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Joint spectra

by

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Abstract. In this paper we discuss classification of joint spectra of commuting n-tuples of operators on a complex Hilbert space analogous to the single operator case. Further, if φ is an element of H^∞ (Hardy space of the unit circle), it is well-known that the spectrum of the analytic Toeplitz operator T_φ is the closure of the image of the open unit disc Δ under the mapping $\hat{\varphi}$; that is, $\sigma(T_\varphi) = \overline{\hat{\varphi}(\Delta)}$. Here $\hat{\varphi}$ is the analytic extension of φ to the interior of the open unit disc. We generalize this assertion to n-tuples of analytic Toeplitz operators.

1. Introduction. There is a successful extension of the notion of spectrum, valid in any commutative algebra $\mathfrak{A}$ with identity. If $A_1, A_2, \ldots, A_n$ are elements of $\mathfrak{A}$, the joint spectrum $\sigma(A_1, A_2, \ldots, A_n)$ of $\{A_i\}_{1\leqslant i\leqslant n}$ relative to $\mathfrak{A}$ is the set of all points $(z_1, z_2, \ldots, z_n)$ of $\boldsymbol{C}^n$ (the n-dimensional complex space) such that $A_1-z_1, A_2-z_2, \ldots, A_n-z_n$ belong to the same proper maximal ideal in $\mathfrak{A}$; or

$$\sigma(A_1, A_2, \ldots, A_n) = \{(\varphi(A_1), \varphi(A_2), \ldots, \varphi(A_n)): \varphi\in M\},$$

where M is the maximal ideal space of $\mathfrak{A}$ [1]. This definition of joint spectrum has the disadvantage that the greater the ambient algebra, the smaller the joint spectrum. Our interest here is in n-tuples of operators on complex Hilbert space; in this context it seems preferable to avoid such ambiguities.

For example: Let U be a bilateral shift. If $\mathfrak{A}$ is taken as the algebra generated by U and 1, one can easily show that the spectrum $\sigma(U)$ of U relative to $\mathfrak{A}$ is the closed unit disc. This is quite unnatural from our point of view.

On the other hand, if $\mathfrak{A}$ is the double *commutant* of U, then $\sigma(U)$ relative to $\mathfrak{A}$ is the unit circle. This latter approach is applicable, and seems natural in our general situation too (cf. Definition 2.1).

In §2, we prove certain property of the joint spectrum analogous to that known in the single operator case, such as Proposition 2.3. §3 pertains to the classification of joint spectra of commuting sets of operators analogous to the single operator case. §5 is devoted to the well known representation of the joint spectrum of a commuting set of normal

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operators in terms of *joint essential range* (Definition 5.1) and (Theorem 5.2). An analogous representation of the joint spectrum of a set of analytic Toeplitz operators is also given in §6 (cf. Theorem 6.2).

2. Joint spectra. The notion of joint spectrum of a family of elements in a commutative Banach algebra was first introduced and studied by Arens and Calderón [1]. Our interest here is to give a suitable definition of the joint spectra of commuting families of operators on a Hilbert space. This is not immediately provided by the above reference [cf. §1].

Here and in what follows we shall need the following notations and terminologies unless otherwise mentioned. Let C^n be the n-dimensional complex space and $A = (A_1, A_2, \ldots, A_n)$ be the n-tuple of commuting operators on a complex Hilbert space H. Then the *double commutant* $\mathfrak{A}$ of the set $S = \{A_1, A_2, \ldots, A_n\}$ is a weakly closed abelian algebra containing the set S and the identity [7].

DEFINITION 2.1. Let $A = (A_1, A_2, \ldots, A_n)$ be an n-tuple of commuting operators on H. Then the point $z = (z_1, z_2, \ldots, z_n)$ of C^n is in the *joint spectrum* $\sigma(A)$ of A relative to $\mathfrak{A}$ if and only if for all $B_1, B_2, \ldots, B_n$ in $\mathfrak{A}$

$$\sum_{i=1}^{n} B_i(A_i - z_i) \neq 1.$$

Equivalently, z is in $\sigma(A)$ if and only if the ideal in $\mathfrak{A}$ generated by $\{A_i - z_i\}_{1 \leqslant i \leqslant n}$ is proper.

It is well known, in fact, it can be easily shown that $\sigma(A)$ is nonempty and compact. Here it would be appropriate to remark that the possibility that it would be better to replace $\mathfrak{A}$ by some different algebra has been discussed by Taylor [11]. If T is an operator on H, then it is well known that $\sigma(T^*) = \sigma(T)^*$, where asterik on the right represents complex conjugates. The analogous assertion for commuting n-tuple of operators can be readily verified. We state it without proof.

PROPOSITION 2.2. *Let $A = (A_1, \ldots, A_n)$ be an n-tuple of commuting operators in H. Then $\sigma(A^*) = \sigma(A)^*$,* where $A^* = (A_1^*, \ldots, A_n^*)$.

The following theorem is an easy consequence of the properties of the commutants and Fuglede's Theorem [8]. We omit its proof. This we shall need later in §5.

THEOREM 2.3. *Let $S = \{A_1, \ldots, A_n\}$ be a commuting set of normal operators. Then:*

(1) *The commutant S' of S is a weakly closed self-adjoint subalgebra of the algebra of the bounded operators $\mathscr{L}(H)$ containing S, S^* and 1.*

(2) *The double commutant $\mathfrak{A} = S''$ is a weakly closed Abelian self--adjoint algebra containing S, S^* and 1; and hence $\mathfrak{A}$ is an Abelian von Neumann algebra.*

3. Classification of joint spectra. We now consider the classification of the joint spectrum of a set of commuting operators analogous to that of a single operator. For clarity, we restrict the discussions in §3 and §4 to a pair of operators.

For a single operator A, $0 \in \sigma_\pi(A)$ if and only if $BA \neq 1$ for all B (here arbitrary bounded operators are allowed.)

The reason the second condition $AB \neq 1$ is not satisfied in the case where A is the backward unilateral shift is that B could be chosen to be A^* (the forward shift), when $AB = 1$ although $BA \neq 1$. Note that $BA \neq 1$ for all B in $\mathscr{L}(H)$ if and only if $A^*B \neq 1$ for all B in $\mathscr{L}(H)$. Furthermore, if there exist B_1, B_2 in $\mathscr{L}(H)$ such that $B_1A = 1$ and $B_2A^* = 1$ (or $AB_1 = 1$ and $A^*B_2 = 1$), then A is invertible. The following discussions will be an interplay of these notions.

For a pair of operators A_1 and A_2, we write $A = (A_1, A_2)$. $B_1A_1 + B_2A_2$ and $A_1B_1 + A_2B_2$ are abbreviated respectively as BA and AB. Several definitions below concern parts $\sigma_.(A)$ of the joint spectrum. In each case, it is implied that a point $z = (z_1, z_2)$ of C^2 is in a set $\sigma_.(A)$ if and only if $0 = (0, 0)$ is in $\sigma_.(A_1 - z_1, A_2 - z_2)$; so in definitions and proofs we will often confine attention, without loss of generality, to the question of whether $0 \in \sigma_.(A)$.

DEFINITION 3.1. 0 is in the *joint approximate point spectrum* $\sigma_\pi(A)$ of A if and only if for all B_1, B_2 in $\mathscr{L}(H)$, $BA \neq 1$.

PROPOSITION 3.2. *0 is in $\sigma_\pi(A)$ of $A = (A_1, A_2)$ if and only if there exists a sequence $\{f_n\}$ of unit vectors in H such that*

$$\|A_1 f_n\| \to 0 \quad \textit{and} \quad \|A_2 f_n\| \to 0.$$

Proof. If 0 is in $\sigma_\pi(A)$, then $B_1A_1 + B_2A_2 \neq 1$ for all B_1, B_2 in $\mathscr{L}(H)$. This implies that 0 is in the spectrum of $A_1^*A_1 + A_2^*A_2$. Since $A_2^*A_1 + A_1^*A_2$ is a positive bounded operator, then there exists a sequence $\{f_n\}$ of unit vectors in H such that

$$\|(A_1^*A_1 + A_2^*A_2)f_n\| \to 0.$$

But

$$\begin{aligned}\|A_1f_n\|^2 + \|A_2f_n\|^2 &= \langle A_1f_n, A_1f_n\rangle + \langle A_2f_n, A_2f_n\rangle\\ &= \langle A_1^*A_1f_n, f_n\rangle + \langle A_2^*A_2f_n, f_n\rangle\\ &= \langle (A_1^*A_1 + A_2^*A_2)f_n, f_n\rangle\\ &\leqslant \|(A_1^*A_1 + A_2^*A_2)f_n\|.\end{aligned}$$

Therefore

$$\|A_1f_n\| \to 0 \quad \text{and} \quad \|A_2f_n\| \to 0.$$

Conversely, suppose there exists $\{f_n\}$ in H with $\|f_n\| = 1$ and

$$\|A_1f_n\| \to 0 \quad \text{and} \quad \|A_2f_n\| \to 0.$$

We must show that $0 \in \sigma_\pi(A)$. If not, then there are operators B_1, B_2 in $\mathscr{L}(H)$ such that $B_1A_1+B_2A_2 = 1$. This implies that

$$\|f_n\| = \|(B_1A_1+B_2A_2)f_n\| = \|B_1(A_1f_n)+B_2(A_2f_n)\|$$
$$\leqslant \|B_1\|\;\|A_1f_n\|+\|B_2\|\;\|A_2f_n\| \to 0$$

which is absurd. Thus 0 is in $\sigma_\pi(A)$.

Definition 3.3. The point 0 is in the *joint approximate compression spectrum* $\sigma_\varrho(A)$ of A if and only if for all B_1 and B_2 in $\mathscr{L}(H)$, $AB \neq 1$.

Proposition 3.4. *The point 0 is in $\sigma_\varrho(A)$ if and only if 0 is in $\sigma_\pi(A^*)$.*

Proof. This follows immediately from Definitions 3.1 and 3.3.

Definition 3.5. The point 0 is in the *joint approximate point-approximate compression spectrum* $\sigma_{\pi\varrho}(A)$ of A if and only if $B_1A_1+B_2A_2^* \neq 1$ for all B_1 and B_2 in $\alpha(H)$.

Proposition 3.6. *0 is in $\sigma_{\pi\varrho}(A)$ if and only if there exists a sequence $\{f_n\}$ of unit vectors in H such that*

$$\|A_1f_n\| \to 0 \quad \textit{and} \quad \|A_2^*f_n\| \to 0 .$$

Proof. The conclusion follows immediately from the relation

$$\sigma_{\pi\varrho}(A_1, A_2) = \sigma_\pi(A_1, A_2^*)$$

and from Proposition 3.2. This line of argument was suggested by the referee.

Definition 3.7. The point 0 is in the *joint approximate compression-approximate point spectrum* $\sigma_{\varrho\pi}(A)$ of A if and only if $B_1A_1^*+B_2A_2 \neq 1$ for all B_1, B_2 in $\alpha(H)$.

Proposition 3.8. *0 is in $\sigma_{\varrho\pi}(A)$ if and only if there exists $\{f_n\}$ in H with $\|f_n\| = 1$ such that*

$$\|A_1^*f_n\| \to 0 \quad \textit{and} \quad \|A_2f_n\| \to 0 .$$

Proof. This follows by imitating the proof of Proposition 3.6.

Proposition 3.9. *The point 0 is in $\sigma_{\pi p}(A)$ if and only if 0 is in $\sigma_{\varrho\pi}(A^*)$.*

Proof. follows immediately from Proposition 3.6 and 3.8 and the corresponding definitions.

These definitions and propositions above could very well be chosen as the basis of the definition of the joint spectrum of a non-commuting set of operators. If so, it is natural to ask whether the joint spectrum of a non-commuting set of operators is non-vacuous. The following example shows that this is not generally true. Let

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then $\sigma(A_1) = \sigma(A_2) = \{1, -1\}$ and $\sigma(A_1)\times\sigma(A_2) = \{(1, 1),\ (-1, -1),\ (-1, 1),\ (1, -1)\}$. Clearly, none of the points of $\sigma(A_1)\times\sigma(A_2)$ are in $\sigma(A_1, A_2)$. Thus $\sigma(A_1, A_2)$ is empty. However, we shall show later that there are a large class of non-commuting operators known as Toeplitz operators for which the joint approximate point spectrum of any finite number of them is non-vacuous (cf. Theorem 6.1).

Next we exhibit the existence of different kinds of spectrum by means of the following example.

Let $H = H_1 \oplus H_2 \oplus \ldots$, where each co-ordinate space is H^2 (Hardy space of the unit circle (cf. §6)). Further, let

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & \ldots \\ 0 & 0 & 1 & \ldots \\ 0 & 0 & 0 & \ldots \\ & & & \cdot \\ & & & \cdot \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} V & 0 & 0 & \ldots \\ 0 & V & 0 & \ldots \\ 0 & 0 & V & \ldots \\ & & & \cdot \\ & & & \cdot \end{bmatrix},$$

where V is the simple unilateral shift, regarded as multiplication by z on H^2. Alternatively, we could regard A_1 as $V^*\oplus 1$ and A_2 as $1 \oplus V$. Obviously A_1 and A_2 commute.

Theorem 3.10. *The joint spectrum of A_1 and A_2 is the cartesian product of the closed unit disc with itself.*

Proof. This result is proved in [4]. Consult also [5].

Thus we have seen that a point (z_1, z_2) of C^2 is in the joint spectrum of $\{A_1, A_2\}$ if $|z_1| \leqslant 1$ and $|z_2| \leqslant 1$. Now one would be interested to know the type of the spectrum each point (z_1, z_2) belongs to.

(I) If $|z_1| < 1$ and $|z_2| < 1$, then $(A_1 - z_1)f = 0$, where

$$f = \begin{bmatrix} 1 \\ z_1 \\ z_1^2 \\ \vdots \end{bmatrix}.$$

Similarly,

$$(V^* - z_2^*)g = 0 \quad \text{if} \quad g = \begin{bmatrix} 1 \\ z_2^* \\ z_2^{*2} \\ \vdots \end{bmatrix}.$$

Choose

$$h = \begin{bmatrix} g \\ z_1 g \\ z_1^2 g \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ z_1 \\ z_1^2 \\ \vdots \end{bmatrix} \otimes \begin{bmatrix} 1 \\ z_2^* \\ z_2^{*2} \\ \vdots \end{bmatrix}.$$

Then $(A_1 - z_1)h = 0$ and $(A_2^* - z_2^*)h = 0$. Thus (z_1, z_2) is in $\sigma_{\pi\varrho}(A)$.

In terms of the new definitions which will be introduced in a moment we can even make the somewhat sharper assertion that

$$\sigma_{pr}(A_1, A_2) = \Delta \times \Delta;$$

that is, the interior of the bicylinder. Here Δ is the open unit disc.

DEFINITION 3.10.1. 0 is in the *joint point spectrum* $\sigma_p(A)$ if and only if there exists f in H such that

$$A_1 f = 0 = A_2 f.$$

DEFINITION 3.10.2. 0 is in the *joint residual spectrum* $\sigma_r(A)$ if and only if there exists f in H such that

$$A_1^* f = 0 = A_2^* f.$$

DEFINITION 3.10.3. 0 is in the *joint point-residual spectrum* $\sigma_{pr}(A)$ if and only if there exists f in H such that

$$A_1 f = 0 = A_2^* f.$$

DEFINITION 3.10.4. 0 is in the *joint residual-point spectrum* $\sigma_{rp}(A)$ if and only if there exists f in H such that

$$A_1^* f = 0 = A_2 f.$$

It is clear that $\sigma_p(A) \subset \sigma_\pi(A)$, $\sigma_r(A) \subset \sigma_\varrho(A)$, $\sigma_{pr}(A) \subset \sigma_{\pi\varrho}(A)$ and $\sigma_{rp}(A) \subset \sigma_{\varrho\pi}(A)$.

PROPOSITION 3.10.5. *0 is in $\sigma_{rp}(A)$ if and only if 0 is in $\sigma_{rp}(A^*)$.*

Proof. This follows immediately from the respective definitions.

We close this section by completing the discussion about the above example.

(II) For the general point (z_1, z_2) with $|z_1| \leqslant 1$ and $|z_2| \leqslant 1$, there exists $\{f_n\}$, $\|f_n\| = 1$ and $\{g_n\}$, $\|g_n\| = 1$ such that

$$\|(V^* - z_1) f_n\| \to 0 \quad \text{and} \quad \|(V^* - z_2^*) g_n\| \to 0.$$

Choose $\{h_n\} = \{f_n \otimes g_n\}$. Clearly, $\|h_n\| = 1$. Thus we have

$$\begin{aligned}\|(A_1 - z_1) h_n\| &= \big\|\big((V^* - z_1) \otimes 1\big)(f_n \otimes g_n)\big\| \\ &= \|(V^* - z_1) f_n \otimes g_n\| \\ &= \|(V^* - z_1) f_n\| \to 0\end{aligned}$$

and

$$\begin{aligned}\|(A_2^* - z_2^*) h_n\| &= \big\|\big(1 \otimes (V^* - z_2^*)\big)(f_n \otimes g_n)\big\| \\ &= \|f_n \otimes (V^* - z_2^*) g_n\| \\ &= \|(V^* - z_2^*) g_n\| \to 0.\end{aligned}$$

This implies that (z_1, z_2) is in $\sigma_{\pi p}(A)$. Thus $\sigma(A) = \sigma_{\pi p}(A)$. Comparing (I) above, we see that

$$\sigma_{pr}(A) \subset \sigma_{\pi\varrho}(A) = \overline{\sigma_{pr}(A)}.$$

4. Some open questions. Let $A = (A_1, A_2)$ be a pair of commuting operators on H. Then it is clear that

$$\sigma_\pi(A) \cup \sigma_\varrho(A) \cup \sigma_{\pi\varrho}(A) \cup \sigma_{\varrho\pi}(A) \subset \sigma(A).$$

But the most formidable part seems to be the converse of this assertion, the answer to which is not known. We conclude this section with the following:

PROBLEM 1. Is it always true that

$$\sigma(A) = \sigma_\pi(A) \cup \sigma_\varrho(A) \cup \sigma_{\pi\varrho}(A) \cup \sigma_{\varrho\pi}(A)?$$

PROBLEM 2. Prove that the joint approximate point spectrum is non-empty. This is well-known in the case of a single operator. Consult [4] and [6].

5. Joint essential range. It is well known that if A is a normal operator on a Hilbert space H, then there is a suitable measure space (X, μ), an identification of H with $L^2(\mu)$, and a bounded measurable function φ such that

$$Af = \varphi f \quad \text{for all } f \text{ in } L^2(\mu)$$

and

$$\sigma(A) = \text{essential range of } \varphi = \sigma_\pi(A).$$

Let $A = (A_1, A_2, \ldots, A_n)$ be an n-tuple of commuting normal operators on H. Then the double commutant $\mathfrak{A}$ of A is an abelian Von Neumann algebra (Theorem 2.3), and every such algebra $\mathfrak{A}$ is contained in some maximal abelian self-adjoint algebra, say $\mathfrak{D}$. It is known that every maximal abelian self-adjoint algebra is unitarily equivalent to a multiplication algebra. That is, there exists a suitable measure space (X, μ) such that $\mathfrak{D}$ is unitarily equivalent to $L^\infty(\mu)$, the algebra of multiplication operators on the Hilbert space $L^2(\mu)$, and the algebra $\mathfrak{A}$ is unitarily equivalent to a subalgebra, say $\mathfrak{B}$, of $L^\infty(\mu)$. Thus there exist bounded measurable functions φ_i, $i = 1, 2, \ldots, n$ such that

$$A_i f = \varphi_i f \quad \text{for all } f \text{ in } L^2(\mu),\ 1 \leqslant i \leqslant n.$$

DEFINITION 5.1. Let $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ be an n-tuple of functions in $L^\infty(\mu)$. Then we define the *joint essential range* of φ to be the set $\mathscr{E}(\varphi)$ consisting of all $z = (z_1, z_2, \ldots, z_n)$ of C^n such that for every $\varepsilon > 0$

$$\mu\Big\{t \in X: \sum_{i=1}^{n} |\varphi_i(t) - z_i| < \varepsilon\Big\} > 0.$$

It is clear that if $\varphi_1, \varphi_2, \ldots, \varphi_n$ are continuous, then the joint essential range of φ is the range of the vector-valued function $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$; that is,

$$\mathscr{E}(\varphi) = \varphi(X) = \{(\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t)) : t \in X\}.$$

The following theorem is well-known. However, we have been unable to find its reference. Since this is crucial in many respects, we give its proof for the continuity of the discussion and benefit of the reader.

THEOREM 5.2. *The joint spectrum of a commuting n-tuples* $A = (A_1, A_2, \ldots, A_n)$ *of normal operators is the joint essential range of* $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$, *where* $A_i f = \varphi_i f$ *for all* f *in* $L^2(\mu)$ *and* $\varphi_i \in L^\infty(\mu)$.

Proof. It will be enough to show that the joint spectrum $\sigma(\varphi)$ of φ relative to $\mathfrak{B}$ is equal to the joint essential range $\mathscr{E}(\varphi)$ of φ. Assume $z = (z_1, z_2, \ldots, z_n)$ is not in $\mathscr{E}(\varphi)$; it will be shown that there exist functions $\psi_1, \psi_2, \ldots, \psi_n$ in $\mathfrak{B} \subset L^\infty(\mu)$ such that

$$\sum_{i=1}^n \psi_i(\varphi - z_i) = 1.$$

Suppose not. Then for all $\psi_1, \psi_2, \ldots, \psi_n$ in $\mathfrak{B}$, $\sum_{i=1}^n \psi_i(\varphi_i - z_i)$ is not invertible in $L^\infty(\mu)$. Thus for every $\varepsilon > 0$ the set

$$E(\psi, \varepsilon) = \left\{t : \left|\sum_{i=1}^n \psi_i(t)(\varphi_i(t) - z_i)\right| < \varepsilon\right\}$$

will have positive measure; that is, $\mu(E(\psi, \varepsilon)) > 0$ for each $\varepsilon > 0$ and for all n-tuples $\psi = (\psi_1, \psi_2, \ldots, \psi_n)$ of functions in $\mathfrak{B}$. In particular, consider the functions $\psi_i = (\varphi_i - z_i)^*$, the complex conjugates of the function $\varphi_i - z_i$. Then we have

$$E(\psi, \varepsilon) = \left\{t : \sum_{i=1}^n |\varphi_i(t) - z_i|^2 < \varepsilon\right\}$$

and $\mu(E(\psi, \varepsilon)) > 0$ for each $\varepsilon > 0$. But z is not in $\mathscr{E}(\varphi)$. Therefore, for some $\varepsilon' > 0$, $\mu(E(\varepsilon')) = 0$, where $E(\varepsilon') = \left\{t : \sum_{i=1}^n |\varphi_i(t) - z_i| < \varepsilon'\right\}$. Furthermore, it is clear that

$$\sum_{i=1}^n |\varphi_i(t) - z_i| \leqslant \left\{n \sum_{i=1}^n |\varphi_i(t) - z_i|^2\right\}^{1/2}.$$

We now let $\varepsilon' = \sqrt{n\varepsilon}$, then we obtain $E(\psi, \varepsilon) \subset E(\varepsilon')$. This implies that $\mu(E(\psi, \varepsilon)) = 0$. This contradiction proves that there are functions $\psi_1, \psi_2, \ldots, \psi_n$ in $\mathfrak{B}$ such that $\sum_{i=1}^n \psi_i(\varphi_i - z_i) = 1$. Thus z is not in $\sigma(\varphi)$. Therefore, $\sigma(\varphi) \subset \mathscr{E}(\varphi)$.

Conversely, if z is in $\mathscr{E}(\varphi)$, then we must show that for no $\psi_1, \psi_2, \ldots, \psi_n$ in $\mathfrak{B}$ is $\sum_{i=1}^n \psi_i(\varphi_i - z_i) = 1$. Assume the contrary; that is, there exist $\psi_1, \psi_2, \ldots, \psi_n$ in $\mathfrak{B}$ such that $\sum_{i=1}^n \psi_i(\varphi_i - z_i) = 1$. Let $M = \max\{\|\psi_1\|, \|\psi_2\|, \ldots, \|\psi_n\|\}$. Since z is in $\mathscr{E}(\varphi)$, then $\mu\{t : \sum_{i=1}^n |\varphi_i(t) - z_i| < \varepsilon\} > 0$. If we let $\varepsilon = \dfrac{1}{2M}$, then

$$\left|\sum_{i=1}^n \psi_i(\varphi_i - z_i)\right| \leqslant M \sum_{i=1}^n |\varphi_i - z_i| \leqslant M . \varepsilon = \tfrac{1}{2} \quad \text{on } E(\varepsilon),$$

which is a contradiction. Hence z is in $\sigma(\varphi)$. This implies that $\mathscr{E}(\varphi)$ is a subset of $\sigma(\varphi)$. Thus the theorem is proved.

THEOREM 5.3. *Let* $A = (A_1, A_2, \ldots, A_n)$ *be an n-tuple of commuting normal operators. Then* $\sigma(A) = \sigma_\pi(A)$; *that is,* $0 = (0, 0, \ldots, 0)$ *is in* $\sigma(A)$ *if and only if there exists a sequence of unit vectors* $\{f_k\}$ *in* H *such that*

$$\|A_i f_k\| \to 0 \quad \text{as } k \to \infty \quad \textit{for each } i, 1 \leqslant i \leqslant n.$$

Proof. 0 is in $\sigma(A)$ if and only if $\sum_{i=1}^n B_i A_i$ is not invertible for all $B_1, B_2, \ldots, B_n$ in $\mathfrak{A}$. Since $\mathfrak{A}$ is a selfadjoint algebra (Theorem 2.4), this means in particular that $\sum_{i=1}^n A_i^* A_i$ is not invertible. Thus, if 0 is in $\sigma(A)$, then 0 is in the spectrum of $\sum_{i=1}^n A_i^* A_i$. Thus it follows that there exists a sequence $\{f_k\}$ of unit vectors in H such that $\left\|\left(\sum_{i=1}^n A_i^* A_i\right) f_k\right\| \to 0$ as $k \to \infty$. As in the proof of Proposition 3.2, $\sum_{i=1}^n \|A_i f_k\|^2 \to 0$ as $k \to \infty$. Therefore, $\|A_i f_k\| \to 0$ as $k \to \infty$ for each i, $1 \leqslant i \leqslant n$.

COROLLARY 5.4. *Under the hypotheses of Theorem* 5.3, $\sigma(A) = \sigma_\varrho(A)$.

Proof. We have shown that $\sigma(A^*) = \sigma(A)^*$ (Proposition 2.3). Further, $A^* = (A_1^*, A_2^*, \ldots, A_n^*)$ is also an n-tuple of commuting normal operators. Thus, Theorem 5.3 gives $\sigma(A^*) = \sigma_\pi(A^*)$. But $\sigma_\pi(A^*) = \sigma_\pi(A)^*$ (Proposition 3.4). Therefore

$$\sigma(A) = \sigma(A^*)^* = \sigma_\pi(A^*)^* = \sigma_\varrho(A).$$

6. Toeplitz operators. As before, let Δ be the open unit disc in the complex plane C, and let ν be the normalized Lebesgue measure on the Borel subsets of the unit circle Γ. We denote by $P(\Gamma)$ the sup norm algebra of all functions on Γ that can be uniformly approximated by polynomials.

Let L^2 be the usual Hilbert space of ν-square-integrable functions on Γ. Further, let H^2 be the closure of $P(\Gamma)$ in L^2 and H^∞ the weak-star closure of $P(\Gamma)$ in L^∞ (the space of essentially-bounded ν-measurable functions in L^2). Equivalently, $H^\infty = H^2 \cap L^\infty$. If φ is an L^∞, we define a bounded operator L_φ on L^2 by

$$L_\varphi f = \varphi f \quad \text{for all } f \text{ in } L^2.$$

Let P be the orthogonal projection of L^2 onto H^2. Then for each φ in L^∞ we define the Toeplitz operators T_φ on H^2 by

$$T_\varphi f = PL_\varphi f \quad \text{for all } f \text{ in } H^2. \tag{2}$$

If φ is in H^∞, then T_φ is called analytic Toeplitz operator.

It is known that $\sigma(L_\varphi) \subset \sigma_\pi(T_\varphi)$ for all φ in L^∞ [10]. In the following section we generalize these ideas to an arbitrary finite collection of Toeplitz operators

THEOREM. (BROWN AND HALMOS). *If φ is in L^∞, then*

$$W^{*k} T_\varphi P W^k \to L_\varphi$$

in the strong operator topology, where W is the bilateral shift and P is the orthogonal projection of L^2 onto H^2.

Proof. ([2], Theorem 5).

Proceeding further we introduce the following notations. If $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ is an n-tuple of functions in L^∞ or H^∞, then we shall denote the n-tuple $(L_{\varphi_1}, L_{\varphi_2}, \ldots, L_{\varphi_n})$ of operators by L_φ and $(T_{\varphi_1}, T_{\varphi_2}, \ldots, T_{\varphi_n})$ by T_φ whenever there is no confusion.

THEOREM 6.1. *Let $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ be an n-tuple of functions in L^∞. Then the joint spectrum of L_φ is a subset of the joint approximate point spectrum of T_φ. In other words*

$$\sigma(L_\varphi) \subset \sigma_\pi(T_\varphi).$$

Proof. It is enough to show that if 0 is in the joint spectrum $\sigma(L_\varphi)$ of L_φ, then 0 is in the joint approximate point spectrum $\sigma_\pi(T_\varphi)$ of T_φ. We now let 0 be in $\sigma(L_\varphi)$. Then from Theorem 5.3 we have for every $\varepsilon > 0$ there exists a unit vector f_ε in H such that $\|L_{\varphi_i} f_\varepsilon\| < \varepsilon$ for all i, $1 \leqslant i \leqslant n$. Furthermore, from the previous theorem we have $W^{*k} PW^k f_\varepsilon \to f_\varepsilon$ and $W^{*k} T_{\varphi_i} PW^k f_\varepsilon \to L_{\varphi_i} f_\varepsilon$ for all i, $1 \leqslant i \leqslant n$. This implies that $\|PW^k f_\varepsilon\| \to 1$ and $\|T_{\varphi_i} PW^k f_\varepsilon\| \to 0$ for each i. Thus 0 is in $\sigma_\pi(T_\varphi)$.

The following notions and notations are needed for further discussions. For each f in H^2, let $\hat{f}$ be the analytic extension of f to the interior of the open unit disc Δ with square-summable Taylor series. Denote this class of functions by $\hat{H}^2$. Of course there is a natural isometry of H^2 onto $\hat{H}^2$. $\hat{H}^\infty$ is defined similarly. For each $\hat{\varphi}$ in H^∞, let $\hat{T}_\varphi$ be the replica of T_φ in $\hat{H}^2$. It is clear that $\hat{T}_\varphi \hat{f} = \hat{\varphi}\hat{f}$ for all $\hat{f}$ in $\hat{H}^2$ and for all $\hat{\varphi}$ in $\hat{H}^\infty$. If $\hat{\varphi}_1, \hat{\varphi}_2, \ldots, \hat{\varphi}_n$ are in $\hat{H}^\infty$, then we abbreviate $\hat{\varphi} = (\hat{\varphi}_1, \ldots, \hat{\varphi}_n)$ and $\hat{\varphi}(\Delta) = \{(\hat{\varphi}_1(z), \hat{\varphi}_2(z), \ldots, \hat{\varphi}_2(z)) : z \text{ in } \Delta\}$, which is a subset of $\boldsymbol{C}^n$. It is well known that $\sigma(T_\psi) = \overline{\hat{\psi}(\Delta)}$ for all ψ in H^∞ [9]. We generalize this to n-tuples.

THEOREM 6.2. *If $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ is an n-tuple of functions in H^∞, then the joint spectrum of T_φ is the closure of the image of the open unit disc Δ under the mapping $\hat{\varphi}$; that is,*

$$\sigma(T_\varphi) = \overline{\hat{\varphi}(\Delta)}.$$

Proof. If 0 is not in $\overline{\hat{\varphi}(\Delta)}$, then there is a $\delta > 0$ such that

$$\sum_{i=1}^{n} |\hat{\varphi}_i(z)| \geqslant \delta$$

for all z in Δ. Thus by the solution to the Corona problem [3] there exist functions $\psi_1, \psi_2, \ldots, \psi_n$ in H^∞ such that

$$\sum_{i=1}^{n} \psi_i \varphi_i = 1.$$

This implies that

$$\sum_{i=1}^{n} T_{\psi_i} T_{\varphi_i} = 1.$$

Hence 0 is not in $\sigma(T_\varphi)$. Therefore $\sigma(T_\varphi)$ is in the closure of $\hat{\varphi}(\Delta)$.

Conversely, if λ is a complex number with $|\lambda| < 1$, then it is clear that $\hat{f}(\lambda) = \langle \hat{f}, k_\lambda \rangle$ for all $\hat{f}$ in $\hat{H}^2$, where $k_\lambda = \dfrac{1}{1-\lambda^* z}$. For fixed λ, let $\alpha_i = \hat{\varphi}_i(\lambda)$ for each i, $1 \leqslant i \leqslant n$,

$$\begin{aligned}
\langle \hat{f}, (\hat{T}^*_{\varphi_i} - \alpha_i^*) k_\lambda \rangle &= \langle (\hat{T}_{\varphi_i} - \alpha_i)\hat{f}, k_\lambda \rangle \\
&= \langle (\hat{\varphi}_i - \alpha_i)\hat{f}, k_\lambda \rangle \\
&= \langle \hat{\varphi}_i \hat{f}, k_\lambda \rangle - \alpha_i \langle \hat{f}, k_\lambda \rangle \\
&= \hat{\varphi}_i(\lambda)\hat{f}(\lambda) - \varphi_i(\lambda)\hat{f}(\lambda) = 0
\end{aligned}$$

for all $\hat{f}$ in $\hat{H}^2$. Thus

$$(\hat{T}^*_{\varphi_i} - \alpha_i^*) k_\lambda = 0$$

for each i. This implies that $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is in the joint residual spectrum $\sigma_r(T_\varphi)$ of T_φ. Hence it follows that $\hat{\varphi}(\Delta) \subset \sigma(T_\varphi)$. But the joint spectrum of any commuting set of operators is a closed set. Therefore $\overline{\hat{\varphi}(\Delta)} \subset \sigma(T_\varphi)$. This proves the theorem.

In passing we have proved the following fact:

$$\overline{\sigma_r(T_\varphi)} = \sigma(T_\varphi).$$

There is still another way to express the result. Recall that if $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ is an n-tuple with φ_i in L^∞, the joint spectrum of L_φ is the joint essential range of φ (Theorem 5.2). Analogously, if each φ_i is in H^∞, then Theorem 6.2 says that the joint spectrum of T_φ is the joint essential range $\overline{\hat{\varphi}(\Delta)}$ of φ. Furthermore, $\hat{\varphi}$ is continuous (since each $\hat{\varphi}_i$ is continuous and Δ is connected. This implies that $\hat{\varphi}(\Delta)$ is connected, and hence $\overline{\hat{\varphi}(\Delta)}$ is connected). Thus it follows that the joint spectrum of an n-tuple of analytic Toeplitz operators is connected.

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Note added in proof. Since the time this paper was written for publication, there have been some works done by others on this subject. In particular, Problem 2 of this paper has been solved by J. Bunce, Proc. Amer. Math. Soc. 29 (1971), pp. 499–505; and also by W. Żelazko, Studia Math, this volume.

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