set of all complex numbers of absolute value one. Let $X_1 = \prod_{i=1}^{\infty} S^1_i$. A point $\bar{z} \in X_1$ can be written coordinatewise as $\bar{z} = (z_1, z_2, \ldots)$, where $|z_i| = 1$ for every $i \geqslant 1$. Define a retraction $r_1\colon X_1 \to r(X_1)$ as follows: $r_1(\bar{z}) = r_1(z_1, z_2, \ldots) = (z_2^2, z_2, z_3, \ldots)$. Note that the map r_1 leaves every coordinate of $\bar{z}$ after the first unchanged. Then let $X_2 = r(X_1) = \{\bar{z} \in X_1 | \ z_1 = z_2^2\}$. Similarly, if X_k has been defined for $k \geqslant 2$, define $r_k\colon X_k \to r_k(X_k) = X_{k+1}$ by $r_k(\bar{z}) = (z_{k+1}^{2k}, z_{k+1}^{2k-1}, \ldots, z_{k+1}^2, z_{k+1}, z_{k+2}, \ldots)$, for every $\bar{z} \in X_k$. The map r_k modifies only the first k coordinates of $\bar{z}$. We have then $X_{k+1} = \{\bar{z} \in X_1 | \ z_i = z_{i+1}^2, 1 \leqslant i \leqslant k\}$. Observe too that $X_{k+1} \subseteq X_k$ and each map $r_k\colon X_k \to X_{k+1}$ is a retraction.

Now $X = \bigcap_{k=1}^{\infty} X_k$ is homeomorphic to the 2-adic solenoid, defined as the inverse limit of the spaces $\{S^1_k | \ k \geqslant 1\}$, with bonding maps $h_k\colon S^1_{k+1} \to S^1_k$, given by $h_k(z) = z^2$. That is, $X = \{\bar{z} \in X_1 | \ z_k = z_{k+1}^2 \text{ for every } k \geqslant 1\}$. In [3] Borsuk proves that the solenoid X is not movable. Borsuk establishes in the same paper that any (compact) ANR is movable, and that the product of a countable number of movable compacta is movable. Thus the compactum X_1 is movable. He also proves the following result which establishes our theorem: If X_1 is movable and X is a fundamental retract of X_1 (in fact if X is only fundamentally dominated by X_1) then X is movable.

3. For our third problem we turn to the question of an addition theorem for movable compacta. The general problem is to find conditions under which $X = X_1 \cup X_2$ is movable, given that X_1, X_2, and $X_0 = X_1 \cap X_2$ are movable, the result being false in general [4]. Borsuk has defined in [4] the notion of shape for pointed compacta $(X, x_0) \subseteq (Q, x_0)$, the definitions being the same with the exception that all homotopies are understood to be relative to the basepoint x_0. Thus a *pointed compactum* $(X, x_0) \subseteq (Q, x_0)$ is said to be *movable* if for every neighborhood U of X in Q there exists a neighborhood U_0 of X, such that for every neighborhood V of X there exists a deformation $\varphi\colon U_0 \times I \to U$ of U_0 into V in U, such that $\varphi(x_0, t) = x_0$ for every $t \in I$. The main results of [3] hold true for this notion, and Borsuk is also able to prove (Theorem 2.8 of [4]) that if (X_1, x_0), $(X_2, x_0) \subseteq (Q, x_0)$ are movable and $X_1 \cap X_2 = \{x_0\}$ then $(X_1 \cup X_2, x_0)$ is movable. He then asks whether the union of two movable compacta having only one point in common is movable. Our goal is to answer this question in the negative, with another counterexample.

THEOREM 3. *Let $X \subseteq Q$ be a movable compactum and $x_0 \in X_0$, a point of a component of X. Suppose for every neighborhood U' of X there exists a neighborhood U_0' of X which is deformable into any neighborhood W' of X by a deformation $\varphi\colon U_0' \times I \to U'$, satisfying $\varphi(x_0, 1) \in W$, where W is the component of W' containing X_0. Then X_0 is movable.*

Proof. We must show that given any neighborhood U of X_0 (in Q) there is a neighborhood U_0 of X_0 which is deformable in U into any neighborhood W of X_0. We may assume that U is connected, and that $\mathrm{Bd}_Q U \cap X = \emptyset$, since X_0 has arbitrarily small neighborhoods in Q with these properties. Let U' be a neighborhood of X in Q which has U as a component. By assumption there exists a neighborhood U_0' of X, which is deformable into any neighborhood W' of X in U', by a deformation satisfying the hypothesized condition. Let U_0 be the component of U_0' containing X_0, so that $U_0 \subseteq U$. Let W be any connected neighborhood of X_0, where we again assume $\mathrm{Bd}_Q W \cap X = \emptyset$, so that there is a neighborhood W' of X, such that W is a component of W'. By hypothesis there is a deformation $\varphi\colon U_0' \times I \to U'$ of U_0' into W', such that $\varphi(x_0, 1) \in W$. This implies $\varphi(U_0, 1) \subseteq W$ and $\varphi(U_0, I) \subseteq U$, since W is a component of W', U is a component of U', and U_0 is connected. The restriction of φ to $U_0 \times I$ is the desired deformation.

As a consequence of the proof of Theorem 3 we have

COROLLARY 1. *Suppose $(X, x_0) \subseteq (Q, x_0)$ is a movable pointed compactum and $x_0 \in X_0$, where X_0 is a component of X. Then (X_0, x_0) is a movable pointed compactum.*

This result is false for movable compacta (see for example the space on p. 140 of [3], which is used below).

We do not work with Theorem 3 but its negation: Suppose $X \subseteq Q$ is movable and X_0 is a non-movable component of X. Then for some neighborhood U' of X, if U_0' is a neighborhood of X deformable in U' into any neighborhood of X, there exists a neighborhood W' of X (depending on U_0'), such that if $\varphi\colon U_0' \times I \to U'$ deforms U_0' into W', then $\varphi(X_0, 1) \cap W = \emptyset$, where W is the component of W' containing X_0. This statement may now be strengthened by noting that we may replace W' by any neighborhood contained in it, and U' by any neighborhood contained in it.

COROLLARY 2. *Let $X \subseteq Q$ be movable and X_0 a non-movable component of X. There exists a neighborhood U' of X, such that, if $U^* \subseteq U'$ and U_0^* is any neighborhood deformable in U^* into any neighborhood of X, then there exists a neighborhood W' (depending on U_0^*), such that if $W^* \subseteq W'$, and $\varphi\colon U_0^* \times I \to U^*$ deforms U_0^* into W^* in U^*, then $\varphi(X_0, 1) \cap W = \emptyset$, where W is the component of W^* containing X_0.*

DEFINITION. A compactum X is said to be *movable in* a compactum Y containing it if for every neighborhood U of X in Y there is a neighborhood U_0 of X in Y which is deformable into any neighborhood of X in Y.

THEOREM 4. *Let X be a compactum. Then in order that X be movable it is necessary that X be movable in every* ANR *Y containing it, and it is sufficient that X be movable in some* ANR *containing it.*

Proof of necessity. Assume X movable and contained in an ANR Y, which we consider embedded in Q. Let V be a neighborhood of Y in Q which retracts to Y, and $r\colon V \to Y$ a retraction. Given a neighborhood U of X in Y let $U' = r^{-1}(U)$. Then U' is a neighborhood of X in Q, and by definition of movability there exists a neighborhood U_0' of X in Q which is deformable into any neighborhood W' of X in Q. Let $U_0 = U_0' \cap Y$. Then U_0 is a neighborhood of X in Y which we claim is deformable into any neighborhood W of X in Y. Given W let $W' = r^{-1}(W)$. Then W' is a neighborhood of X in Q, and there is a deformation $\varphi\colon U_0' \times I \to U'$ of U_0' into W'. $r\varphi\colon U_0 \times I \to U$ then deforms U_0 into W in U.

Proof of sufficiency. We assume that X is movable in an ANR Y which is again embedded in Q. Let U be any neighborhood of X in Q. By a standard theorem (Corollary 3.5, p. 104 of [5]) there exists a neighborhood K of Y and a strong deformation retraction $\Lambda\colon K \times I \to Q$ of K to Y in Q. Now $U' = U \cap Y$ is a neighborhood of X in Y, and there exists by hypothesis a neighborhood U_0' of X in Y which is deformable into any neighborhood W' of X in Y. Clearly there exists a neighborhood U_0 of X in Q, such that $\Lambda(U_0, I) \subseteq U$ and $\Lambda(U_0, 1) \subseteq U_0'$. Given any neighborhood W of X in Q, let $W' = W \cap Y$. Then there exists a deformation $\varphi\colon U_0' \times I \to U'$ of U_0' into W' and we may deform U_0 into W in U by

$$\Gamma(x, t) = \begin{cases} \Lambda(x, 2t) & \text{if} \quad 0 \leqslant t \leqslant \frac{1}{2}, \\ \varphi(\Lambda(x, 1), 2t-1) & \text{if} \quad \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Therefore X_0 is movable.

Since no particular properties of Q were used to prove Theorem 3, this theorem and its second corollary remain true if Q is replaced by an ANR Y, and movability by movability in Y. We call the corresponding version of Corollary 2 the ANR version of Corollary 2.

We now construct the counterexample which will answer Borsuk's question. Let X be any of the uncountably many examples described on p. 140 of [3]. That is, we first define the non-movable solenoid S ($\not\cong S^1$) as the intersection of an appropriately nested sequence of solid tori $\{T_i|\ i \geqslant 1\}$ in E^3. Then we let $X = S \cup \bigcup_{i=1}^{\infty} \operatorname{Bd} T_i$, where $\operatorname{Bd} T_i$ denotes the boundary of T_i in E^3. (As part of the definition we require that $T_i \subseteq \operatorname{Int} T_{i-1}$ for every $i \geqslant 2$.) That X is movable may be seen directly or as a consequence of Theorem 5.3 of [3] (deforming $\bigcup_{i=1}^{n} \operatorname{Bd} T_i \cup T_{n+1}$ into $\bigcup_{i=1}^{n+1} \operatorname{Bd} T_i \cup T_{n+2}$ in $T_{n+1} \cup \bigcup_{i=1}^{n} \operatorname{Bd} T_i$ in the obvious way). Let $I^3 = [0,1]\times[0,1]\times[0,1] \subseteq E^3$ and embed $X \subseteq \operatorname{Int} I^3$. Let $h\colon I^3 \to [2,3]\times[2,3]\times[2,3] = J^3$ be the linear homeomorphism. Attach I^3 to J^3 by identifying x_0 with $h(x_0)$ for some $x_0 \in S$. Let $P\colon I^3 + J^3 \to P(I^3+J^3) = Z$ be the identification. Then $P|I^3$ and $P|J^3$ are homeomorphisms, the spaces $P(X)$ and $\operatorname{Ph}(X)$ are movable subsets of Z (homeomorphic to X), and Z itself is an ANR. We may now establish

THEOREM 5. *The subspace $Y = P(X) \cup Ph(X)$ of Z is an example of a union of two movable compacta, having one point in common, which is not movable.*

Proof. Suppose Y is movable. For convenience we no longer write the projection P. Note that while S is contractible in I^3, there exists a "toroidal" neighborhood V of X in I^3 such that S is not contractible in V. Let $U = V \cup h(V)$, so that U is a neighborhood of Y in Z. Let V' be a $1/n$ neighborhood of Y in Z, where n is chosen large enough that V' is not connected. Then V' has exactly one component which intersects both I^3 and J^3, namely the one containing the connected set $S \cup h(S)$. The same is true of any neighborhood of Y contained in V'. Note too that $S \cup h(S)$ is non-movable, since it retracts to S and movability is preserved by fundamental domination, of which retraction is a special case [3]. We apply the ANR version of Corollary 2 to the pair $\big(Y, S \cup h(S)\big)$ to obtain the neighborhood U'. Let $U^* = U \cap U'$. Since Y is movable, by Theorem 4 there exists a neighborhood U_0^* of Y in Z which is deformable in U^* into any neighborhood of Y in U^*. We conclude, by the ANR version of Corollary 2 and the movability of Y, that there exists a neighborhood W' of Y, such that for $W^* = W' \cap V'$, there is a deformation $\varphi\colon U_0^* \times I \to U^*$ of U_0^* into W^* in U^*, satisfying $\varphi\big(S \cup h(S), 1\big) \cap W = \emptyset$, where W is the component of W^* containing $S \cup h(S)$. Therefore $\varphi\big(S \cup h(S), 1\big)$ is entirely contained in either I^3 or J^3, we assume J^3. Let $r\colon Z \to I^3$ be the retraction which sends J^3 to x_0. Then the deformation $r\varphi|S\times I \to V$ contracts S to a point in V, a contradiction. We conclude that Y is non-movable.

By an isolated subset of a compactum X, we mean any compact subset whose compliment in X is compact.

THEOREM 7. *An isolated subset of a movable compactum is movable.*

Proof. Let $X_0 \subseteq X$ be an isolated subset of a movable compactum X, and consider X as embedded in Q. Let U be a neighborhood of X_0 in Q such that $U \cap X \setminus X_0 = \emptyset$. Then there is a neighborhood U' of $X \setminus X_0$ whose closure is disjoint from the closure of U. Given $U \cup U'$, a neighborhood of X, there is a neighborhood U_0' of X which is deformable into any neighborhood W of X in $U \cup U'$. Then $U_0 = U_0' \cap U$ is a neighborhood of X_0 which can be deformed into any neighborhood of X_0 in U.

COROLLARY. *If X_0 is an isolated subset of a movable compactum X and (X_0, x_0) is a movable pointed compactum, then (X, x_0) is a movable pointed compactum.*

From the above corollary it is easy to see, since every pointed ANR (X, x_0) is movable for every $x_0 \in X$, that the previous example $X = S \cup \bigcup_{i=1}^{\infty} \operatorname{Bd} T_i$ has the property that (X, x) is a movable pointed compactum for any x belonging to the dense subset $\bigcup_{i=1}^{\infty} \operatorname{Bd} T_i$ of X. However (X, x) is not movable for any $x \in S$. Thus we have another example of a compactum X such that $\operatorname{Sh}(X, x)$ depends on the choice of $x \in X$ (see Section 5 of [4]).

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