(1) $D_i \cap S_t, D_i' \cap S_t'$ are simple closed curves for $t<1$ and p_i, p_i' respectively for $t=1$.

(2) Each component of $C \setminus D_i$, $C \setminus D_i'$ contains exactly one component of $K \setminus p_i$, $K' \setminus p_i'$ respectively.

(3) D_i, D_i' are locally tame except possibly at p_i, p_i' respectively.

(4) For $i \neq j$ there is a number t such that $0 \leqslant t < 1$ and

$$C_t \cap D_i \cap D_j = \emptyset = C_t' \cap D_i' \cap D_j'.$$

(5) If i, j, and t are as in (4), $t<s<1$ and W, W' are the closures of $C_t \setminus C_s$, $C_t' \setminus C_s'$ respectively, then the closures of the components of $W \setminus (D_i \cup D_j)$ and $W' \setminus (D_i' \cup D_j')$ are two tame 3-cells and a tame solid torus.

Let $t_1, t_2, \ldots$ be a monotone increasing sequence of positive numbers with limit 1 such that for $m \neq n$ and $m, n \leqslant i+1$,

$$C_{t_i} \cap D_m \cap D_n = \emptyset = C_{t_i}' \cap D_m' \cap D_n'.$$

The S_{t_i}'s, D_i's, S_{t_i}''s and D_i''s form isomorphic decompositions of C into tame 3-cells, tame solid tori, and points of K and K'. Now it is not difficult to extend the homeomorphism h inductively over elements of the decompositions to obtain a homeomorphism of C onto itself. This completes the outline of the proof of Theorem 5.

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Monotone decompositions of continua into generalized arcs and simple closed curves

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Abstract. For compact, Hausdorff continua that are irreducible between two points, and those that are separated by no subcontinuum, sufficient conditions are given in order for the continuum to have a monotone, upper semi-continuous decomposition onto a generalized arc and generalized simple closed curve, respectively. The conditions involve the use of saturated and bi-saturated collections of continua. For metric continua the conditions are both necessary and sufficient. It is also shown that the elements Q in the decomposition with void interiors are of the form $T(T(x)), x \in Q$, where T is the aposyndetic set function. The structure of the elements with non-void interiors is described and two open questions relating to the paper are discussed.

1. Introduction. A compact, Hausdorff continuum M that is irreducible between a pair of its points is *type A* [4] if M has a monotone, upper semi-continuous decomposition whose quotient space is a generalized arc (a continuum with exactly two non-separating points). If M is separated by no subcontinuum let us say M is also type A if there exists a monotone, upper semi-continuous decomposition whose quotient space is a generalized simple closed curve (a continuum in which every set of two distinct points separates). The primary theorems of this paper establish sufficient conditions for these two kinds of continua to be type A. If the continuum is metric the conditions are both necessary and sufficient where, of course, the quotient space is now a simple arc or a simple closed curve. Whether in the Hausdorff setting these conditions characterize type A continua is not known.

Prior work for non-metric continua has been done by Gordh [4], and FitzGerald and Swingle [3] in the case where the continuum is irreducible. Gordh generalizes the work of Thomas on metric continua [7] to prove that a compact, Hausdorff continuum M is type A with the elements of the decomposition nowhere dense if and only if M contains no region-containing indecomposable continuum. FitzGerald and Swingle give sufficient conditions for the continuum to be type A and give results concerning the nature of the elements of the minimal decomposition.

For metric continua separated by no subcontinua, Stratton [6] has developed enough machinery to characterize the required decompositions onto simple closed curves *provided* there exist two points x and y in M such that $L_x \cap L_y = \emptyset$ ($L_x = T(x)$ in our definition and notation used later). Our approach makes use of a modification of the notion of a saturated collection of subsets due to Whyburn [8, p. 45]. A collection $\{G\}$ of subsets of a connected topological space X is *saturated* if for each $G \in \{G\}$ and $p \in X - G$, there exists $G' \in \{G\}$ such that G' separates p from G in X, i.e., $X - G' = A \cup B$ where $p \in A$, $G \subset B$ and $\bar{A} \cap B = \emptyset = A \cap \bar{B}$. A *decomposition* of a topological space X is a collection $\{Q\}$ of subsets of X whose union is X and such that if $Q, Q' \in \{Q\}$ then $Q = Q'$ or $Q \cap Q' = \emptyset$. Each subset in the collection is called an *element* and a decomposition is *monotone* if each element is a closed connected set (continuum). It is *upper semi-continuous* provided that if $Q \in \{Q\}$ and U is an open set containing Q there is an open set V containing Q and lying in U such that if $Q' \in \{Q\}$ and $Q' \cap V \neq \emptyset$, then $Q' \subset U$. Finally a subset H of X is *region-containing* if there is an open set U in X such that $U \subset H$.

2. Decompositions into generalized arcs.

THEOREM 1. *Let M be a compact, Hausdorff continuum that is irreducible between a pair of its points. A sufficient condition that M have a monotone, upper semi-continuous decomposition such that the quotient space is a generalized arc is that there exists a saturated collection $\{G\}$ of subsets of M such that each $G \in \{G\}$ is a non-region-containing continuum.*

Proof. Let G be a saturated collection of non-region-containing continua of M. We first need to establish that no element of $\{G\}$ separates two points belonging to any other element of $\{G\}$ (such a collection is called non-separating [8, p. 42]). Suppose $x, y \in G \in \{G\}$ and $G' \in \{G\}$ separates x from y, i.e., $M - G' = A_x \cup A_y$, a separation, with $x \in A_x$ and $y \in A_y$. The sets A_x and A_y are connected and $A_x \cup G'$ and $A_y \cup G'$ are continua. There is an element $G'' \in \{G\}$ such that G'' separates G' from x. It is clear that $G'' \subset A_x$ and if B_x and B' are the connected open sets containing x and G', respectively, in the separation of M by G'', it is also clear that $G' \cup A_y \subset B'$ and $G'' \cup B_x \subset A_x$. The continuum G must intersect both G' and G'' so let us denote by H an irreducible subcontinuum of G from G' to G''. The continuum $(B_x \cup G'') \cup H \cup (A_y \cup G') = M$ since M is irreducible between some pair of its points. This means that H and consequently G have non-empty interiors in M. This contradiction establishes that $\{G\}$ is a non-separating collection.

It is an immediate consequence of this proof that if $G, G' \in \{G\}$ then either $G \cap G' = \emptyset$ or $G = G'$. As in [8, p. 128] let $Q_x = \{y \in M | x$ and y cannot be separated by any $G \in \{G\}\}$. First notice that if $x \in G \in \{G\}$ then $Q_x = G$ so that $\{G\} \subset \{Q_x | x \in M\}$. If $Q_x \cap Q_y \neq \emptyset$ then $Q_x = Q_y$ for if $p \in Q_y - Q_x$ then p can be separated from x by an element $G \in \{G\}$. But $Q_x \cap G = \emptyset$ so that G separates Q_x from p. Because $Q_x \cap Q_y \neq \emptyset$, G separates two points of Q_y which is impossible. Also if $p \notin Q_x$ then p can be separated from Q_x by an element of $\{G\}$, which implies that Q_x is a closed set. So $\{Q_x | x \in M\}$ is a closed decomposition of M.

At this point it is useful to introduce a definition. If X is a topological space and $A \subset X$ then $T(A) = A \cup \{x \in X |$ there does not exist an open set U and continuum H such that $x \in U \subset H \subset X - A\}$. We denote $T(T(A))$ by $T^2(A)$ and in general $T^i(A) = T(T^{i-1}(A))$. If $T(x) = \{x\}$ for every $x \in X$ we say X is *semi-locally connected*. The function T was first defined by Jones [5]. For more information on T and T^i, see [2].

By the definition of $\{Q_x | x \in M\}$ it follows that $T(Q_x) = Q_x$ for all $x \in M$. For if $y \notin Q_x$ there is $G \in \{G\}$ such that $M - G = A \cup B$, a separation of M, such that $Q_x \subset A$, $y \in B$. Then $y \in B \subset B \cup G$, a continuum; so $y \notin T(Q_x)$. Let $\{Q'_x | x \in M\}$ be the collection of components of members of $\{Q_x | x \in M\}$. This collection is a monotone decomposition of M. Since Q'_x is a component of Q_x and $T(Q'_x)$ is a continuum [3, p. 35], we have $T(Q'_x) \subset T(Q_x) = Q_x$ and consequently $T(Q'_x) = Q'_x$. For each $x \in M$ we now have $T(Q'_x) = Q'_x$. From this it is not hard to show that $\{Q'_x | x \in M\}$ is an upper semi-continuous decomposition of M [3, p. 45] and that the quotient space M' is semi-locally connected [3, p. 37]. Because M is irreducible between two points so is M' and consequently, because of semi-local connectedness, every point of M' except for two separates M'. Therefore M' is a generalized arc.

The hypothesis could be slightly weakened in the theorem in that the saturated collection $\{G\}$ could be taken to consist of non-region-containing connected sets since the fact that they would be closed is an immediate consequence of the definition.

3. Decompositions into generalized simple closed curves. The essential concept needed here is another modification of the notion of a saturated collection of subsets. A collection $\{G\}$ of subsets of a connected topological space X is *bi-saturated* if for each $G \in \{G\}$ and $p \in X - G$, there exist G', $G'' \in \{G\}$ such that $G' \cup G''$ separates p from G in X.

THEOREM 2. *A compact, Hausdorff continuum M that is separated by no subcontinuum has a monotone, upper semi-continuous decomposition whose quotient space is a generalized simple closed curve if there exists a bi-saturated collection $\{G\}$ of subsets of M such that each $G \in \{G\}$ is a non-region-containing continuum.*

Proof. Let $\{G\}$ be as in the hypothesis. We need to show first that for any two elements $G_1, G_2 \in \{G\}$, $G_1 \cup G_2$ does not separate two points belonging to any other element of $\{G\}$ (this is a modification of the notion of a non-separating collection used earlier). Suppose $x, y \in G$ and there exist $G_1, G_2 \in \{G\}$ such that $G_1 \cup G_2$ separates x from y, i.e., $M-(G_1 \cup G_2) = A_x \cup A_y$, a separation, with $x \in A_x$ and $y \in A_y$. Since no subcontinuum separates M, A_x and A_y are connected open sets and $\overline{A}_x$, $\overline{A}_y$ are each irreducible from G_1 to G_2. Clearly the continuum G must intersect G_1 or G_2, but not both. Let $G \cap G_1 \neq \emptyset$. Because $\{G\}$ is bi-saturated there exist $H_1, H_2 \in \{G\}$ such that $M-(H_1 \cup H_2) = B_x \cup B_1$, a separation, with $x \in B_x$ and $G_1 \subset B_1$. Either $H_1 \subset A_x$ or $H_2 \subset A_x$ (say $H_1 \subset A_x$) or else $H_1 \cup H_2$ would not separate x from G_1 in $G_1 \cup A_x \cup G_2$ let alone in M. Let us consider the two cases: $H_2 \cap G_2 \neq \emptyset$ and $H_2 \cap G_2 = \emptyset$. If $H_2 \cap G_2 \neq \emptyset$ then H_1 must separate G_1 from $H_2 \cup G_2$ in $\overline{A}_x$ or else $H_1 \cup H_2$ does not separate M. So $\overline{A}_x - H_1 = C \cup D$, a separation, with $\overline{A}_x \cap G_1 \subset C$ and $\overline{A}_x \cap (H_2 \cup G_2) \subset D$. In order for $H_1 \cup H_2$ to separate x from G_1 in M, we must have $x \in D$. Let G' be an irreducible subcontinuum of G from H_1 to $G_1 (G \cap H_1 \neq \emptyset)$. Since $\overline{C}$ is irreducible from H_1 to G_1 and $G' \subset \overline{C}$ then $G' = \overline{C}$. Thus $C \subset G' \subset G$ and since C contains an open set of M so does G, a contradiction. If $H_2 \cap G_2 = \emptyset$ and $H_2 \subset A_y$ then a similar argument leads to a contradiction. Finally suppose $H_2 \cap G_2 = \emptyset$ and $H_2 \subset A_x$. Then $\overline{B}_x$ and $\overline{B}_1$ are irreducible continua from H_1 to H_2. Since $G_1 \subset B_1$ and $G_1 \cup A_y \cup G_2$ is a continuum lying in $M-(H_1 \cup H_2)$ then it lies in B_1. So $B_x \cup H_1 \cup H_2 \subset A_x$ and, because B_x is a non-empty open set, $B_x \cup H_1 \cup H_2$ separates $G_1 \cap \overline{A}_x$ from $G_2 \cap \overline{A}_x$ in $\overline{A}_x$. If G' is an irreducible subcontinuum of G from G_1 to $B_x \cup H_1 \cup H_2$ then again G' must have a non-empty interior. But then so must G and this is impossible. Both cases $H_2 \cap G_2 \neq \emptyset$ and $H_2 \cap G_2 = \emptyset$ have led to contradictions so if $x, y \in G \in \{G\}$ then they cannot be separated from each other in M by the union of any two elements of $\{G\}$.

It is evident that if $G, G' \in \{G\}$ and $G \cap G' \neq \emptyset$ then $G = G'$. Now let $Q_x = \{y \in M|\ x$ and y cannot be separated by any $G_1 \cup G_2$, where $G_1, G_2 \in \{G\}\}$. If $x \in G \in \{G\}$ then it is clear that $Q_x = G$ so $\{Q_x|\ x \in M\} \supset \{G\}$. As in Theorem 1 the elements of $\{Q_x\}$ are closed sets and if $Q_x \cap Q_y \neq \emptyset$ then $Q_x = Q_y$ so $\{Q_x|\ x \in M\}$ is a closed decomposition of M. Again it follows that $T(Q_x) = Q_x$ for all $x \in M$. For if $y \notin Q_x$ there exist $G_1, G_2 \in \{G\}$ such that $M-(G_1 \cup G_2) = A \cup B$, a separation of M, such that $Q_x \subset A$ and $y \in B$. But $G_1 \cup B \cup G_2$ is a continuum and B is an open set; thus $y \notin T(Q_x)$. Let $\{Q'_x|\ x \in M\}$ be the collection of components of members of $\{Q_x|\ x \in M\}$. By the same argument as before we have $T(Q'_x) = Q'_x$ for all $x \in M$. By a similar argument to the one in [3, p. 45] it follows that $\{Q'_x|\ x \in M\}$ is upper semi-continuous. As before by [3, p. 37] M' is semi-locally connected. Because M is separated by no subcontinuum, neither is M' and hence M' is loc lly connected [1, Th. 2]. Therefore every pair of distinct points of M' separates M' and so M' is a generalized simple closed curve.

4. Decompositions into simple arcs and simple closed curves. In this section M will be a metric continuum and here the conditions in Theorems 1 and 2 characterize type A continua for irreducible continua and those in which no subcontinuum separates, respectively. Since Theorems 1 and 2 prove the sufficiency, only the necessity needs to be established and, because the proofs are similar, the argument will be given only for simple closed curves.

THEOREM 1′. *Let M be a compact, metric continuum that is irreducible between a pair of its points. A necessary and sufficient condition that M have a monotone, upper semi-continuous decomposition such that the quotient space is an arc is that there exist a saturated collection $\{G\}$ of subsets of M such that each $G \in \{G\}$ is a non-region-containing continuum.*

THEOREM 2′. *A compact, metric continuum M that is separated by no subcontinuum has a monotone, upper semi-continuous decomposition whose quotient space is a simple closed curve if and only if there exists a bi-saturated collection $\{G\}$ of subsets of M such that each $G \in \{G\}$ is a non-region-containing continuum.*

Proof of necessity. Assuming that there exists the decomposition of the hypothesis, let M' be the quotient space, f the associated map of the decomposition, and $\{G\}$ the collection of point inverses $f^{-1}(x)$, $x \in M'$, that have void interiors. Since M has a countable basis and $\{f^{-1}(x)|\ x \in M'\}$ is a decomposition of M, at most a countable number of these elements have non-void interiors. Thus $\{G\}$ is clearly uncountable and is a collection of non-region-containing continua. It remains to show that $\{G\}$ is bi-saturated. Let $G \in \{G\}$ and $p \in M-G$. There exist elements $x, y \in M'$ such that $f(G) = x$ and $f(p) = y$. Since M' is a simple closed curve there exist $u, v \in M'$ such that $M'-\{u, v\} = A_x \cup A_y$, a separation of M', with $x \in A_x$, $y \in A_y$. Then $M-\big(f^{-1}(u) \cup f^{-1}(v)\big) = f^{-1}(A_x) \cup f^{-1}(A_y)$, a separation of M. Clearly u and v can be chosen so that $f^{-1}(u)$, $f^{-1}(v)$ have void interiors. Thus $f^{-1}(u), f^{-1}(v) \in \{G\}$. But $G = f^{-1}(x) \subset f^{-1}(A_x)$ and $p \in f^{-1}(y) \subset f^{-1}(A_y)$, so $f^{-1}(u) \cup f^{-1}(v)$ separates G from p in M. Hence $\{G\}$ is a bi-saturated collection.

5. Structure of the elements in the decompositions of type A Hausdorff continua. The next theorem tells something about the structure of the elements that have void interiors in the decompositions.

THEOREM 3. *Let M be a compact, Hausdorff continuum that is either irreducible between two points or is separated by no subcontinuum. Furthermore let M be type A and let $\{D\}$ be a monotone, upper semi-continuous decomposition whose quotient space is a generalized arc or generalized simple closed curve, respectively. If $D \in \{D\}$ and $D^0 = \emptyset$, then $D = T^2(x)$ for every $x \in D$.*

Proof. The argument will be presented only for the case where M is separated by no subcontinuum. Let $D \in \{D\}$ where $D^0 = \emptyset$. First we will show that if $x \in D$ then $D \subset T^2(x)$. Suppose $y \notin T^2(x)$. There is a continuum H such that $y \in H^0 \subset H \subset M - T(x)$. For each $z \in H$ there is a continuum H_z such that $z \in H_z^0 \subset H_z \subset M - \{x\}$. The set $\bigcup_{z \in H} H_z^0$ contains H and, by compactness, the union of a finite number of the H_z's contains H. This union is a continuum K with $H \subset K^0$. We have the inclusions: $y \in H^0 \subset H \subset K^0 \subset K \subset M - \{x\}$. Because K does not separate M, $M - K$ is an open, connected set whose closure is a continuum L with the property that $L \cap H = \emptyset$. Also $M - (H \cup L) = A \cup B$, a separation of M, since otherwise $\overline{M - (H \cup L)}$ is a continuum separating M. Moreover $\bar{A}$ and $\bar{B}$ are irreducible continua from H to L. Now $x \in L$ so $D \cap L \neq \emptyset$. But $D \cap H = \emptyset$ for otherwise an irreducible subcontinuum of D from H to L must be $\bar{A}$ or $\bar{B}$ each of which contains an open set of M. However D contains no such open set, so $D \cap H = \emptyset$ and $y \notin D$ which shows that $D \subset T^2(x)$.

Next assume $y \notin D$ and let f be the natural map from M onto the quotient space M'. There are points $x', y' \in M'$ such that $f(D) = x'$ and $f(y) = y'$. Since M' is a generalized simple closed curve, $M' - \{x', y'\} = P \cup Q$, a separation of M', where $\{x'\} \cup P \cup \{y'\}$ and $\{x'\} \cup Q \cup \{y'\}$ are generalized arcs. In the order of both arcs from x' to y' there are points a, a', b, b' such that $x' < a < a' < y'$ in the first arc and $x' < b < b' < y'$ in the second We have $M' - \{a', b'\} = A' \cup B'$ and $M' - \{a, b\} = A \cup B$, both separations of M' with $x' \in A'$, $y' \in B'$ and $x' \in A$, $y' \in B$, respectively. The following inclusions are valid:

$$y' \in B' \subset \{a'\} \cup B' \cup \{b'\} \subset B \subset \{a\} \cup B \cup \{b\} \subset M' - \{x'\} .$$

Consequently

$$f^{-1}(y') \subset f^{-1}(B') \subset \overline{f^{-1}(B')} \subset f^{-1}(B) \subset \overline{f^{-1}(B)} \subset M - f^{-1}(x') .$$

But $y \in f^{-1}(y')$, $D = f^{-1}(x')$ and $\overline{f^{-1}(B')}$ and $\overline{f^{-1}(B)}$ are continua. Therefore for $x \in D$, $y \notin T^2(x)$, so $T^2(x) \subset D$, and hence $T^2(x) = D$.

In general the structure of the elements with non-void interior is much more complicated. However, if the decomposition is minimal (our construction in Theorems 1 and 2 does not necessarily produce the minimal one) in the sense of refinement and having the same properties, then some information about the elements with non-void interior can be obtained. The next theorem concerns this. It should be remarked that if a continuum is type A thereby having a monotone, upper semi-continuous decomposition whose quotient space is a generalized arc or generalized simple closed curve it does have a unique minimal such decomposition [3, p. 37].

THEOREM 4. *Let M be a compact, Hausdorff continuum that is either irreducible between two points or is separated by no subcontinuum. Furthermore let M be type A and $\{H\}$ be the unique minimal monotone upper semi-continuous decomposition whose quotient space is a generalized arc or generalized simple closed curve, respectively. If $H \in \{H\}$ and $H^0 \neq \emptyset$, then H^0 is a subset of the union of a collection of region-containing indecomposable subcontinua or a subset of the closure of such a union.*

Proof. Let $H \in \{H\}$ and suppose $H^0 \neq \emptyset$. Then every subcontinuum K of H containing an open set in M contains a region-containing indecomposable subcontinuum. For if K is a subcontinuum of H with a non-void interior in M and K contains no region-containing indecomposable subcontinuum of M, then because we can assume without loss of generality that K is irreducible between a pair of its points, K has a monotone, upper semi-continuous decomposition each element of which has void interior and whose quotient space is a generalized arc [4, Th. 2.7]. But this decomposition of K will result in a refinement of $\{H\}$ since K is a subset of a single element of $\{H\}$. This is impossible because $\{H\}$ is minimal. Therefore K contains a region-containing indecomposable subcontinuum. Using this together with the fact that M is irreducible between two points or is separated by no subcontinuum, the conclusion of the theorem follows readily.

For a method that generates the elements of the minimal decomposition by an iterated chaining technique the reader is referred to [3]

6. Final remarks. Two open questions relating to this paper are interesting and worth mentioning. First, what are necessary and sufficient conditions for a Hausdorff continuum to be type A for either the irreducible continuum or the one that is separated by no subcontinuum? Theorems 1′ and 2′ answer this when the continua are metric but the Hausdorff case is not known.

The second question pertains only to metric continua. After Thomas [7], define an irreducible continuum or one that is separated by no subcontinuum to be *type A'* if it has a monotone, upper semi-continuous decomposition whose quotient space is a simple arc or simple closed curve, respectively, *and* if the elements of the decomposition have void interior. For a continuum M that is irreducible between two points, Thomas obtained a necessary and sufficient condition that M be type A'. His condition is simply that M contain no region-containing indecomposable subcontinuum. Theorems 1′ and 2′ characterize type A continua for both the simple arc and simple closed curve where, of course, the elements of the decomposition do not necessarily have void interior. But the type A' characterization for simple closed curves is not yet satisfactorily solved. Specifically: If M is a compact, metric continuum that is separated by no subcontinuum, what is a necessary and sufficient condition (or conditions) in order that M have a monotone, upper semi-continuous decomposition, each element of which has void interior and such that the quotient space is a simple closed curve?

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The Whitehead Theorem in the theory of shapes

by

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Abstract. The purpose of this paper is to establish in the theory of shapes a theorem, which is an analogue of the Whitehead Theorem. We start with proving some statements concerning category theory (Section 1); they could not be found by the author in the literature. These statements enable us to prove the exactness property for homotopy systems (§ 1 of Section 2). Next, we establish some propositions on inverse systems of polyhedra (§ 2 of Section 2); they are needed in a proof of Theorem 3.5, which is referred to as the Whitehead Theorem for inverse sequences of polyhedra (§ 3 of Section 2). At last, applying the Freudental Theorem, the Holsztyński theorem on the fundamental dimension and the results of [6] and [10]-[13], we obtain the main theorem (Th.4.3).

Contents

Introduction. As proved by J. H. C. Whitehead in [16] (see also [15]), if two spaces X and Y are homotopically dominated by some connected CW-complexes of dimension $\leqslant n_0$ (of infinite dimension), then for a map $f\colon X\to Y$ to be a homotopy equivalence it is sufficient that f induces isomorphisms of homotopy groups, $f_n\colon \pi_n(X)\to\pi_n(Y)$, for $n = 1, \ldots, n_0$ (for $n = 1, 2, \ldots$).

Thus, for spaces with nice local properties (e.g. ANR's) the homotopy groups are the most important homotopy invariants. However, for arbitrary compact metric spaces, the homotopy groups lose their validity. For this reason K. Borsuk introduced the notion of fundamental groups. As proved in [1], p. 253, the fundamental groups are shape invariants; for ANR's they are isomorphic to the homotopy groups.