denote a point of $f^{-1}(y)$. For each positive integer n let $z_n = (z_{1n}, z_{2n}, z_{3n}, \ldots)$ denote the point of Y' such that $z_{kn} = y_k$ for $k = 1, 2, 3, \ldots, n$ and $z_{k+1,n} = (z_{kn}, z_{kn})$ for $k = n, n+1, n+2, \ldots$; then the sequence $z_1, z_2, z_3, \ldots$ converges to y in Y. For each positive integer n, let $w_n = (w_{1n}, w_{2n}, w_{3n}, \ldots)$ denote a point of $f^{-1}(z_n)$ such that $w_{kn} = x_n$ for $k = 1, 2, 3, \ldots, n$ and w_{kn} belongs to $f_k^{-1}(z_{kn})$ for $k = n+1, n+2, n+3, \ldots$; then the sequence $w_1, w_2, w_3, \ldots$ converges to x in $\prod_{n=1}^{\infty} X_n$. Thus, $f^{-1}(Y')/G'$ is dense in $f^{-1}(Y)/G$. It is easily shown that if every open subset of a dense subspace of a topological space contains a copy of X, then every open subset of the space itself contains a copy of X. Hence, every open subset of $f^{-1}(Y)/G$ contains a copy of X.

If $X = S$, the space described in [6], then $f^{-1}(Y)/G$ is a closed, continuous image of a complete metric space such that the set of all points at which $f^{-1}(Y)/G$ is not first countable is dense in $f^{-1}(Y)/G$. It follows from Corollary 2 that $f^{-1}(Y)/G$ is not a countable union of closed, metrizable subspaces. Therefore, closed, continuous images of complete metric spaces need not be countable unions of closed, metrizable subspaces.

References

[1] B. Fitzpatrick, Jr., *Some topologically complete spaces*, General Topology and its Applications 1 (1971), pp. 101–103.

[2] N. Lašnev, *Continuous decompositions and closed mappings of metric spaces*, Sov. Math. Dokl. 6 (1965), pp. 1504–1506.

[3] K. Morita and S. Hanai, *Closed mappings and metric spaces*, Proc. Japan Acad. 32 (1956), pp. 10–14.

[4] A. H. Stone, *Metrizability of decomposition spaces*, Proc. Amer. Math. Soc. 7 (1956), pp. 690–700.

[5] I. A. Vaĭnsteĭn, *On closed mappings of metric spaces*, Dokl. Akad. Nauk. USSR 57 (1947), pp. 319–321.

[6] K. R. Van Doren, *Concerning non-metrizable images of metrizable spaces under closed, continuous mappings*, Proceedings of the University of Houston Point Set Topology Conference (1971), pp. 32–36.

[7] — *Inverse limits and closed mappings* (to appear).

AUBURN UNIVERSITY
SOUTHWEST TEXAS STATE UNIVERSITY

Reçu par la Rédaction le 21. 3. 1972

X^m is homeomorphic to X^n iff $m \sim n$ where $\sim$ is a congruence on natural numbers

by

Věra Trnková (Prague)

Abstract. If a congruence $\sim$ on the additive semigroup of all natural numbers is given then a locally compact separable metric space X is constructed such that X^m is homeomorphic to X^n iff $m \sim n$.

Let X be a topological space. Define an equivalence $\sim$ on the set N of all natural numbers such that $m \sim n$ iff X^m is homeomorphic to X^n. Clearly, $\sim$ is a congruence on the additive semigroup $(N, +)$. In the paper, the following theorem is proved:

THEOREM. *For every congruence $\sim$ on the additive semigroup of all natural numbers there exists a locally compact separable metric space X such that X^m is homeomorphic to X^n iff $m \sim n$.*

The analogical results for Abelian groups and modules are shown in [3] and [1] respectively, but the proofs are quite different. Concerning the terminology, see [4].

1. Productively independent spaces.

CONVENTION. Denote by N the set of all natural numbers. If X is a topological space then, as usual, $X^1 = X$, $X^{n+1} = X \times X^n$, X^0 is a one-point space.

DEFINITION. A set $\boldsymbol{X}$ of topological spaces is said to be *productively independent* if, whenever $\{k_X;\ X \in \boldsymbol{X}\}$, $\{h_X;\ X \in \boldsymbol{X}\}$ are two collections of non-negative integers and $\prod_{X \in \boldsymbol{X}} X^{k_X}$ is homeomorphic to $\prod_{X \in \boldsymbol{X}} X^{h_X}$, then $k_X = h_X$ for all $X \in \boldsymbol{X}$. We recall that a set $\boldsymbol{X}$ of topological spaces is said to be *rigid* if, whenever $f\colon X \to Y$ is a continuous mapping, $X, Y \in \boldsymbol{X}$, then either f is a constant or $X = Y$ and f is the identity. Every element of a rigid set is called a *rigid space*.

LEMMA 1. *Let $\{X, Y\}$ be a rigid set, $m, n \in N$, and $f\colon X^n \to Y^m$ a continuous mapping. Then f is a constant.*

Proof. Let $g\colon X^n \to Y$ be a continuous mapping. Choose $x \in X$. Denote by x^i the point of X^i whose coordinates are all x. Put $a = g(x^n)$. Since every continuous mapping of X into Y is a constant one can prove by induction that $g(z) = a$ whenever $z \in X^i \times \{x^{n-i}\}$, $i = 1, \ldots, n$. q.e.d.

LEMMA 2. *Let X be a rigid space, $m, n \in N$. Let $h\colon X^m \to X^n$ be a homeomorphism into X^n. Then $m \leqslant n$.*

Proof. Let $\pi_i\colon X^m \to X$, $p_j\colon X^n \to X$, $i = 1, \ldots, m$, $j = 1, \ldots, n$ be the projections. Put $I = \{1, \ldots, m\}$, $J = \{1, \ldots, n\}$ and choose $a \in X$. If $S \subset I$, define a mapping $\Delta_S\colon X \to X^m$ such that $\pi_i \circ \Delta_S(x) = x$ for $i \in S$, $\pi_i \circ \Delta_S(x) = a$ for $i \notin S$. For every $j \in J$ the mapping $X \xrightarrow{\Delta_S} X^m \xrightarrow{h} X^n \xrightarrow{p_j} X$ is either the identity or a constant. Denote by S' the set of all $j \in J$ such that it is the identity. Obviously $S = \emptyset \Leftrightarrow S' = \emptyset$. Now we prove that $\varphi\colon \exp I \to \exp J$ defined by $\varphi(S) = S'$ is a one-to-one mapping. Take $S_1, S_2 \subset I$, $S_1 \neq S_2$; we may suppose $S_1 \neq \emptyset \neq S_2$. If $j \in J - (S_1' \cup S_2')$, then $p_j \circ h \circ \Delta_{S_1} = p_j \circ h \circ \Delta_{S_2}$ because both the mappings are constant and $\Delta_{S_1}(a) = \Delta_{S_2}(a)$. The equality $S_1' = S_2'$ implies $p_j \circ h \circ \Delta_{S_1} = p_j \circ h \circ \Delta_{S_2}$, for all $j \in J$. Then necessarily $h \circ \Delta_{S_1} = h \circ \Delta_{S_2}$, hence $\Delta_{S_1} = \Delta_{S_2}$ and consequently $S_1 = S_2$, which is a contradiction. q.e.d.

PROPOSITION 1. *Every rigid set of spaces is productively independent.*

Proof. Let $\boldsymbol{X}$ be a rigid set, and let $\{k_X;\ X \in \boldsymbol{X}\}$, $\{h_X;\ X \in \boldsymbol{X}\}$ be collections of non-negative integers. Put $K = \prod_{X \in \boldsymbol{X}} X^{k_X}$, $H = \prod_{X \in \boldsymbol{X}} X^{h_X}$. Let $h\colon K \to H$ be a homeomorphism. For every $X \in \boldsymbol{X}$ denote by $\pi_X\colon K \to X^{k_X}$, $p_X\colon H \to X^{h_X}$ the projections. Let $Y \in \boldsymbol{X}$; we prove $k_Y \leqslant h_Y$. Denote by $f\colon Y^{k_Y} \to K$ such a mapping that $\pi_Y \circ f$ is the identity and $\pi_X \circ f$ is a constant for all $X \in \boldsymbol{X}$, $X \neq Y$. Lemma 1 implies that $p_X \circ h \circ f$ is a constant for all $X \neq Y$ and consequently $p_Y \circ h \circ f$ is a homeomorphism. Now use Lemma 2. q.e.d.

PROPOSITION 2. *There exists a countable productively independent set of metric continua.*

Proof. In [2] a metric continuum M_1 with the following property is constructed:

if H is a subcontinuum of M_1 and $f\colon H \to M_1$ is a continuous mapping, then either f is a constant or $f(x) = x$ for all $x \in H$.

Let $\boldsymbol{X}$ be a countable set of pairwise disjoint subcontinua of M_1. Clearly, $\boldsymbol{X}$ is a rigid set of metric continua. q.e.d.

PROPOSITION 3. *For every cardinal number m there exists a productively independent set $\boldsymbol{X}$ of metric semicontinua such that* card $\boldsymbol{X} = m$.

Proof. For every cardinal number m there exists a rigid set of metric semicontinua, as follows from [5]. q.e.d.

2. Topological representation of semigroups. All semigroups are supposed to be commutative. Their composition will usually be denoted by $\circ$.

DEFINITION. Let S be a semigroup, and $\boldsymbol{T}$ a class of topological spaces. We say that S *has a representation in* $\boldsymbol{T}$ or S *is representable in* $\boldsymbol{T}$ if there exists a mapping $r\colon S \to \boldsymbol{T}$ such that

(a) $r(s_1)$ is homeomorphic to $r(s_2) \Leftrightarrow s_1 = s_2$;

(b) $r(s_1 \circ s_2)$ is homeomorphic to $r(s_1) \times r(s_2)$.

Then r is called a *representation of S in $\boldsymbol{T}$*.

Remark. Clearly, if S is representable in $\boldsymbol{T}$, so is each of its subsemigroups.

CONVENTION. We denote by $\boldsymbol{MC}$ the class of all metric continua.

PROPOSITION 4. *Every free unitary semigroup (with the unity) has a representation in the class of all metric semicontinua. Moreover, if it is countable then it has a representation in $\boldsymbol{MC}$.*

Proof. The proposition follows immediately from Propositions 2, 3. q.e.d.

PROPOSITION 5. *Let a semigroup S be a product of a collection of free unitary semigroups with one generator. Then S has a representation in the class of all uniformizable semicontinua. Moreover, if the collection is countable then S has a representation in $\boldsymbol{MC}$.*

Proof: Let $\{S_l;\ l \in I\}$ be a collection of free semigroups with the unity 1_l and the generator a_l. If $s_l \in S_l$, put $s_l^0 = 1_l$. Every $s_l \in S_l$ can be uniquely expressed as $s_l = a_l^{k_l}$. Let $\boldsymbol{X}$ be a productively independent set of metric semicontinua (or metric continua) such that card $\boldsymbol{X} =$ card I, let $\psi\colon \{a_l;\ l \in I\} \to \boldsymbol{X}$ be a bijection. If $S = \prod_{l \in I} S_l$, $s = \{a_l^{k_l};\ l \in I\} \in S$, put $r(s) = \prod_{l \in I} \psi(a_l)^{k_l}$. q.e.d.

CONVENTION. a) Let S be a semigroup. If $A, B \subset S$, put $A \circ B = \{a \circ b;\ a \in A,\ b \in B\}$. So, all subsets of S form a (commutative) semigroup again. Denote it by $\exp S$.

b) Let $\boldsymbol{T}$ be a class of topological spaces, and m a cardinal number. Denote by $\bigvee_m \boldsymbol{T}$ the class of all topological spaces that are sums of collections $\{X_l;\ l \in I\}$ such that card $I \leqslant m$ and $X_l \in \boldsymbol{T}$ for all $l \in I$. The sum of the collection $\{X_l;\ l \in I\}$ will be denoted by $\bigvee_{l \in I} X_l$.

PROPOSITION 6. *Let $\boldsymbol{C}$ be a class of connected topological spaces, and S a semigroup representable in $\boldsymbol{C}$. Then $\exp S$ is representable in $\bigvee_m \boldsymbol{C}$, where $m = \max(\aleph_0, \operatorname{card} S)$.*

Proof. Let $r\colon S \to \boldsymbol{C}$ be a representation of S in $\boldsymbol{C}$. If $s \in S$, put $X_s = \bigvee_{l \in I} P_l$, where card $I = \max(\aleph_0, \operatorname{card} S)$ and every P_l is homeomorphic to $r(s)$. If $A \subset S$, put $\varrho(A) = \bigvee_{a \in A} X_a$. One can easily see that $\varrho\colon \exp S \to \bigvee_{\max(\aleph_0, \operatorname{card} S)} T$ is a representation. q.e.d.

3. The semigroups S_m and $\exp S_m$. 1. Let m be a natural number. Put $M = \{0, 1, \ldots, m\}$. Denote by $M^{\aleph_0}$ the set of all sequences of elements of M. Denote by $p_n\colon M^{\aleph_0} \to M^n$ the projection to the first n coordinates.

LEMMA 3. *There exists a countable set $D \subset M^{\aleph_0}$ such that for every $n \in N$, $a \in M^n$ the set $\{d \in D;\ p_n(d) = a\}$ is infinite.*

Proof. Let D be a countable dense subset of the set of all irrational numbers of the interval $(0, 1)$. Consider the $(m+1)$-adic presentations of the numbers of D. q.e.d.

2. Put $\boldsymbol{M} = \bigcup_{n=1}^{\infty} M^n$. If $a \in M^n$, write $D_a = \{d \in D;\ p_n(d) = a\}$. Clearly, all D_a are infinite and $D_a = D_{\langle a,0\rangle} \cup \ldots \cup D_{\langle a,m\rangle}$. Let B be an element, different from all D_a, $a \in \boldsymbol{M}$. Let T_m be a free (commutative) semigroup such that $D' = \{D_a;\ a \in \boldsymbol{M}\} \cup \{B\}$ is the set of all its generators. Let S_m be its factor-semigroup given by the equalities:

$$D_{\langle a,0\rangle} \circ D_{\langle a,1\rangle} \circ \ldots \circ D_{\langle a,m\rangle} = D_a\,. \tag{1}$$

Then every element of S_m can be written in the form

$$D_{a_1}^{k_1} \circ \ldots \circ D_{a_l}^{k_l} \circ B^h\,, \tag{2}$$

where $k_1, \ldots, k_l$ are natural numbers, h, l are non-negative integers, $h + l > 0$ (if $h = 0$, we mean the word $D_{a_1}^{k_1} \circ \ldots \circ D_{a_l}^{k_l}$, if $l = 0$ — the word B^h) and the sets $D_{a_1}, \ldots, D_{a_l}$ are pairwise disjoint.

LEMMA 4. *The semigroup S_m is isomorphic to a subsemigroup of $\prod_{d \in D'} F_d$, where every F_d is a free unitary semigroup with the generator d.*

Proof. If $x = D_{a_1}^{k_1} \circ \ldots \circ D_{a_l}^{k_l} \circ B^h$, $d \in D$, put $k_{x,d} = \sum_{d \in D_{a_i}} k_i$ and $l(x) = \{d^{k_{x,d}};\ d \in D\} \cup \{B^h\}$. Then $l\colon S_m \to \prod_{d \in D'} F_d$ is an isomorphism. q.e.d.

PROPOSITION 7. *The semigroup S_m has a representation in the class $\boldsymbol{MC}$.*

Proof. The proposition follows immediately from Lemma 4 and Proposition 5.

3. Let p be a natural number. Denote by $A_{m,p} \subset S_m$ the set of all elements of S_m that can be written in the form (2) such that

$$l > 0\,, \quad \sum_{i=1}^{l} k_i \equiv 1 \bmod m\,, \tag{a}$$

$$h \in \{0, 1\} \cup \{p+1, p+2, \ldots\}\,. \tag{b}$$

LEMMA 5. *Let t be a natural number. Then $A_{m,p}^t$ is the set of all elements of S_m that can be written in the form (2) such that*

$$l > 0\,, \quad \sum_{i=1}^{l} k_i \equiv t \bmod m\,, \tag{a$'$}$$

$$h \in \{0, 1, \ldots, t\} \cup \{p+1, p+2, \ldots\}\,. \tag{b$'$}$$

Proof. The lemma is true for $t = 1$. Let $t = r+1$, $r > 0$ and let the proposition hold for r. Denote by Y the set of all elements of S_m in the form (2) with (a$'$) and (b$'$) satisfied. If

$$x = D_{a_1}^{k_1} \circ \ldots \circ D_{a_l}^{k_l} \circ B^h\,, \qquad x' = D_{a'_1}^{k'_1} \circ \ldots \circ D_{a'_{l'}}^{k'_{l'}} B^{h'}\,, \tag{3}$$

$x \in A_{m,p}^r$, $x' \in A_{m,p}$, then

$$\sum_{i=1}^{l} k_i + \sum_{i=1}^{l'} k'_i = (r+1) \bmod m\,, \qquad h + h' \in \{0, 1, \ldots, r+1\} \cup \{p+1, p+2, \ldots\}.$$

However, the sets $D_{a_1}, \ldots, D_{a_l}, D_{a'_1}, \ldots, D_{a'_{l'}}$ need not be disjoint. Use (1) and find the form (2) for $x \circ x'$, say

$$x \circ x' = D_{a''_1}^{k''_1} \circ \ldots \circ D_{a''_{l''}}^{k''_{l''}} \circ B^{h+h'}\,.$$

One can prove that $l'' > 0$ and $\sum_{i=1}^{l''} k''_i \equiv \left(\sum_{i=1}^{l} k_i + \sum_{i=1}^{l'} k'_i\right) \bmod m$; hence $A_{m,p}^t \subset Y$. Conversely, let $y \in Y$, i.e. let y have the form (2) with (a$'$) and (b$'$) satisfied.

1) Let there exist $i_0 \in \{1, \ldots, l\}$ such that $k_{i_0} > 1$: we may suppose $k_1 > 1$. Put $h' = 1$ whenever $h = t$, and $h' = 0$ otherwise. If we put $x' = D_{a_1} \cdot B^{h'}$, $x = D_{a_1}^{k_1 - 1} \circ D_{a_2}^{k_2} \circ \ldots \circ D_{a_l}^{k_l} \circ B^{h-h'}$, then $x' \in A_{m,p}$, $x \in A_{m,p}^r$ and $x \circ x' = y$.

2) Let $k_1 = \ldots = k_l = 1$. Put $h' = 1$ whenever $h = t$, and $h' = 0$ otherwise. If we put $x' = D_{\langle a_1,0\rangle} \circ B^{h'}$, $x = D_{\langle a_1,1\rangle} \circ \ldots \circ D_{\langle a_1,m\rangle} \circ D_{a_2} \circ \ldots \circ D_{a_l} \circ B^{h-h'}$, then $x' \in A_{m,p}$, $x \in A_{m,p}^r$ and $x \circ x' = y$. Hence $Y \subset A_{m,p}^t$. q. e. d.

LEMMA 6. *Let t, ε be natural numbers. Then $A_{m,p}^t = A_{m,p}^{t+\varepsilon}$ if and only if $t \geqslant p$ and $\varepsilon \equiv 0 \bmod m$.*

Proof. The lemma follows immediately from Lemma 5. q.e.d.

PROPOSITION 8. *Every semigroup with one generator is isomorphic to a subsemigroup of some $\exp S_m$.*

Proof. Let S be a semigroup with one generator a. If S is free, then $i\colon S \to \exp S_m$, defined by $i(a) = \{B\}$ is an isomorphism. If S is given by the equality $a^p = a^{p+m}$, put $i(a) = A_{m,p}$ and use Lemma 6. q.e.d.

PROPOSITION 9. *Every semigroup with one generator is representable in $\bigvee_{\aleph_0} \boldsymbol{MC}$.*

Proof. The proposition follows immediately from Propositions 7, 6 and 8. q.e.d.

Remark. Since every space from $\bigvee_{\aleph_0} MC$ is a locally compact separable metric space, the theorem follows immediately from Proposition 9.

References

[1] P. M. Cohn, *Some remarks on the invariance bases property*, Topology 5 (1966), pp. 215–228.

[2] H. Cook, *Continua which admit only the identity mapping onto non-degenerate subcontinua*, Fund. Math. 60 (1967), pp. 241–249.

[3] A. L. S. Corner, *On a conjecture of Pierce concerning direct decompositions of Abelian groups*, Proc. of the Coll. on Abelian Groups, Tihany 1963, pp. 43–48.

[4] E Čech, *Topological Spaces*, Prague 1966.

[5] V. Trnková, *Non-constant continuous mappings of metric or compact Hausdorff spaces*, Comment. Math. Univ. Carolinae 13 (1972), pp. 289–295.

Reçu par la Rédaction le 20. 6. 1972

Limit mappings and projections of inverse systems

by

E. Puzio (Warszawa)

Abstract. Let $S = \{X_\sigma, \Pi_\varrho^\sigma, \Sigma\}$ and $S' = \{Y_{\sigma'}, \Pi_{\varrho'}^{\sigma'}, \Sigma'\}$ be inverse systems and $\{\varphi, f_{\sigma'}\}$ be a mapping of S into S'. For some classes K of mappings we discuss the problem when $f_{\sigma'} \in K$ implies $\varprojlim \{\varphi, f_{\sigma'}\} \in K$ and when $\Pi_\varrho^\sigma \in K$ implies $\Pi_\sigma \in K$.

In this paper we are concerned with limits of inverse systems, their projections and limit mappings induced by mappings of inverse systems. More precisely, we show how the projections depend on bonding mappings and how the limit mapping depends on the mapping of systems inducing it.

To begin with, we recall some definitions and simple facts about inverse systems and give two auxiliary examples. Our terminology and notations are consistent with those used in [3]; except that by a mapping of an inverse system $S = \{X_\sigma, \Pi_\varrho^\sigma, \Sigma\}$ into $S' = (Y_{\sigma'}, \Pi_{\varrho'}^{\sigma'}, \Sigma'\}$ we understand a system $\{\varphi, f_{\sigma'}\}$ satisfying besides the usual commutativity conditions also the condition that $\varphi(\Sigma')$ is cofinal in Σ.

The diagram

(1)
$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ T & \xrightarrow{k} & Z \end{array}$$

is said to be *exact* (see [8], p. 19) if it is commutative and the following implication is true:

$$\text{if} \quad h(y) = k(t)\,, \quad \text{then} \quad g^{-1}(t) \cap f^{-1}(y) \neq \emptyset\,.$$

The diagram (1) is exact (see p. 19 of [8]) if and only if

(2) $$fg^{-1}(B) = h^{-1}k(B) \quad \text{for} \quad B \subset T$$

or, equivalently,

(2′) $$gf^{-1}(A) = k^{-1}h(A) \quad \text{for} \quad A \subset Y.$$

Obviously, the diagram (1) is commutative if and only if

(3) $$fg^{-1}(B) \subset h^{-1}k(B) \quad \text{for} \quad B \subset T$$