Bibliographie

[1] N. Bourbaki, *Espaces vectoriels topologiques*, Chap. I et II, Paris.

[2] H. G. Eggleston, *Convexity*, Cambridge Tracts in Math. and Math. Phys. 47.

[3] C. Fenske, E. Schock, *Nuklearität und lokale Konvexität von Folgenraümen*, Math. Nach. 45 (1970), p. 327–335.

[4] H. Hogbe-Nlend, *Théorie des Bornologies et Applications*, Lectures Notes 213 (1971).

[5] — *Ultranucléarité et bornologie à décroissance très rapide*, Séminaire Schwartz, Ecole Polytechnique (1970–71).

[6] J. P. Ligaud, *Sur les rapports entre topologies et bornologies pseudo-convexes*, C. R. Acad. Sc. Sér. A T. 271, (1970), p. 1058–1060.

[7] — *Solution d'un problème de S. Rolewicz sur les espaces nucléaires*, C. R. Acad. Sc. Sér. A T. 273, (1971), p. 113–114.

[8] — *Sur les limites inductives de suites d'espaces localement pseudo-convexes*, Colloque d'Analyse Fonctionnelle Bordeaux (1971). Bull. Soc. Math. Fr. (à paraître).

[9] S. Mazur, W. Orlicz, *Sur les espaces métriques linéaires (I)*; Studia Math. 10 (1948), p. 184–208.

[10] B. S. Mityagin, *Approximate dimension and bases in nuclear spaces*, Russian Math. Surveys 16 (1961), p. 59–127.

[11] A. P. Robertson, W. Robertson, *Topological vector spaces*, Cambridge 1966.

[12] S. Rolewicz, Colloquium on Nuclear Spaces and Ideals in Operator Algebras, III. *Unsolved problems*, Studia Math. 38 (1969), p. 477–478.

[13] — *Open problems on linear metric spaces*, Colloque d'Analyse Fonctionnelle Bordeaux (1971). Bull. Soc. Math. Fr. (à paraître).

[14] L. Waelbroeck, *Fonctions différentiables et petite bornologie*, C. R. Acad. Sci. Sér. A T. 267 (1968), p. 220–222.

UNIVERSITÉ BORDEAUX I
DÉPARTEMENT DE MATHÉMATIQUES

Received November 5, 1971 (444)

STUDIA MATHEMATICA T. XLV. (1973)

Factorization of compact operators and applications to the approximation problem*

by

T. FIGIEL (Warszawa)

Abstract. In the present paper we give necessary and sufficient conditions for a Banach space Z to have one of the following factorization properties: (i) every compact operator, which can be uniformly approximated by finite dimensional operators, admits a factorization through Z, (ii) any compact operator admits a factorization through a subspace of Z.

As a consequence we obtain, for example, that every compact operator admits a compact factorization through a reflexive space. Hence the approximation problem can be reduced to the case of reflexive spaces. (It is a negative answer to one of Grothendieck's conjectures.)

Some related problems concerning L_p spaces and the traces of nuclear operators are also considered.

1. Introduction. Factorization problems for compact operators have recently been treated by Johnson [4]. He discussed, however, only the case of those operators $T\colon X\to Y$, which admit an approximation by finite dimensional operators in the norm topology of $B(X, Y)$. We recall that if either X^* or Y has the approximation property (abbreviated a.p.), then every compact operator in $B(X, Y)$ admits such an approximation. Since the approximation problem, i.e. the question "Does every Banach space have the a.p.?", is still open, it is not known whether Johnson's restriction is essential (1)

This restriction can, however, be avoided if, instead of factorization through a given space, one considers factorization through its subspaces. Moreover, this approach permits us to obtain some new information concerning the approximation problem. In particular, we obtain the result that the approximation problem and the question "Does every reflexive Banach space have the a.p.?" are equivalent. This shows that not both of the conjectures formulated in [2] (chap. II, p. 135) and [7] can be true.

* This paper is a part of the author's Ph. D. thesis prepared under the supervision of Professor A. Pełczyński at the Warsaw University.

(1) Added in proof. The approximation problem has recently been solved (in the negative) by P. Enflo. (His remarkable paper *A counterexample to the approximation problem* will appear in Acta Math.) Some later related results are mentioned at the end of the present paper.

This is a consequence of the theorem that an arbitrary compact operator factors compactly through a reflexive space. These facts are proved in Section 3, where the simplest version of our results is presented. In order to shorten the proofs we refer there to Johnson's theorem, although we need only a weaker fact, whose direct proof is somewhat simpler. (It may be found in Section 5.)

Sections 5 and 6 contain the more detailed exposition of the two factorization properties we consider. We characterize Banach spaces possessing these properties in terms of projections onto finite dimensional subspaces. Some consequences of this characterization are deduced.

In Section 7 we briefly study factorization problems for compact operators whose ranges are subspaces of L_p spaces (where p, $1 \leqslant p \leqslant \infty$, is fixed). This also leads to corollaries concerning the approximation problem for corresponding classes of spaces. For example if every subspace of l_p, $1 \leqslant p < \infty$, has the a.p., then so has every subspace of every $L_p(\mu)$. Hence, if $1 \leqslant p < q \leqslant 2$, and every subspace of l_p has the a.p., then so has every subspace of l_q (Corollary 7.6).

Proofs of our characterization theorems use a certain lemma on finite dimensional projections, which is proved in Section 4.

The main result of Section 8 may be treated as an analogue of Corollary 7.6 for $2 \leqslant q < p \leqslant \infty$. We restate it in a slightly stronger version which also follows from our argument.

Let $2 \leqslant p \leqslant \infty$, and suppose that every subspace of l_p (where l_∞ denotes the space c_0) has the a.p. (it is certainly satisfied if $p = 2$).

Let $u = (u_{ij})_{i,j=1}^{\infty}$ be an infinite matrix such that

$$\sum_{j=1}^{\infty} \sup_i |u_{ij}|^{\frac{p}{p+1}} < \infty, \quad \text{and} \quad u^2 = 0.$$

Then

$$\sum_{i=1}^{\infty} u_{ii} = 0.$$

The extreme cases $p = 2$ and $p = \infty$ were proved in [2]. Our proof combines a factorization of matrices with the idea used by Grothendieck in the case $p = \infty$. In particular for $p = 2$ we avoid using the theory of entire functions.

2. Preliminaries and notation. All spaces we shall consider are supposed to be Banach spaces over the same, either real or complex field. We shall deal only with norm topologies.

The word "subspace" (resp. "operator") will always mean "a closed linear manifold" (resp. "a continuous linear mapping"). All isomorphisms, isometries, projections etc. will be also linear and continuous.

Letters X, Y, Z will always stand for Banach spaces, and letters E, F, G for finite dimensional Banach spaces. The conjugate space of X will be denoted by X^*. The symbols $B(X, Y)$, $K(X, Y)$, $F(X, Y)$ will denote the spaces of all operators mapping X into Y which are, respectively, bounded, compact and finite dimensional. These spaces are equipped with the operator norm. The closure of $F(X, Y)$ in $B(X, Y)$ will be denoted by $\overline{F}(X, Y)$.

A pair (T_1, T_2) is called a *factorization* of an operator $T \in B(X, Y)$ through Z, provided $T_1 \in B(X, Z)$, $T_2 \in B(Z, Y)$ and $T_2 T_1 = T$. A factorization (T_1, T_2) is said to be a *K-factorization* (resp. *$\overline{F}$-factorization*) whenever T_1, T_2 are compact (resp. admit approximation by finite rank operators).

A family $(P_i)_{i\in I}$ of elements of $B(X, X)$ is said to be a *family of disjoint projections* (abbreviated f.d.p.) whenever

$$P_i P_j = \delta_{ij} P_j \quad \text{for } i, j \in I,$$

where δ_{ij} is the usual Kronecker's symbol.

For every pair X, Y of Banach spaces we define the distance coefficient $d(X, Y)$ putting

$$d(X, Y) = \inf(\|T\| \, \|T^{-1}\|),$$

where the infimum is taken over all invertible operators from X onto Y. Hence, $d(X, Y) = \infty$ whenever X and Y are not isomorphic.

Let $(X_i)_{i\in I}$ be a family of Banach spaces, and let $1 \leqslant p \leqslant \infty$. Let $\Sigma_p X_i$ denote the space of all functions $f\colon I \to \bigcup_{i\in I} X_i$ such that $f(i) \in X_i$, for $i \in I$, and the norm of f defined as follows

$$\|f\|_p = \begin{cases} \Big(\sum_{i\in I} \|f(i)\|_{X_i}^p\Big)^{\frac{1}{p}} & \text{if } p < \infty, \\ \sup_i \|f(i)\|_{X_i} & \text{if } p = \infty \end{cases}$$

is finite. If $p = \infty$, then we require additionally that for every $\varepsilon > 0$ the set

$$\{i \in I\colon \|f(i)\|_{X_i} > \varepsilon\}$$

is finite.

The space $\Sigma_p X_i$ equipped with the norm $\|\cdot\|_p$ will be called the *l_p-sum of the family* $(X_i)_{i\in I}$. (Hence by l_∞-sum we mean the c_0-sum.)

If n is a non-negative integer, $I = \{1, \ldots, n\}$, and X_i is a one-dimensional Banach space for $i \in I$, then $\Sigma_p X_i$ is an n-dimensional Banach space denoted more simply l_p^n.

We recall the definition of $\mathscr{L}_p$-spaces introduced in [8].

Let $1 \leqslant p \leqslant \infty$, and let $1 \leqslant \lambda < \infty$. A Banach space X is called an *$\mathscr{L}_{p,\lambda}$-space*, provided that for every finite dimensional subspace E of X

there is a subspace F of X such that $E \subseteq F$, $\dim F = n < \infty$ and $d(F, l_p^n) \leqslant \lambda$. X is said to be an $\mathscr{L}_p$-space whenever it is an $\mathscr{L}_{p,\lambda}$-space for some $\lambda \geqslant 1$.

Since we consider only Banach spaces, it will be convenient to use another definition of the approximation property, equivalent to the original one.

A Banach space Y has the a.p., if for every Banach space X we have

$$K(X, Y) = \overline{F}(X, Y).$$

All facts concerning the approximation property we use can be found in [2].

3. Primary version. In this section we shall base on results obtained by Johnson in [4]. Another, more general, treatment, independent of those results, will be given in the subsequent sections.

We recall the definition of the spaces C_p, $1 \leqslant p \leqslant \infty$, introduced by Johnson.

Let $(G_i)_{i=1}^{\infty}$ be a sequence of finite dimensional Banach spaces such that

(i) for every finite dimensional Banach space E and every $\varepsilon > 0$ there is an i such that $d(E, G_i) < 1+\varepsilon$,

(ii) for every $i = 1, 2, \ldots,$ the set

$$\{j\colon d(G_i, G_j) = 1\}$$

is infinite.

Let $1 \leqslant p \leqslant \infty$. We shall, following Johnson, denote by C_p the space $\Sigma_p G_i$ (recall that by Σ_∞ we mean the c_0-sum).

Johnson proved that for every p, $1 \leqslant p \leqslant \infty$, every operator $T \in \overline{F}(X, Y)$ admits an $\overline{F}$-factorization (A, B) through C_p.

This result implies that the spaces C_p have also another factorization property.

PROPOSITION 3.1. *Let $1 \leqslant p \leqslant \infty$, and let $T \in K(X, Y)$. Then there exists a K-factorization (A, B) of T through a subspace Z of C_p.*

Proof. Let $j\colon Y \to C(S)$ be a linear isometric embedding, S being suitable compact space. Since $C(S)$ has the a.p., we have

$$jT \in K\big(X, C(S)\big) = \overline{F}\big(X, C(S)\big).$$

By Johnson's result, there is an $\overline{F}$-factorization, say $(\tilde{A}, \tilde{B})$, of jT through C_p. Let Z be a subspace of C_p such that

$$\tilde{A}(X) \subseteq Z \subseteq \tilde{B}^{-1}\big(j(Y)\big).$$

Define operators $A \in B(X, Z)$, $B \in B(Z, Y)$ putting $Ax = \tilde{A}x$ for $x \in X$, and $Bz = j^{-1}(\tilde{B}z)$ for $z \in Z$. The factorization (A, B) has the required properties.

Remark 3.2. Observe that the proof of 3.1 remains valid, if C_p is replaced by any space possessing the property described in Johnson's theorem.

COROLLARY 3.3. *Every compact operator admits a compact factorization through a reflexive space.*

Proof. If we take $1 < p < \infty$, then C_p is reflexive, hence so is Z.

Remark 3.4. It is not known whether every weakly compact operator can be factorized through a reflexive space.

COROLLARY 3.5. *Suppose that for some p, $1 \leqslant p \leqslant \infty$, every subspace of C_p has the a.p. Then every Banach space has the a.p.*

Proof. Let $T\colon X \to Y$ be an arbitrary compact operator. Let (A, B) be a K-factorization of T through a subspace Z of C_p. If we assume that Z has the a.p., then $K(X, Z) = \overline{F}(X, Z)$, hence A is a limit of a sequence $(A_n)_{n=1}^{\infty} \subset F(X, Z)$. Clearly $BA_n \in F(X, Y)$ for $n = 1, 2, \ldots,$ and since

$$\|T - BA_n\| = \|BA - BA_n\| \leqslant \|B\|\,\|A - A_n\|,$$

we infer that $T \in \overline{F}(X, Y)$. This concludes the proof.

Remark 3.6. The proof of 3.5 remains valid if C_p is replaced by any space possessing the property described in Proposition 3.1.

COROLLARY 3.7. *If every reflexive Banach space has the a.p., then the approximation problem has a positive solution.*

Proof. Since the spaces C_p, where $1 < p < \infty$, are reflexive, it is an immediate consequence of Corollary 3.5.

Remark 3.8. Since the space C_∞ is isomorphic to a subspace of c_0, Corollary 3.5 implies Grothendieck's result that the approximation problem has a positive solution if every subspace of c_0 has the a.p.

Remark 3.9. In the formulation of Proposition 3.1 and Corollary 3.5 the space C_p can be replaced by $\Sigma_p l_\infty^n$.

4. Auxiliary lemma. This section is devoted to the proof of the following known lemma, which will be repeatedly needed in the sequel.

LEMMA 4.1. *Let Z be a Banach space, and let $S \in F(Z, Z)$, $\dim S(Z) = m < \infty$. Let E be an r-dimensional Banach space, and let $1 \leqslant p \leqslant \infty$. Denote by F the l_p-sum of $(mr+1)^2$ copies of E. Suppose that P is a projection in Z such that $\|P\| \leqslant K$, and $d(P(Z), F) \leqslant L$.*

Then there is a projection Q in Z such that

$$QS = SQ = 0, \quad \|Q\| \leqslant KL, \quad d\big(Q(Z), E\big) \leqslant L.$$

Proof. Put $k = mr+1$. We have

$$F = E_1 \times \ldots \times E_{k^2},$$

where each E_i is a copy of E. Let $I_i\colon E \to F$ be the natural embedding, and $P_i\colon F \to E$ the natural projection, both onto the ith summand, $1 \leqslant i \leqslant k^2$. Let $J\colon F \to Z$ be an isomorphism onto $P(Z)$ such that

$$\|J\| \leqslant 1, \qquad \|J^{-1}\| \leqslant L.$$

Fix j, $1 \leqslant j \leqslant k$, and let $G = S(Z)$.

Since $\dim B(G, E) = mr < k$, then the set

$$\{P_{(j-1)k+1}J^{-1}P|_G, \ldots, P_{jk}J^{-1}P|_G\} \subset B(G, E)$$

is linearly dependent. Hence there is a $\underline{b} = (b_1, \ldots, b_k) \in l_q^k = (l_p^k)^*$ with $\|\underline{b}\|_q = 1$ such that

$$(b_1 P_{(j-1)k+1} + \ldots + b_k P_{jk}) J^{-1}P|_G = 0.$$

Let $\underline{c} = (c_1, \ldots, c_k)$ be an element of l_p^k such that $\|\underline{c}\|_p = 1$ and

$$\langle \underline{b}, \underline{c} \rangle = \sum_{s=1}^{k} b_s c_s = 1.$$

Put

$$\varepsilon_i = \begin{cases} c_{i-(j-1)k} & \text{if } (j-1)k < i \leqslant jk, \\ 0 & \text{otherwise,} \end{cases}$$

and define an operator B_j in F by

$$B_j(e_1, \ldots, e_{k^2}) = (\varepsilon_1 x, \ldots, \varepsilon_{k^2} x),$$

where $x = \sum_{s=1}^{k} b_s e_{(j-1)k+s} \in E$. Obviously

$$(*) \qquad B_j J^{-1} P S = 0.$$

Let $\underline{e} = (e_1, \ldots, e_{k^2}) \in F$. Since

$$B_j(B_j(e_1, \ldots, e_{k^2})) = B_j(\varepsilon_1 x, \ldots, \varepsilon_{k^2} x) = (\varepsilon_1 y, \ldots, \varepsilon_{k^2} y),$$

where

$$y = \sum_{s=1}^{k} b_s \varepsilon_{(j-1)k+s} x = \sum_{s=1}^{k} b_s c_s x = x,$$

we infer that $B_j B_j = B_j$, i.e. B_j is a projection. Further, we have

$$\begin{aligned} \|B_j \underline{e}\| &= \|B_j(e_1, \ldots, e_{k^2})\| = \|(\varepsilon_1 x, \ldots, \varepsilon_{k^2} x)\| \\ &= \|(\varepsilon_i)\|_p \|x\|_E = \|x\|_E = \Big\|\sum_{s=1}^{k} b_s e_{(j-1)k+s}\Big\| \\ &\leqslant \sum_{s=1}^{k} |b_s| \, \|e_{(j-1)k+s}\| \leqslant \Big(\sum_{s=1}^{k} |b_s|^q\Big)^{\frac{1}{q}} \Big(\sum_{s=1}^{k} \|e_{(j-1)k+s}\|^p\Big)^{\frac{1}{p}} = \Big(\sum_{s=(j-1)k+1}^{jk} \|e_s\|^p\Big)^{\frac{1}{p}}. \end{aligned}$$

Observe that the operator $U_j\colon E \to F$, defined by

$$U_j(e) = (\varepsilon_1 e, \ldots, \varepsilon_{k^2} e),$$

is an isometric mapping of E onto $B_j(F)$. Consider operators

$$SJU_1, \ldots, SJU_k \in B(E, G),$$

and choose a $\underline{d} = (d_1, \ldots, d_k) \in l_p^k$ such that

$$\|\underline{d}\|_p = 1, \qquad \sum_{j=1}^{k} d_j SJU_j = 0.$$

Define the operator $U\colon E \to F$ by

$$U(e) = \sum_{j=1}^{k} d_j U_j e.$$

Clearly, u is an isometric embedding and

$$(**) \qquad SJU = 0.$$

Now choose $\underline{g} = (g_1, \ldots, g_k) \in l_q^k$ so that

$$\|\underline{g}\|_q = 1, \quad \text{and} \quad \langle \underline{g}, \underline{d} \rangle = \sum_{i=1}^{k} g_i d_i = 1,$$

and define for $x \in F$

$$\tilde{Q}x = U\Big(\sum_{i=1}^{k} g_i U_i^{-1}(B_i x)\Big).$$

Let $x = Ue \in U(E)$. Then we have

$$\begin{aligned} \tilde{Q}x &= U\Big(\sum_{i=1}^{k} g_i U_i^{-1}(B_i Ue)\Big) \\ &= U\Big(\sum_{i=1}^{k} g_i U_i^{-1}(d_i U_i e)\Big) = U\Big(\sum_{i=1}^{k} g_i d_i e\Big) \\ &= U(e) = x. \end{aligned}$$

Hence $\tilde{Q}$ is a projection of F onto $U(E)$. Since for each $\underline{e} = (e_1, \ldots, e_{k^2}) \in F$ we have

$$\begin{aligned} \|\tilde{Q}\underline{e}\| &= \Big\|\sum_{i=1}^{k} g_i U_i^{-1}(B_i \underline{e})\Big\| \leqslant \sum_{i=1}^{k} |g_i| \|B_i \underline{e}\| \\ &\leqslant \Big(\sum_{i=1}^{k} |g_i|^q\Big)^{\frac{1}{q}} \Big(\sum_{i=1}^{k} \|B_i \underline{e}\|^p\Big)^{\frac{1}{p}} \leqslant \Big(\sum_{i=1}^{k}\Big(\sum_{s=(i-1)k+1}^{ik} \|e_s\|^p\Big)\Big)^{\frac{1}{p}} = \|\underline{e}\|, \end{aligned}$$

we infer that $\|\tilde{Q}\| \leqslant 1$.

Let $Q = J\tilde{Q}J^{-1}P$. Then $\|Q\| \leqslant KL$, and, by (*) and (**), we have for $z \in Z$

$$SQZ = SJ\tilde{Q}J^{-1}Pz = SJU\Big(\sum_{i=1}^{k} g_i U_i^{-1}\big(B_i(J^{-1}Pz)\big)\Big) = 0,$$

$$QSz = J\tilde{Q}J^{-1}PSz = JU\Big(\sum_{i=1}^{k} g_i U_i^{-1}(B_i J^{-1}PSz)\Big) = 0.$$

Further, since $PJ = J$ and $\tilde{Q}^2 = \tilde{Q}$, we get

$$Q^2 = J\tilde{Q}J^{-1}PJ\tilde{Q}J^{-1}P = J\tilde{Q}J^{-1}P = Q.$$

Finally, observe that JU is an isomorphism of E onto $Q(Z)$. Since for every $e \in E$ we have

$$L^{-1}\|e\| = L^{-1}\|Ue\| \leqslant \|JUe\| \leqslant \|Ue\| \leqslant \|e\|,$$

we infer that

$$d\big(Q(Z), E\big) \leqslant L,$$

which completes the proof of the lemma.

5. Factorization through a given space. In view of the results of Section 3 we introduce the following definition.

Definition 5.1. A Banach space Z is said to have the *factorization property* (resp. the *subspace factorization property*) if every $T \in \overline{F}(X, Y)$ (resp. every $T \in K(X, Y)$) admits a factorization through Z (resp. through a subspace of Z). We shall use abbreviations f.p. and s.f.p.

In this section we give a characterization of the class of Banach spaces possessing the f.p. and deduce some of its consequences. This characterization is contained in the following theorem.

Theorem 5.2. *Each of the following properties of a Banach space Z is equivalent to the f.p.*

(i) *Every operator $T \in \overline{F}(X, Y)$ admits an $\overline{F}$-factorization, say (A, B), through Z.*

(ii) *There exists a constant $K > 0$ such that A and B in* (i) *can be chosen so that $\|A\|\,\|B\| \leqslant K\|T\|$.*

(iii) *There exists a p, with $1 \leqslant p \leqslant \infty$, such that every $T \in \overline{F}(C_p, C_p)$ admits a factorization, say (A, B), through Z.*

(iv) *There exists a constant $K > 0$ such that for every finite dimensional Banach space E there exists a projection P in Z with*

$$\|P\| \leqslant K, \quad d\big(E, P(Z)\big) \leqslant K.$$

(v) *There exists a constant $K > 0$ such that for every sequence $(E_i)_{i=1}^{\infty}$ of finite dimensional Banach spaces there exists a f.d.p. $(P_i)_{i=1}^{\infty}$ in Z such that*

$$\|P_i\| \leqslant K, \quad d\big(E_i, P_i(Z)\big) \leqslant K$$

for $i = 1, 2, \ldots$

(vi) *There exists a sequence $(a_i)_{i=1}^{\infty}$ of positive reals such that for every sequence $(E_i)_{i=1}^{\infty}$ of finite dimensional Banach spaces there is a f.d.p. $(P_i)_{i=1}^{\infty}$ in Z such that the sequences*

$$(a_i\|P_i\|)_{i=1}^{\infty} \quad \big(a_i d\big(E_i, P_i(Z)\big)\big)_{i=1}^{\infty}$$

are bounded.

Proof. Clearly (ii) $\Rightarrow$ (i) $\Rightarrow$ (f.p.) $\Rightarrow$ (iii) and (v) $\Rightarrow$ (vi). Hence it suffices to prove that (i) $\Rightarrow$ (ii), (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v), and (vi) $\Rightarrow$ (i).

(vi) $\Rightarrow$ (i). Let $T \in \overline{F}(X, Y)$. T can be decomposed in the form $T = \sum_{i=1}^{\infty} T_i$, where $T_i \in F(X, Y)$ for $i = 1, 2, \ldots$, and

$$\|T_1\| \leqslant (1 + \tfrac{1}{8}a_1^2)\|T\|,$$
$$\|T_i\| \leqslant 4^{-i}a_i^2\|T\| \quad \text{for } i = 2, 3, \ldots$$

Indeed, it suffices to choose for $i = 1, 2, 3 \ldots$ an element $S_i \in F(X, Y)$ such that

$$\|T - S_i\| < \tfrac{1}{2}\cdot 4^{-i}\cdot\min(a_i^2, a_{i+1}^2),$$

and put for $i = 1, 2, \ldots$

$$T_i = S_i - S_{i-1},$$

where $S_0 = 0$.

Consider a sequence $(E_i)_{i=1}^{\infty}$ of finite dimensional Banach spaces defined by

$$E_i = T_i(X), \quad i = 1, 2, \ldots,$$

and let $(P_i)_{i=1}^{\infty}$ be a f.d.p. in Z existing by (vi). For every $i = 1, 2, \ldots$, choose an isomorphism $I_i\colon E_i \to P_i(Z)$ such that

$$\|I_i\| \leqslant 1, \quad \|I_i^{-1}\| \leqslant Ka_i^{-1},$$

K being a suitable constant independent of i. Define $A \in \overline{F}(X, Z)$, $B \in \overline{F}(Z, Y)$ by the formulae

$$Ax = \sum_{i=1}^{\infty} 2^i a_i^{-2} I_i T_i x \quad \text{for } x \in X,$$

$$Bz = \sum_{i=1}^{\infty} 2^{-i} a_i^2 I_i^{-1}(P_i z) \quad \text{for } z \in Z.$$

Since $P_i I_j = \delta_{ij} I_j$, for $i, j = 1, 2, \ldots$, and the series are absolutely convergent, we have for $x \in X$,

$$BAx = \sum_{i=1}^{\infty} 2^{-i} a_i^2 I_i^{-1} \Big(P_i \sum_{j=1}^{\infty} 2^j a_j^{-2} I_j T_j x\Big)$$
$$= \sum_{i=1}^{\infty} 2^{-i} a_i^2 I_i^{-1} (2^i a_i^{-2} I_i T_i x) = \sum_{i=1}^{\infty} T_i x = Tx.$$

Hence (A, B) is a desired factorization.

(i) $\Rightarrow$ (ii). Suppose that Z satisfie (i) but no (ii).

For any X, Y and any $T \in \overline{F}(X, Y) \setminus \{0\}$ denote

$$c(T) = \inf(\|A\| \, \|B\| \, \|T\|^{-1}),$$

where the infimum is taken over all $\overline{F}$-factorizations (A, B) of T through Z.

By our assumption, there exist sequences

$$(X_n)_{n=1}^{\infty}, \quad (Y_n)_{n=1}^{\infty}, \quad \text{and } (T_n)_{n=1}^{\infty}$$

such that

$$T_n \in \overline{F}(X_n, Y_n), \quad \|T_n\| = 1, \quad c(T_n) \geqslant 4^n,$$

for $n = 1, 2, \ldots,$

Let X (resp. Y) be the c_0-sum of the sequence (X_n) (resp. (Y_n)). (The l_p-sums, $1 \leqslant p < \infty$, may be used as well.) Let $I_j: X_j \to X$ be the natural embedding, and P_j: $Y \to Y_j$ the natural projection. The operator T: $X \to Y$ defined as follows

$$T\big((x_n)_{n=1}^{\infty}\big) = (2^{-n} T_n x_n)_{n=1}^{\infty}$$

belongs to $\overline{F}(X, Y)$. Let (A, B) be an $\overline{F}$-factorization of T through Z. It is easy to check that, for every $n = 1, 2, \ldots$ $(AI_n, 2^n P_n B)$ is an $\overline{F}$-factorization of T_n. Hence

$$4^n \leqslant c(T_n) \leqslant \|AI_n\| \, \|2^n P_n B\| \leqslant 2^n \|A\| \, \|B\|.$$

Since n is arbitrary, we get a contradiction.

(iii) $\Rightarrow$ (iv). Since C_p is isometrically isomorphic to the l_p-sum of a countable family of its copies, using a similar argument as that in the proof of (i) $\Rightarrow$ (ii), we infer that there exists a $K > 0$ independent of T, such that A and B in (iii) can be chosen so that $\|A\| \, \|B\| \leqslant K\|T\|$.

Now fix $\varepsilon > 0$ and choose an i such that $d(E, G_i) < 1+\varepsilon$.

Let I: $G_i \to C_p$ be the natural embedding, and let Q be the natural projection of C_p onto its summand $I(G_i)$. Let (A, B) be a factorization of Q through Z such that

$$\|A\| \, \|B\| \leqslant K\|Q\| = K.$$

Then $AB: Z \to Z$ is a projection of norm $\leqslant K$, and, since $AI: E \xrightarrow[\text{onto}]{} AB(Z)$ is an isomorphism with

$$(AI)^{-1} = I^{-1}B|_{AB(Z)},$$

we have

$$d\big(G_i, (AB)(Z)\big) \leqslant \|AI\| \, \|I^{-1}B\|$$
$$\leqslant \|A\| \cdot \|B\| \cdot \|I\| \, \|I^{-1}\| \leqslant K.$$

Hence

$$d\big(E, (AB)(Z)\big) \leqslant d(E, G_i)\, d\big(G_i, (AB)(Z)\big) \leqslant (1+\varepsilon)K$$

which completes the proof of (iii) $\Rightarrow$ (iv).

(iv) $\Rightarrow$ (v). Let $(E_i)_{i=1}^{\infty}$ be an arbitrary sequence of finite dimensional Banach spaces. We shall define recursively a f.d.p. $(P_i)_{i=1}^{\infty}$ in Z such that

$$(*) \qquad \|P_i\| \leqslant K^2, \quad d\big(E_i, P_i(Z)\big) \leqslant K,$$

for $i = 1, 2, \ldots,$ where K denotes the constant appearing in (iv).

Let P_1 be an arbitrary projection in Z such that

$$\|P_1\| \leqslant K, \quad d\big(E_1, P_1(Z)\big) \leqslant K.$$

Suppose that we have defined $P_1, \ldots, P_{n-1}$ so that $(*)$ is satisfied and

$$P_i P_j = P_j P_i = 0 \quad \text{for } 1 \leqslant i < j < n.$$

Let $r = \dim E_n$, $m = \sum_{i=1}^{n-1} \dim E_i$, $k = mr+1$. Let F be the c_0-sum of k^2 copies of E_n, and let P be a projection in Z such that

$$\|P\| \leqslant K, \quad d\big(P(Z), F\big) \leqslant K.$$

Putting $S = P_1 + \ldots + P_{n-1}$ we have $\dim S(Z) = r$, hence we may apply Lemma 4.1 to obtain a projection Q in Z such that

$$QS = SQ = 0, \quad \|Q\| \leqslant K^2, \quad d\big(E_n, Q(Z)\big) \leqslant K.$$

Since, for every $i = 1, \ldots, n-1$, we have

$$P_i Q = (P_i S)Q = P_i SQ = 0,$$
$$QP_i = Q(SP_i) = (QS)P_i = 0,$$

we may put $P_n = Q$ and continue.

This completes the proof of the theorem.

COROLLARY 5.3 (Johnson). *The spaces C_p, $1 \leqslant p \leqslant \infty$, have the f.p.*

Proof. Use condition (iii).

COROLLARY 5.4. *A Banach space Z has the f.p. if and only if its dual Z^* has.*

Proof. Suppose that Z satisfies condition (ii) of the theorem, and let E be a finite dimensional Banach space.

The identity operator $I_{E^*}\colon E^* \to E^*$ admits, by (ii), a factorization (A, B) through Z such that

$$\|A\| \leqslant 1, \quad \|B\| \leqslant K.$$

Since the adjoint operators satisfy

$$\|A^*\| = \|A\| \leqslant 1, \quad \|B^*\| = \|B\| \leqslant K,$$
$$A^*B^* = (BA)^* = (I_{E^*})^* = I_E,$$

we get easily that $d(E, B^*(E)) \leqslant K$, and B^*A^* is projection of Z^* onto $B^*(E)$ of norm $\leqslant K$. Hence Z^* satisfies condition (iv).

Conversely, suppose that Z^* has the f.p. Then, by the first part of the proof, so has Z^{**}. The proof that this implies that Z satisfies condition (iv) follows readily from the strengthening of the principle of local reflexivity given in [5] (Th. 3.3).

Indeed, it states that if E is a subspace of Z^{**}, $\dim E < \infty$, and P is a projection of Z^{**} onto E, then, for every $\varepsilon > 0$, there is a subspace E_1 of Z and a projection P_1 of Z onto E_1 such that

$$d(E, E_1) < 1+\varepsilon, \quad \|P_1\| \leqslant \|P\|(1+\varepsilon).$$

COROLLARY 5.5. *If Z has the f.p. and $S \in F(Z, Z)$, then the kernel of S has the f.p.*

Proof. Use condition (iv) of the theorem and Lemma 4.1.

PROPOSITION 5.6. *If Z has the f.p., then it has the s.f.p.*

Proof. It is a consequence of 5.1 and 3.2. A direct proof can also be given.

Remark 5.7. An interesting example of a Banach space possessing the f.p. is that discovered by Szankowski [9]. Namely, he constructed a separable reflexive Banach space Z such that for every finite dimensional Banach space E there is a norm 1 projection in Z, whose range is isometric to E.

6. Factorization through subspaces. Now we pass to the characterization of Banach spaces possessing the s.f.p. For every Banach space Z and every positive integer n it will be convenient to introduce an abbreviation, say $c_n(Z)$, for the quantity

$$\inf\{d(E, l_\infty^n)\colon E \subset Z, \dim E = n\}.$$

In this notation our result reads as follows.

THEOREM 6.1. *The following properties of a Banach space Z are equivalent to the s.f.p.*

(i) *Every $T \in K(X, Y)$ admits a compact factorization through a subspace of Z.*

(ii) *There exists a non-increasing sequence $(a_i)_{i=1}^{\infty}$ of positive reals such that*

(a) $\lim_n a_n = 0$,

(b) $\lim_n a_n n^c = \infty$ *for every* $c > 0$,

(c) *the operator $T\colon c_0 \to c_0$ given by $T((x_i)_{i=1}^{\infty}) = (a_i x_i)_{i=1}^{\infty}$ admits a factorization through a subspace of Z.*

(iii) $\lim_n c_n(Z) n^{-c} = 0$ *for every* $c > 0$.

(iv) $c_n(Z) = 1$ *for* $n = 1, 2, \ldots$

(v) *For every sequence $(\varepsilon_n)_{n=1}^{\infty}$ of positive numbers there exists a f.d.p. $(P_n)_{n=1}^{\infty}$ in Z such that*

$$\|P_n\| < 1+\varepsilon_n, \quad d(P_n(Z), l_\infty^n) < 1+\varepsilon_n,$$

for $n = 1, 2, \ldots$.

(vi) *There exists a sequence $(a_n)_{n=1}^{\infty}$ of positive reals such that for every sequence $(k_n)_{n=1}^{\infty}$ of positive integers there exists a f.d.p. $(P_n)_{n=1}^{\infty}$ in Z such that the sequences*

$$(a_n \|P_n\|)_{n=1}^{\infty}, \quad (a_n d(P_n(Z), l_\infty^{k_n}))_{n=1}^{\infty}$$

are bounded.

Proof. The implications (i) $\Rightarrow$ (s.f.p.) $\Rightarrow$ (ii) and (v) $\Rightarrow$ (vi) are obvious.

In order to prove that (ii) $\Rightarrow$ (iii) we fix a factorization (A, B) of T through a subspace Z' of Z.

Let n be an arbitrary positive integer. Define $I\colon l_\infty^n \to c_0$ and $J\colon c_0 \to l_\infty^n$ by

$$I(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, 0, \ldots),$$
$$J(x_1, x_2, \ldots) = (a_1^{-1}x_1, \ldots, a_n^{-1}x_n).$$

Let $E = AI(l_\infty^n)$. Then we have

$$(JB)(AI)(x) = J(BA)(Ix) = JT(Ix) = x$$

for all $x \in l_\infty^n$, and

$$\|JB\| \leqslant \|J\|\,\|B\| \leqslant a_n^{-1}\|B\|, \quad \|AI\| \leqslant \|A\|\,\|I\| = \|A\|.$$

Hence

$$c_n(Z) \leqslant c_n(Z') \leqslant d(E, l_\infty^n) \leqslant \|JB\|\,\|AI\| \leqslant a_n^{-1}\|A\|\,\|B\|.$$

Since n was arbitrary, the conclusion follows immediately.

The implication (iii) $\Rightarrow$ (iv) is a consequence of the estimate

$$(*)\qquad c_{kl}(Z) \geqslant 2c_l(Z)\big(1+c_k(Z)^{-1}\big)^{-1} = a_k c_l(Z), \quad \text{for } k, l = 1, 2, \ldots$$

Indeed, if $c_k(Z) > 1$ for some $k > 1$, then, by $(*)$, $c_{k^n}(Z) \geqslant a_k^n = (k^n)^b$, for $n = 1, 2, \ldots$, with $\log_k a_k > 0$, which contradicts (iii). We omit the proof of $(*)$, since similar facts are well known (cf. [1], [3]).

The proof that (iv) $\Rightarrow$ (v) uses Lemma 4.1. Being similar to the analogous one in Theorem 5.2 it can be left to the reader.

It remains to prove that (vi) $\Rightarrow$ (i).

Let $j\colon Y \to C(S)$ be an isometrical embedding. Then $jT \in \overline{F}(X, C(S))$, hence we can write $jT = \sum_{i=1}^{\infty} T_i$, where

$$T_i \in F(X, C(S)) \quad \text{for } i = 1, 2, \ldots,$$
$$\|T_1\| \leqslant 2\|T\|,$$
$$\|T_i\| \leqslant 4^{-i} a_i^2 \|T\| \quad \text{for } i = 2, 3, \ldots$$

Choose for every $i = 1, 2, \ldots$, a subspace $F_i \subset C(S)$ such that

$$T_i(X) \subseteq F_i, \quad \dim F_i = n_i < \infty, \quad d(F_i, l_\infty^{n_i}) < 2.$$

Take a f.d.p. $(P_i)_{i=1}^{\infty}$ in Z such that for some $K > 0$

$$\|P_i\| \leqslant K a_i^{-1}, \quad d\big(P_i(Z), l_\infty^{n_i}\big) \leqslant K a_i^{-1}, \ i = 1, 2, \ldots,$$

and choose isomorphisms $I_i\colon F_i \to P_i(Z)$ such that

$$\|I_i\| \leqslant 1, \quad \|I_i^{-1}\| \leqslant K a_i^{-1}, \quad i = 1, 2, \ldots$$

Define operators $\tilde{A}\colon X \to Z$ and $\tilde{B}\colon Z \to C(S)$ by

$$\tilde{A}x = \sum_{i=1}^{\infty} 2^i a_i^{-2} I_i T_i x \quad \text{for } x \in X,$$

$$\tilde{B}z = \sum_{i=1}^{\infty} 2^{-i} a_i^2 I_i^{-1}\big(P_i(z)\big) \quad \text{for } z \in Z.$$

Obviously $(\tilde{A}, \tilde{B})$ is an $\overline{F}$-factorization of jT. Let Z' be subspace of Z such that

$$\tilde{A}(X) \subseteq Z' \subseteq B^{-1}\big(j(Y)\big).$$

Then (A, B), where A, B are suitable restrictions, is a required K-factorization of T.

This completes the proof of the theorem.

Corollary 6.2. *The spaces $\Sigma_p l_\infty^n$, $1 \leqslant p < \infty$, and c_0 have the s.f.p.*

Proof. They obviously satisfy (iv).

Corollary 6.3. *If the approximation problem has a negative solution, then every space possessing the s.f.p. contains a subspace which does not have the a.p.*

Proof. Use Theorem 6.1 and Remark 3.6.

Corollary 6.4. *If Z has the s.f.p., then some subspace Z' of Z has the f.p.*

Proof. Let $(P_n)_{n=1}^{\infty}$ be a f.d.p. in Z such that

$$\|P_n\| \leqslant 1 + \frac{1}{n}, \quad d\big(P_n(Z), l_\infty^n\big) \leqslant 1 + \frac{1}{n},$$

and let $(G_i)_{i=1}^{\infty}$ be a sequence appearing in the definition of the spaces C_p. Choose an increasing sequence $(n_i)_{i=1}^{\infty}$ of positive integers such that, for every $i = 1, 2, \ldots$, there exists a subspace $F_i \subseteq P_{n_i}(Z)$ such that $d(F_i, G_i) < 1 + \frac{1}{i}$. It is easy to see that the subspace

$$Z' = \{z \in Z\colon P_{n_i}(z) \in F_i \text{ for } i = 1, 2, \ldots\}$$

has the f.p.

7. L_p factorization. Factorization problems similar to these we considered, can be stated also for other classes of operators (cf. [4]). In this section we briefly discuss the factorization of compact operators whose range is a subspace of a certain $L_p(\mu)$ space. We indicate some applications of these results to the approximation problem for this class of spaces. It will be convenient to denote by l_∞ the space usually called c_0.

We start with the following definitions of two factorization properties.

Definition 7.1. Let $1 \leqslant p \leqslant \infty$. A Banach space Z is said to have the *factorization property for L_p* (resp. the *subspace factorization property for L_p*), provided every compact operator $T\colon l_p \to l_p$ admits a factorization through Z (resp. through a subspace of Z). We shall use abbreviations f.p. and s.f.p. as before.

Obviously the space l_p and more generally, an arbitrary infinite dimensional $\mathscr{L}_p$-space, has these properties. (The latter fact follows from Proposition 7.3 of [8]). A characterization of spaces possessing the f.p. for L_p is contained in the following

Theorem 7.2. *Let $1 \leqslant p \leqslant \infty$. The following properties of a Banach space Z are equivalent.*

(i) *Every operator $T \in \overline{F}(X, Y)$, where Y is an $\mathscr{L}_p$-space, admits an $\overline{F}$-factorization, say (A, B), through Z.*

(ii) *There exists a real function $g(\lambda)$ such that whenever Y is an $\mathscr{L}_{p,\lambda}$-space and $T \in \overline{F}(X, Y)$ $(= K(X, Y))$, then A and B in (i) can be chosen so that*

$$\|A\|\,\|B\| \leqslant g(\lambda)\,\|T\|.$$

(iii) *Z has the factorization property for L_p.*

(iv) *There is a constant $K > 0$ such that for every n there is a projection P in Z such that*

$$\|P\| \leqslant K, \quad d(P(Z), l_p^n) \leqslant K.$$

(v) *There is a f.d.p. $(P_n)_{n=1}^\infty$ in Z such that the sequences*

$$(\|P_n\|)_{n=1}^\infty, \quad \big(d(P_n(Z), l_p^n)\big)_{n=1}^\infty$$

are bounded.

(vi) *Analogue of* (vi) *in Theorem* 5.2 *with E_i replaced by $l_p^{n_i}$.*

The proof of this theorem, being similar to that of 5.2, may be left to the reader. We remark only that the proof of (i) ⇒ (ii) uses the following simple lemma. (We state it here in more precise form than we need.)

LEMMA 7.3. *Let $1 \leqslant p \leqslant \infty$. If $(Y_i)_{i\in I}$ is a family of Banach spaces, which are $\mathscr{L}_{p,\lambda+\varepsilon}$-spaces for every $\varepsilon > 0$, then so is the space $\Sigma_p Y_i$.*

This lemma may be proved by a usual stability argument. We omit the details.

Theorem 7.2 implies that Z has the f.p. for L_p if and only if Z^* has the f.p. for L_q where $q = p/(p-1)$.

Obviously the f.p. for L_p implies the s.f.p. for L_p. On the other hand, the proofs of Section 6 can be modified to show that every space possessing the s.f.p. for L_p contains a subspace which has the f.p. for L_p. We do not present this proof here. We give, however, an independent proof of an important consequence of this fact (Theorem 7.4). It should be remarked that in this proof, instead of Theorem 7.2, one can use the following consequence of another result from [4].

If Y is an $\mathscr{L}_p$-space, $1 \leqslant p \leqslant \infty$, then every $T \in K(X, Y)$ factors compactly through l_p.

THEOREM 7.4. *Let $1 \leqslant p \leqslant \infty$. Suppose that Z has the s.f.p. for L_p and Y is isomorphic to a subspace of a certain $L_p(\mu)$. Then every $T \in K(X, Y)$, X being an arbitrary Banach space, admits a compact factorization through a subspace of Z.*

Proof. Let $j\colon Y \to L_p(\mu)$ be an isomorphic embedding. Then we have $jT \in \overline{F}(X, L_p(\mu))$, hence, by Theorem 7.2 or Johnson's result, we get an $\overline{F}$-factorization (A_1, A_2) of T through l_p. The same argument shows that A_1 admits an $\overline{F}$-factorization (A_3, A_4) through l_p.

Let (A_5, A_6) be a factorization of A_4 through a subspace of Z. Let Z' be a subspace of Z such that

$$A_5A_3(X) \subseteq Z' \subseteq (A_2A_6)^{-1}\big(j(Y)\big).$$

The operators $A\colon X \to Z'$, $B\colon Z \to Y$ given by

$$A(x) = A_5A_3(x) \qquad \text{for } x \in X,$$
$$B(z) = j^{-1}\big(A_2A_6(z)\big) \qquad \text{for } z \in Z',$$

form a required K-factorization.

COROLLARY 7.5. *Let $1 \leqslant p \leqslant \infty$. Suppose that every subspace of l_p has the a.p. Then every subspace Y of every $L_p(\mu)$ has the a.p.*

Proof. It follows from Theorem 7.4 that every compact operator into Y factors compactly through a subspace of l_p. So we can proceed as in the proof of 3.5.

COROLLARY 7.6. *If $1 \leqslant p < q \leqslant 2$, and every subspace of l_p has the a.p. then so has every subspace of l_q.*

Proof. By a result of Kadec [6], l_q is in this case isometric to a subspace of $L_p([0, 1])$, hence the result follows from Corollary 7.5.

8. Some factorizations of matrices. We do not know whether the last corollary is still valid if p, q are supposed to satisfy $2 \leqslant q < p \leqslant \infty$. However, here also the hypothesis that the subspaces of l_p have the a.p. leads to a stronger result than the one for the subspaces of l_q. The result we mentioned is closely connected with Grothendieck's Proposition 37 (f″) and Remarque 14 of [2]. It will be convenient to formulate it in terms of matrices.

Let M denote the set of all infinite matrices $u = (u_{ij})_{i,j=1}^\infty$ such that

$$u_j = (u_{ij})_{i=1}^\infty \in c_0 \quad \text{for } j = 1, 2, \ldots,$$
$$\sum_{j=1}^\infty \|u_j\|_{c_0} < \infty,$$
$$u^2 = 0.$$

The subset of M consisting of those u such that $\sum_{j=1}^\infty \|u_j\|_{c_0}^q < \infty$, where $0 < q \leqslant 1$, will be denoted M_q. Let

$$N = \Big\{u \in M\colon \sum_{i=1}^\infty u_{ii} = 0\Big\}.$$

Grothendieck proved that the approximation problem is equivalent to the statement "$M = N$". Using some properties of entire functions and Fredholm determinants he showed that $M_{2/3} \subseteq N$.

We show that one can, in a sense, interpolate between these two results. (We assume below that $p < \infty$, only for simplyfying of the notation.)

PROPOSITION 8.1. *Suppose that every subspace of l_p, $2 \leqslant p < \infty$, has the a.p. Then $M_{p/(p+1)} \subseteq N$.*

Proof. Let $u \in M_{p/(p+1)}$. Then there exists a sequence $(b_j)_{j=1}^{\infty}$ of positive numbers such that $\sum_{j=1}^{\infty} b_j < \infty$ and

$$|u_{ij}| \leqslant b_j^{(p+1)/p} \quad \text{for } i, j = 1, 2, \ldots$$

Let

$$a_{ij} = u_{ij}\left(\frac{b_i}{b_j}\right)^{\frac{1}{p}}, \quad i, j = 1, 2, \ldots$$

Then

$$a_j = (a_{ij})_{i=1}^{\infty} \in l_p \quad \text{and} \quad \sum_{j=1}^{\infty} \|a_j\|_p < \infty.$$

Let $e_i^* = (\delta_{ij})_{j=1}^{\infty} \in (l_p)^*$. Consider a nuclear operator A in l_p defined by

$$A = \sum_{j=1}^{\infty} e_j^* \otimes a_j.$$

It is easy to check that $A^2 = 0$. Now we shall repeat Grothendieck's argument.

Let X be the kernel of A. Then $a_j \in X$, for $j = 1, 2, \ldots$, hence the expression

$$\sum_{j=1}^{\infty} (e_j^*|_X) \otimes a_j \in X^* \hat{\otimes} X$$

is a nuclear representation of the zero operator in X. Since, by hypothesis, X has the a.p., we get, in virtue of Proposition 35 of [2], that

$$0 = \mathrm{Tr}.0 = \sum_{j=1}^{\infty} (e_j^*|_X)(a_j) = \sum_{j=1}^{\infty} a_{jj}.$$

To conclude the proof it remains to observe that $u_{ii} = a_{ii}$ for $i = 1, 2, \ldots,$

Remark 8.2. As far as we know, the problem whether the space l_p, $2 < p < \infty$, possesses closed infinite dimensional subspaces non-isomorphic to the whole space remains still open.

Added in proof. Enflo's result has completely changed many opinions about the approximation property. Now it seems very likely that each Banach space non-isomorphic to any Hilbert space contains a subspace failing to have the a.p.

In any case, we have proved, using Enflo's method, that the hypothesis of Proposition 8.1 is satisfied only for $p = 2$ (cf. Remark 8.2). (This proof yields, however, only that $\bigcap_{p>3} M_{p/(p+1)} \setminus N \neq \emptyset$.)

Observe that, by Theorem 7.4, if l_p contains a subspace failing to have the a.p., then so does each space with the s.f.p. for L_p (cf. the proof of Corollary 6.3). Since it appeared that the former could happen also for $p \neq \infty$, we decided to include a sketch of the proof of the characterization we had mentioned in Section 7.

THEOREM 7.7. *Let $1 \leqslant p \leqslant \infty$. The following properties of a Banach space Z are equivalent to the s.f.p. for L_p.*

(i) *There exist a $K \geqslant 1$ such that for each $n = 1, 2, \ldots$ there is a subspace E of Z with $d(E, l_p^n) \leqslant K$.*

(ii) *Z has a subspace possessing the f.p. for L_p.*

Proof. The implication (ii) $\Rightarrow$ (s.f.p. for L_p) is obvious.

(i) $\Rightarrow$ (ii). It follows easily from Lemma 4.1 that

(*) *each subspace Z' of Z with $\dim Z/Z' < \infty$ satisfies* (i) *with the same K as Z does.*

Using this and the following well known fact

(**) *given $\varepsilon > 0$ and a subspace $E \subset Z$ with $\dim E < \infty$, there is a subspace $Z' \subseteq Z$ and an operator $P\colon Z' \to E$ such that $\dim Z/Z' < \infty$, $Z' \supseteq E$, $Pe = e$ for $e \in E$, and $\|P\| < 1+\varepsilon$,*

we can define recursively sequences $(E_n)_{n=1}^{\infty}$, $(Z_n)_{n=1}^{\infty}$, $(Q_n)_{n=1}^{\infty}$ so that $E_n, Z_n \subseteq Z$, $d(E_n, l_n^p) \leqslant K$, $Z_n \supseteq \sum_{i=1}^{n} E_i = F_n$, $\dim Z/Z_n < \infty$, $Q_n \in B(Z_n, F_n)$, $Q_n e = e$ for $e \in F_n$, $\|Q_n\| < 1+\varepsilon$, $E_{n+1} \subset \mathrm{Ker}(Q_n)$, $Z_{n+1} \subseteq Z_n$, for $n = 1, 2, \ldots$ Indeed, it is easy to choose E_1, Z_1, Q_1. If $n \geqslant 1$ and E_i, Z_i, Q_i have been defined for $i = 1, 2, \ldots, n$ so that all conditions are satisfied, we can, by (*), choose a subspace $E_{n+1} \subset \mathrm{Ker}(Q_n)$ so that $d(E_{n+1}, l_p^{n+1}) \leqslant K$ and then, using (**) with $E = \sum_{i=1}^{n} E_i$, define Z_{n+1} as $Z_n \cap Z'$ and put $Q_{n+1} = P|_{Z_{n+1}}$.

The subspace $\tilde{Z} = \bigcap_{n=1}^{\infty} Z_n$ of Z and a sequence $P_n = (Q_{n+1} - Q_n)|\tilde{Z}$, $n = 1, 2, \ldots$ satisfy the condition (iv) of Theorem 7.2 with the constant $\max(2+2\varepsilon, K)$.

(s.f.p. for L_p) $\Rightarrow$ (i). Let, for $n = 1, 2, \ldots,$

$$K(n) = \inf\{d(E, l_p^n)\colon E \subset Z\}.$$

Suppose, on the contrary, that $\sup_n K(n) = \infty$. Choose a sequence $(n_i)_{i=1}^{\infty}$ so that $K(n_i) > 4^i$ for $i = 1, 2, \ldots$ The desired contradiction will be obtained, if we represent l_p as $\Sigma_p l_p^{n_i}$ and factorize through a subspace Z' of Z a diagonal operator T in $\Sigma_p l_p^{n_i}$ defined by the formula $T((x_i)_{i=1}^{\infty}) = (2^{-i}x_i)_{i=1}^{\infty}$. Indeed, if (A, B) were a factorization of T, we would have

$$4^i < K(n_i) \leqslant d(A(l_p^{n_i}), l_p^{n_i}) \leqslant 2^i \|A\| \|B\|,$$

for $i = 1, 2, \ldots,$ a contradiction. This completes the proof of the theorem.

References

[1] D. P. Giesy, *A convexity condition in normed linear spaces*, Trans. Amer. Math. Soc. 125 (1966), pp. 114–146.

[2] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. 16 (1955).

[3] R. C. James, *Uniformly non-square Banach spaces*, Ann. of Math. 80 (1964), pp. 542–550.

[4] W. B. Johnson, *Factoring compact operators*, Israel J. Math. 9 (1971), pp. 337–345.

[5] — H. P. Rosenthal and M. Zippin, *On bases, finite dimensional decompositions and weaker structures in Banach spaces*, Israel J. Math. 9 (1971), pp. 488–506.

[6] M. I. Kadec, *On linear dimension of the spaces L_p* (Russian), Uspehi Mat. Nauk 13 (1958), pp. 95–98.

[7] J. Lindenstrauss, *Some aspects of the theory of Banach spaces*, Advances in Math. 5 (1970), pp. 159–180.

[8] — and A. Pełczyński, *Absolutely summing operators in $\mathscr{L}_p$-spaces and their applications*, Studia Math. 29 (1968), pp. 275–326.

[9] A. Szankowski, *An example of a universal Banach space*, Israel J. Math. 11(1972), pp. 292-296.

Received November 28, 1971 (442)