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Holomorphy types for open subsets of a Banach space

by

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Abstract. In an earlier article in this journal, S. Dineen considered the space of complex-valued entire functions on a complex Banach space E which were of α-holomorphy type. In this note, we extend this study to spaces of holomorphic functions of α-holomorphy type defined on an arbitrary open subset of E.

§ 1. Introduction. In [2] and [3], Dineen defined and studied the space of entire functions of α-holomorphy type θ from one complex Banach space E into another complex Banach space F. We extended these results in [1] to the study of holomorphic functions of α-holomorphy type θ defined on balanced open subsets of E. In this note, we extend the results of Dineen in a somewhat different way to the space of holomorphic functions of α-holomorphy type θ from an arbitrary open subset of E into F.

As is the case with Dineen's work, our definition of holomorphic functions of α-holomorphy type θ gives rise to a smaller, more manageable class of functions than that considered in [5]. In fact, it is unknown if there is an open subset of a Banach space for which our class is strictly smaller.

In [3], [4], [6], and [7], problems involving partial differential and convolution equations for entire functions on a Banach space were considered. Just as in the finite dimensional case, the analogous questions for holomorphic functions on open subsets of a Banach space are much harder, and will be discussed in a future note.

§ 2. Notation and terminology. Our notation will follow that of [3] and [5]. For convenience we give the following main definitions which will be required later.

Let U be an open subset of the complex Banach space E, and let F be a complex Banach space. A function $f\colon U \to F$ is holomorphic on U if for each $\xi \in U$, there is a sequence $\left\{\frac{\hat{d}^n f(\xi)}{n!}\right\}_{n\in N}$ of continuous n-homogeneous polynomials such that $f(x) = \sum_{n=0}^{\infty} \frac{\hat{d}^n f(\xi)}{n!}(x-\xi)$ uniformly in some neighborhood of ξ. $\mathscr{H}(U;F)$ denotes the space of holomorphic functions from U to F. The vector space of continuous n-homogeneous polynomials from E into F is denoted $\mathscr{P}(^nE;F)$, and becomes a Banach space with the norm $P \in \mathscr{P}(^nE;F) \to \|P\| = \sup_{\|x\|\leqslant 1} \|P(x)\|$.

Definition 2.1. (see [5], p. 34). A holomorphy type θ from E to F is a sequence of Banach spaces $(\mathscr{P}_\theta(^nE;F), \|\ \|_\theta)$, for $n \in N$, such that the following hold:

i) Each $\mathscr{P}_\theta(^nE;F)$ is a vector subspace of $\mathscr{P}(^nE;F)$.

ii) $\mathscr{P}_\theta(^0E;F)$ coincides with $\mathscr{P}(^0E;F)$, the vector space of constant mappings from E to F as a normed vector space.

iii) There is a real number $\sigma = \sigma_\theta \geqslant 1$ for which the following is true: Given any m and $n \in \boldsymbol{N}$, $m \leqslant n$, $x \in E$, and $P \in \mathscr{P}_\theta(^nE;F)$, we have

$$\hat{d}^m P(x) \in \mathscr{P}_\theta(^mE;F)$$

and

$$\frac{1}{m!}\|\hat{d}^m P(x)\|_\theta \leqslant \sigma^n \|P\|_\theta \|x\|^{n-m}.$$

This yields the following natural definition of the space of holomorphic functions from U into F of holomorphy type θ.

Definition 2.2. Let $f \in \mathscr{H}(U;F)$. f is of holomorphy type θ at $\xi \in U$ if there exist real numbers $C \geqslant 0$ and $c \geqslant 0$ such that the following hold:

i) $\hat{d}^m f(\xi) \in \mathscr{P}_\theta(^mE;F)$, for all $m \in \boldsymbol{N}$.

ii) $\left\|\frac{\hat{d}^m f(\xi)}{m!}\right\|_\theta \leqslant Cc^m$, for all $m \in \boldsymbol{N}$.

$\mathscr{H}_\theta(U;F)$ is the set of functions $f \in \mathscr{H}(U;F)$ which are of holomorphy type θ at each point of U. The current holomorphy type θ is the holomorphy type for which $(\mathscr{P}_\theta(^mE;F), \|\ \|_\theta) = (\mathscr{P}(^mE;F), \|\ \|)$ for all $m \in \boldsymbol{N}$. In this case $\mathscr{H}_\theta(U;F) = \mathscr{H}(U;F)$.

It is easy to see that the topology τ_0 of uniform convergence on compact sets is not the largest natural locally convex topology on $\mathscr{H}_\theta(U;F)$, when E is infinite dimensional and $F \neq 0$. Definition 2.2 leads to a natural definition of a much larger topology.

Definition 2.3. A seminorm p on $\mathscr{H}_\theta(U;F)$ is said to be ported by a compact set $K \subseteq U$ if either of the following equivalent conditions holds:

i) For any real number $\varepsilon > 0$, there is a $c(\varepsilon) > 0$ such that

$$p(f) \leqslant c(\varepsilon) \sum_{m=0}^{\infty} \varepsilon^m \sup_{x\in K} \left\|\frac{\hat{d}^m f(x)}{m!}\right\|_\theta,$$

for every $f \in \mathscr{H}_\theta(U;F)$.

ii) For any real number $\varepsilon > 0$ and for any open set V with $K \subset V \subset U$, there is a $c(\varepsilon, V) > 0$ such that

$$p(f) \leqslant c(\varepsilon, V) \sum_{m=0}^{\infty} \varepsilon^m \sup_{x\in V} \left\|\frac{\hat{d}^m f(x)}{m!}\right\|_\theta$$

for every $f \in \mathscr{H}_\theta(U;F)$.

When θ is the current holomorphy type, the following condition is equivalent to the two above:

iii) For any open set V, $K \subset V \subset U$, there is a $c(V) > 0$ such that

$$p(f) \leqslant c(V) \sup_{x\in V} \|f(x)\|,$$

for every $f \in \mathscr{H}(U;F)$.

The topology $\tau_{\omega\theta}$ on $\mathscr{H}_\theta(U;F)$ is the locally convex topology generated by all seminorms ported by some compact set $K \subseteq U$; when θ is the current holomorphy type, the topology is denoted by τ_ω.

We note that there is no known example of a $\tau_{\omega\theta}$-continuous seminorm on $\mathscr{H}_\theta(U;F)$ which is not dominated by one of the type

$$f \in \mathscr{H}_\theta(U;F) \to p_{K,\{\alpha_n\}}(f) = \sum_{n=0}^{\infty} \alpha_n^n \sup_{x\in K} \left\|\frac{\hat{d}^n f(x)}{n!}\right\|_\theta,$$

where $K \subseteq U$ is compact, and $\{\alpha_n\} \in c_0^+$, the set of non-negative sequences of real numbers tending to 0. The locally convex topology generated by the family $p_{K,\{\alpha_n\}}$, with K and $\{\alpha_n\}$ varying as described above, is denoted $\tau_{\sigma\theta}$; when θ is the current holomorphy type, we denote this topology by τ_σ.

Proposition 2.4. *$(\mathscr{H}_\theta(U;F), \tau_{\sigma\theta})$ is complete.*

Proof. We will prove the proposition in the case when θ is the current holomorphy type, which is the only case we will use; the proof for arbitrary holomorphy types θ is slightly more complicated.

Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a Cauchy net in $(\mathscr{H}(U;F), \tau_\sigma)$. Since $\mathscr{H}(U;F)$ is closed in the complete space $C(U;F)$ of continuous F-valued functions on U (by Proposition 4, Section 6 of [5]), it follows that there is a function $f \in \mathscr{H}(U;F)$ such that $f_\lambda \to f$ for the topology τ_0. By Proposition 2, Section 6 of [5], we have that $\frac{\hat{d}^n f_\lambda(x)}{n!} \to \frac{\hat{d}^n f(x)}{n!}$ uniformly on compact subsets of U. From this, it is immediate that $f_\lambda \to f$ in τ_σ, and the proof is complete.

The space $\mathscr{P}_\theta(^nE;F)$ is called intrinsic if the algebraic and topological definition of $\mathscr{P}_\theta(^nE;F)$ depends only on the algebraic and topological definition of E. If U is the unit ball for an equivalent norm on E, we denote by $\| \|_{\theta,U}$ the norm on $\mathscr{P}_\theta({}^nE;F)$ corresponding to U. It will be convenient for us to use various equivalent norms on E, whose unit balls will often be expressed in terms of the unit ball B_1 for a fixed norm on E. In [3], Dineen notes that every known natural example of holomorphy type also satisfies the following definition.

Definition 2.5. An α-holomorphy type is a holomorphy type θ which satisfies the following conditions:

i) For each n, the space $(\mathscr{P}_\theta({}^nE;F), \| \|_\theta)$ is intrinsic, and the constant σ in Definition 2.1 is independent of the choice of equivalent norm on E.

ii) If U and V are two open, bounded, convex, balanced sets, and $c \geqslant 0$ is such that $cU \subset V$, then for all $n \in \boldsymbol{N}$ and all $P_n \in \mathscr{P}_\theta({}^nE;F)$

$$c^n \|P_n\|_{\theta,U} \leqslant \|P_n\|_{\theta,V}.$$

For the remainder of this article, every holomorphy type θ will be assumed to be of α-holomorphy type. We will also assume that all holomorphic functions are complex-valued, and will denote $\mathscr{H}_\theta(U;\boldsymbol{C})$ simply by $\mathscr{H}_\theta(U)$. There is no difficulty in generalizing all results to arbitrary Banach space-valued mappings.

§ 3. The space $(H_\theta(U), \tau_\theta)$. Because of the difficulty in proving certain basic properties of $(\mathscr{H}_\theta(E), \tau_{\omega\theta})$ and exhibiting a generating family of seminorms for the topology $\tau_{\omega\theta}$, as well as solving differential and convolution equations in the space $\mathscr{H}_\theta(E)$, Dineen in [3] and Nachbin and Gupta in [7] were led to consider new definitions of entire functions of θ-type and the corresponding locally convex topology. We extend these definitions here to holomorphic functions of θ-type on arbitrary open sets.

Let U be an open subset of E, and let θ be an α-holomorphy type. For a point $\xi \in U$, let $\mathscr{K}(\xi, U)$ denote the family of all compact, convex, ξ-equilibrated subsets of U.

Definition 3.1 (a). The space $H_\theta(U)$ of complex-valued holomorphic functions of α-holomorphy type θ in U is the set of functions f mapping U into $\boldsymbol{C}$ which satisfy the following conditions:

i) $f \in \mathscr{H}(U)$.

ii) For all $\xi \in U$, and for all $n \in \boldsymbol{N}$, $\hat{d}^n f(\xi) \in \mathscr{P}_\theta({}^nE)$.

iii) For all $\xi \in U$ and for all $K \in \mathscr{K}(\xi, U)$, there is an $\varepsilon > 0$ such that

$$\sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(\xi)}{n!} \right\|_{\theta, K-\xi+\varepsilon B_1} < \infty.$$

(b) The topology τ_θ on $H_\theta(U)$ is the locally convex topology generated by all seminorms p on $H_\theta(U)$ which satisfy the following condition for some $\xi \in U$ and $K \in \mathscr{K}(\xi, U)$:

For all $\varepsilon > 0$, there is $c(\varepsilon) > 0$ such that

$$p(f) \leqslant c(\varepsilon) \sum_{m=0}^{\infty} \left\| \frac{\hat{d}^m f(\xi)}{m!} \right\|_{\theta, K-\xi+\varepsilon B_1}$$

for all $f \in H_\theta(U)$.

We shall sometimes say that p is $\theta-\xi-K$ ported if p depends on ξ and K in the above sense.

We remark that when θ is the current holomorphy type, $H_\theta(U) = \mathscr{H}(U)$. Indeed, it is clear from Definition 3.1 that $H_\theta(U) \subseteq \mathscr{H}(U)$. To show the converse inclusion, we let $f \in \mathscr{H}(U)$ be arbitrary, and let $\xi \in U$ and $K \in \mathscr{K}(\xi, U)$. Let $r > 1$ be any number such that for all $\lambda \in \boldsymbol{C}$, $|\lambda| \leqslant r$, $\xi + \lambda(K-\xi) \subseteq U$. Since $\xi + r(K-\xi)$ is compact, there is some $\varepsilon > 0$ such that f is bounded on $\xi + r(K-\xi) + \varepsilon B_1$, say by $M > 0$. By Proposition 2, Section 6 of [5], for all $n \in \boldsymbol{N}$,

$$\left\| \frac{\hat{d}^n f(\xi)}{n!} \right\|_{K-\xi+\frac{\varepsilon}{r}B_1} = \sup_{x \in K-\xi+\frac{\varepsilon}{r}B_1} \left| \frac{1}{2\pi i} \int_{|\lambda|=r} \frac{f(\xi+\lambda x)}{\lambda^{n+1}} d\lambda \right|$$
$$\leqslant \frac{1}{r^n} \sup_{y \in \xi + r(K-\xi)+\varepsilon B_1} |f(y)| \leqslant \frac{1}{r^n} M.$$

Applying Definition 3.1, we see that $H_\theta(U) = \mathscr{H}(U)$.

The following proposition shows that, in fact, Definition 3.1 yields exactly the same space as in [3] when $U = E$ and θ is an arbitrary α-holomorphy type.

Proposition 3.2 a) *A function $f \in H_\theta(E)$ if and only if for all $n \in \boldsymbol{N}$, $\hat{d}^n f(0) \in \mathscr{P}_\theta({}^nE)$, and for all $K \in \mathscr{K}(0, E)$, there is a number $\varepsilon > 0$ such that*

$$\sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{0, K+\varepsilon B_1} < \infty.$$

b) *The topology τ_θ on $H_\theta(E)$ is generated by all seminorms p on $H_\theta(E)$ which are $\theta-0-K$ ported, for some $K \in \mathscr{K}(0, E)$.*

Proof. The conditions in a) and b) are clearly necessary. Conversely, let f satisfy the condition given in a). Then by Lemma 6 of [3], $f \in \mathscr{H}(E)$. Let $\xi \in E$ be arbitrary, and let K be any set in $\mathscr{K}(\xi, E)$. By Proposition 12 of [3], $\hat{d}^n f(\xi) \in \mathscr{P}_\theta({}^nE)$ for each $n \in \boldsymbol{N}$. Let L be any set in $\mathscr{K}(0, E)$ such that $\xi \in L$ and $K-\xi \subset L$. By hypothesis, there is some number $\varepsilon > 0$

such that

$$\sum_{m=0}^{\infty}\left\|\frac{\hat{d}^m f(0)}{m!}\right\|_{\theta,2\sigma L+2\sigma\varepsilon B_1}<\infty.$$

By Corollary 2, Section 10, of [5],

$$\sum_{j=0}^{\infty}\left\|\frac{\hat{d}^j f(\xi)}{j!}\right\|_{\theta,K-\xi+\varepsilon B_1}\leqslant\sum_{m=0}^{\infty}\left\|\frac{\hat{d}^m f(0)}{m!}\right\|_{\theta,2\sigma L+2\sigma\varepsilon B_1}<\infty,$$

since $\|\xi\|_{L+\varepsilon B_1}<1$. This proves that $f\in H_\theta(E)$, completing the proof of a).

Now, let p be any τ_θ-continuous seminorm on $H_\theta(E)$. Suppose that p is $\theta-\xi-K$ ported. Let L be chosen as above and let $\varepsilon>0$ be arbitrary. Then, as above, there is a number $c\left(\frac{\varepsilon}{2\sigma}\right)\geqslant 0$ such that for all $f\in H_\theta(E)$,

$$\begin{aligned}p(f)&\leqslant c\left(\frac{\varepsilon}{2\sigma}\right)\sum_{j=0}^{\infty}\left\|\frac{\hat{d}^j f(\xi)}{j!}\right\|_{\theta,K-\xi+\frac{\varepsilon}{2\sigma}B_1}\\&\leqslant c\left(\frac{\varepsilon}{2\sigma}\right)\sum_{m=0}^{\infty}\left\|\frac{\hat{d}^m f(0)}{m!}\right\|_{\theta,2\sigma L+\varepsilon B_1}.\end{aligned}$$

Hence, every τ_θ—seminorm p on $H_\theta(E)$ is $\theta-0-2\sigma L$ ported, for some compact, convex, equilibrated set L, proving b).

The following Theorem motivates our characterization of the topology τ_θ.

THEOREM 3.3. *Let $f\in\mathscr{H}(U)$ and suppose that for all $\xi\in U$ and for all $n\in\boldsymbol{N}$, $d^n f(\xi)\in\mathscr{P}_\theta({}^nE)$. Then, the following conditions are equivalent:*

i) $f\in H_\theta(U)$.

ii) *For all $\xi\in U$, for all $K\in\mathscr{K}(\xi,U)$, and for all $\{\alpha_n\}_{n\in\boldsymbol{N}}\in c_0^+$,*

$$\sum_{n=0}^{\infty}\left\|\frac{\hat{d}^n f(\xi)}{n!}\right\|_{\theta,K-\xi+\alpha_n B_1}<\infty.$$

iii) *For all $\xi\in U$, for all $K\in\mathscr{K}(\xi,U)$, and for all $\{\alpha_n\}_{n\in\boldsymbol{N}}\in c_0^+$,*

$$\limsup_n\left\|\frac{\hat{d}^n f(\xi)}{n!}\right\|^{1/n}_{\theta,K-\xi+\alpha_n B_1}<1.$$

Proof. The proof of i) $\to$ ii) is immediate.

Assume that ii) is true, and let ξ, K, and $\{\alpha_n\}$ be as in iii). There is a number $r>1$ such that $r(K-\xi)\subset U-\xi$. Let $L=r(K-\xi)+\xi$; $L\in\mathscr{K}(\xi,U)$. By ii),

$$\sum_{n=0}^{\infty}\left\|\frac{\hat{d}^n f(\xi)}{n!}\right\|_{\theta,L-\xi+r\alpha_n B_1}<\infty.$$

Therefore

$$\limsup_n\left\|\frac{\hat{d}^n f(\xi)}{n!}\right\|^{1/n}_{\theta,K-\xi+\alpha_n B_1}\leqslant 1/r<1.$$

Hence, iii) holds.

Finally, if i) is false, then there is a $\xi\in U$ and a set $K\in\mathscr{K}(\xi,U)$ such that for all $n=1,2,\ldots,$

$$\sum_{m=0}^{\infty}\left\|\frac{\hat{d}^m f(\xi)}{m!}\right\|_{\theta,K-\xi+\frac{1}{n}B_1}=\infty.$$

Hence we can choose an increasing sequence of integers $\{m_k\}$ such that for each $k\in\boldsymbol{N}$,

$$\left\|\frac{\hat{d}^{m_k} f(\xi)}{m_k!}\right\|^{1/m_k}_{\theta,K-\xi+\frac{1}{k}B_1}\geqslant 1-\frac{1}{k}.$$

Letting $\{\alpha_n\}\in c_0^+$ be defined by

$$\alpha_m=\begin{cases}1 & \text{if } m\leqslant m_1,\\ \dfrac{1}{k} & \text{if } m_{k-1}<m\leqslant m_k,\end{cases}$$

we get

$$\limsup_m\left\|\frac{\hat{d}^m f(\xi)}{m!}\right\|^{1/m}_{\theta,K-\xi+\alpha_m B_1}\geqslant 1,$$

contradicting iii). Hence, the theorem is proved.

The following two lemmas will be used to produce the promised characterization of the topology τ_θ on $H_\theta(U)$. The proofs for both are straightforward and are omitted.

LEMMA 3.4. *Let $f\in H_\theta(U)$, where $0\in U$. Let $K\in\mathscr{K}(0,U)$. Let q be any τ_θ—continuous seminorm on $H_\theta(U)$ which is $\theta-0-K$ ported. Then the Taylor series of f at 0 converges to f with respect to q; that is, $q\left(f-\sum_{j=0}^{m}\frac{d^j f(0)}{j!}\right)\to 0$ as $m\to\infty$.*

LEMMA 3.5. *Let $\xi\in U$. Let $V=U-\xi$. Then the mapping $\tau_{-\xi}$: $H_\theta(U)\to H_\theta(V)$ defined by $\tau_{-\xi}f(x)=f(x+\xi)$, for $f\in H_\theta(U)$ and $x\in V$, is a topological isomorphism of $(H_\theta(U),\tau_\theta)$ onto $(H_\theta(V),\tau_\theta)$.*

THEOREM 3.6. *The topology τ_θ on $H_\theta(U)$ is generated by all seminorms p on $H_\theta(U)$, of the form*

$$p(f) = \sum_{m=0}^{\infty} \left\| \frac{\hat{d}^m f(\xi)}{m!} \right\|_{\theta, K-\xi+\alpha_m B_1},$$

where $\xi \in U$, $K \in \mathscr{K}(\xi, U)$, and $\{\alpha_n\}_{n \in N} \in c_0^+$.

Proof. By Theorem 3.3, p is a seminorm on $H_\theta(U)$. The fact that every such seminorm p is τ_θ-continuous follows from the definition of α-holomorphy type.

Conversely, let p_1 be a τ_θ-continuous seminorm on $H_\theta(U)$, which is $\theta-\xi-K$ ported. Set $V = U-\xi$. Define the seminorm q on $H_\theta(V)$ by $q(g) = p_1(f)$, for all $g \in H_\theta(V)$, where $f \in H_\theta(U)$ is the unique function such that $\tau_{-\xi} f = g$, by Lemma 3.5. Further, Lemma 3.5 implies that q is τ_θ-continuous on $H_\theta(V)$, since it is $\theta-0-(K-\xi)$ ported.

It follows that for each $\varepsilon > 0$, there is $c(\varepsilon) > 0$ such that for all $n \in N$ and $P_n \in \mathscr{P}_\theta({}^nE)$, $q(P_n) \leqslant c(\varepsilon)\|P_n\|_{\theta, K-\xi+\varepsilon B_1}$. As in [3], for each $n = 0, 1, \ldots$, and $\varepsilon > 0$, define $K_n(\varepsilon)$ to be the smallest number such that

$$q(P_n) \leqslant K_n(\varepsilon)\|P_n\|_{\theta, K-\xi+\varepsilon B_1}.$$

Let now $r > 1$ be such that $r(K-\xi) \subseteq V$. Since $\limsup\limits_{n\to\infty} K_n(\varepsilon)^{1/n} \leqslant 1$ for each $\varepsilon > 0$, we can choose an increasing sequence of integers $\{n_k\}$ such that $K_n\left(\frac{1}{k}\right)^{1/n} \leqslant r$, if $n \geqslant n_k$ for each $k \in N$.

Define $\{\alpha_n\} \in c_0^+$ by

$$\alpha_n = \begin{cases} 1 & \text{if } n \leqslant n_1, \\ \dfrac{1}{k} & \text{if } n_k \leqslant n < n_{k+1}, \quad \text{for } k \geqslant 1. \end{cases}$$

Therefore for some $C \geqslant 0$, $K_n(\alpha_n) \leqslant Cr^n$, for all $n \in N$. By Lemma 3.4 for all $f \in H_\theta(U)$,

$$\begin{aligned} p_1(f) = q(g) &\leqslant \sum_{n=0}^{\infty} q\left(\frac{\hat{d}^n g(0)}{n!}\right) \\ &\leqslant C \sum_{n=0}^{\infty} r^n \left\| \frac{\hat{d}^n g(0)}{n!} \right\|_{\theta, K-\xi+\alpha_n B_1} \\ &= C \sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(\xi)}{n!} \right\|_{\theta, L-\xi+r\alpha_n B_1}, \end{aligned}$$

where $L = r(K-\xi)+\xi \in \mathscr{K}(\xi, U)$. This completes the proof of the Theorem.

We consider now the relationship between the space $(\mathscr{H}_\theta(U), \tau_{\omega\theta})$ defined in Section 2 and $(H_\theta(U), \tau_\theta)$.

PROPOSITION 3.7. *Let $U \subseteq E$ be open. Then $H_\theta(U) \subset \mathscr{H}_\theta(U)$.*

Proof. Let $f \in H_\theta(U)$ and let $\xi \in U$. Let $K = \{\xi\} \in \mathscr{K}(\xi, U)$. Then for some number $\varepsilon > 0$,

$$\sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(\xi)}{n!} \right\|_{\theta, \varepsilon B_1} = \sum_{n=0}^{\infty} \varepsilon^n \left\| \frac{\hat{d}^n f(\xi)}{n!} \right\|_{\theta, B_1} < \infty,$$

from which the result follows.

In fact, there is no known example of an α-holomorphy type θ and an open set U for which the inclusion in Proposition 3.7 is strict. The question of whether the inclusion in this proposition is continuous is discussed below. We first require the following technical lemma.

LEMMA 3.8. *Let $U \subset E$ be open, $\xi \in U$, and $K \in \mathscr{K}(\xi, U)$ such that $2\sigma(K-\xi) \subset U-\xi$, where $\sigma = \sigma_\theta$ as in Definition 2.1. Let p be a $\tau_{\omega\theta}$-continuous seminorm on $\mathscr{H}_\theta(U)$ which is K-ported. Then p is continuous for $(H_\theta(U), \tau_\theta)$.*

Proof. Our proof is similar to Proposition 12 of [3]. We shall show that p is $\theta-\xi-L$ ported on $H_\theta(U)$, where $L = 2\sigma(K-\xi)+\xi$. Let $f \in H_\theta(U)$ be arbitrary, and define $g = \tau_{-\xi} f \in H_\theta(U-\xi)$, by Lemma 3.5. Choose $\varepsilon_0 > 0$ so small that $K-\xi+\varepsilon_0 B_1 \subseteq U-\xi$ and let $\varepsilon > 0$, $\varepsilon < \varepsilon_0$ be arbitrary. Set $r = \sup\limits_{x \in K-\xi} \|x\|_{2(K-\xi)+2\varepsilon B_1}$. By Corollary 1, Section 10 of [5], for all $x \in K-\xi$,

$$\left\| \frac{\hat{d}^m g(x)}{m!} \right\|_{\theta, 2(K-\xi)+2\varepsilon B_1} \leqslant \sum_{n=m}^{\infty} r^{n-m} \left\| \frac{\hat{d}^n g(0)}{n!} \right\|_{\theta, 2\sigma(K-\xi)+2\sigma\varepsilon B_1}.$$

Therefore since p is K-ported, there is $c(\varepsilon) > 0$ such that for all $f \in H_\theta(U)$,

$$\begin{aligned} p(f) &\leqslant c(\varepsilon) \sum_{m=0}^{\infty} \varepsilon^m \sup_{x \in K} \left\| \frac{\hat{d}^m f(x)}{m!} \right\|_\theta \\ &\leqslant c(\varepsilon) \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^m \sup_{x \in K-\xi} \left\| \frac{\hat{d}^m g(x)}{m!} \right\|_{\theta, 2(K-\xi)+2\varepsilon B_1} \\ &\leqslant c(\varepsilon) \sum_{n=0}^{\infty} r^n \sum_{m=0}^{n} \left(\frac{1}{2r}\right)^m \left\| \frac{\hat{d}^n g(0)}{n!} \right\|_{\theta, 2\sigma(K-\xi)+2\sigma\varepsilon B_1} \\ &\leqslant \frac{2}{1-2r} c(\varepsilon) \cdot \sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(\xi)}{n!} \right\|_{\theta, L-\xi+2\sigma\varepsilon B_1}. \end{aligned}$$

It follows that p is τ_θ-continuous on $H_\theta(U)$, which completes the proof.

PROPOSITION 3.9. *$(H_0(U), \tau_0) \to (\mathscr{H}_0(U), \tau_{\sigma 0})$ is continuous.*

Proof. Let p be a $\tau_{\sigma 0}$-continuous seminorm on $\mathscr{H}_0(U)$; say

$$p(f) = \sum_{m=0}^{\infty} \sup_{x\in K} \left\| \frac{\hat{d}^m f(x)}{m!} \right\|_{0, a_m B_1},$$

where $\{a_m\} \in c_0^+$ and $K \subseteq U$ is compact. By compactness of K, we get a finite cover $\{B_{s_i}(\xi_i): i = 1, \ldots, k\}$ of K such that $2\sigma\,\overline{(B_{s_i}(\xi_i) - \xi_i)} \subseteq \overline{U - \xi_i}$ for each i. Let K_i be the convex, ξ_i-equilibrated, compact hull of $K \cap \overline{B_{s_i}(\xi_i)}$. For each $f \in H_0(U)$,

$$p(f) \leqslant \sum_{i=1}^{k} \sum_{m=0}^{\infty} \sup_{x\in K_i} \left\| \frac{\hat{d}^m f(x)}{m!} \right\|_{0, a_m B_1} = \sum_{i=1}^{k} p_i(f),$$

where

$$p_i(f) = \sum_{m=0}^{\infty} \sup_{x\in K_i} \left\| \frac{\hat{d}^m f(x)}{m!} \right\|_{0, a_m B_1}, \quad \text{for } i = 1, \ldots, k.$$

By Lemma 3.8, each p_i is τ_0-continuous on $H_0(U)$, which proves the Proposition.

Proposition 3.9 can be improved for certain important α-holomorphy types θ, provided restrictions are made on the open set U. Two such extensions are described below.

PROPOSITION 3.10. *Let U be ξ-equilibrated for some $\xi \in E$, and let θ be the current holomorphy type. Then the injection $(H(U), \tau_\theta) \to (H(U), \tau_\omega)$ is continuous.*

The proof of Proposition 3.10 is similar to the proof of Theorem 3.6, and is omitted.

PROPOSITION 3.11. *Let $U \subseteq E$ be open, convex, and ξ-equilibrated for some $\xi \in E$. Let $\theta = N$ be the nuclear holomorphy type. (see, for example, [6]). Then the injection mapping $(H_N(U), \tau_N) \to (\mathscr{H}_N(U), \tau_{\omega N})$ is continuous.*

Proof. Without loss of generality, $\xi = 0$. Let p be a $\tau_{\omega N}$-continuous seminorm on $\mathscr{H}_N(U)$ which we may assume to be K-ported for some $K \in \mathscr{K}(0, U)$. Let $\varepsilon > 0$ be arbitrary. Thus, for each $\delta > 0$, there is a number $c_\varepsilon(\delta) > 0$ such that for all $f \in \mathscr{H}_N(U)$,

$$p(f) \leqslant c_\varepsilon(\delta) \sum_{n=0}^{\infty} \delta^n \sup_{x\in K} \left\| \frac{\hat{d}^n f(x)}{n!} \right\|_{N, K+\varepsilon B_1}$$

(where we are letting $K + \varepsilon B_1$ be the unit ball of an equivalent norm on E).

For some $r > 1$, $rK \subseteq U$. Let $t > 0$ be arbitrary, such that $t \leqslant 1 - \frac{1}{r}$. Then, for each $f \in \mathscr{H}_N(U)$,

$$p(f) \leqslant c_t(rt) \sum_{n=0}^{\infty} t^n \sup_{y\in K} \left\| \frac{\hat{d}^n f(y)}{n!} \right\|_{N, rK+rtB_1}$$

$$\leqslant c_t(rt) \sum_{n=0}^{\infty} t^n \sup_{y\in K} \sum_{m=n}^{\infty} \left\| \frac{\hat{d}^n P_m(y)}{n!} \right\|_{N, rK+rtB_1},$$

where $P_m = \dfrac{\hat{d}^m f(0)}{m!}$, by Lemma 2, Section 11 of [5],

$$\leqslant c_t(rt) \sum_{n=0}^{\infty} t^n \sum_{m=n}^{\infty} \sup_{y\in K} \binom{m}{n} \|P_m\|_{N, rK+rtB_1} \|y\|^{m-n}_{rK+rtB_1},$$

by Lemma 7 of [4],

$$\leqslant c_t(rt) \sum_{m=0}^{\infty} \|P_m\|_{N, rK+rtB_1}.$$

This proves the proposition.

The following result extends Proposition 13 of [3]. The proof is similar to that given in [3] and is omitted.

PROPOSITION 3.12. *Let θ be the current holomorphy type and let U be an arbitrary open set in E. Then the identity mapping $(H(U), \tau_\omega) \to (H(U), \tau_\theta)$ is continuous.*

COROLLARY 3.13. *When θ is the current holomorphy type and U is ξ-equilibrated for some $\xi \in E$, then $\tau_\omega = \tau_\theta$. In fact, a generating family of seminorms for τ_ω consists of the set*

$$f \in H(U) \to \sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(\xi)}{n!} \right\|_{K-\xi+a_n B_1},$$

where $K \in \mathscr{K}(\xi, U)$ and $\{a_n\} \in c_0^+$.

§ 4. Topological properties of $(H_0(U), \tau_0)$.

THEOREM 4.1. *$(H_0(U), \tau_0)$ is complete.*

Proof. Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a Cauchy net in $(H_0(U), \tau_0)$. Then, by Proposition 3.9 above and Proposition 1, Section 9 of [5], $\{f_\lambda\}$ is a Cauchy net in $(H(U), \tau_\sigma)$. By Proposition 2.4, $f_\lambda \to f$ in the space $(H(U), \tau_\sigma)$ for some function $f \in H(U)$. Further, for each $n \in N$ and $\xi \in U$, the net $\{\hat{d}^n f_\lambda(\xi)\}_{\lambda\in\Lambda}$ is Cauchy in $\mathscr{P}_0(^nE)$ and so it converges to an element $P \in \mathscr{P}_0(^nE)$. Since the inclusion mapping $(\mathscr{P}_\theta({}^nE), \|\ \|_\theta) \to (\mathscr{P}({}^nE), \|\ \|)$ is continuous, it follows that $\hat{d}^n f(\xi) = P \in \mathscr{P}_\theta({}^nE)$.

Now, let p be a τ_θ-continuous seminorm on $H_\theta(U)$; suppose that p is of the form

$$p(g) = \sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n g(\xi)}{n!} \right\|_{\theta, K-\xi+\alpha_n B_1},$$

where $\xi \in U$, $K \in \mathscr{K}(\xi, U)$, and $\{\alpha_n\}_{n\in\boldsymbol{N}} \in c_0^+$. Let $\varepsilon > 0$. For some $\lambda_0 \in \Lambda$, we have that $p(f_\beta - f_\gamma) < \varepsilon$ whenever $\beta, \gamma \geqslant \lambda_0$. From this it follows that

$$(*) \qquad \sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n (f-f_\lambda)(\xi)}{n!} \right\|_{\theta, K-\xi+\alpha_n B_1} \leqslant \varepsilon \quad \text{if} \quad \lambda \geqslant \lambda_0$$

and

$$(**) \qquad \sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(\xi)}{n!} \right\|_{\theta, K-\xi+\alpha_n B_1} \leqslant \varepsilon + \sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f_{\lambda_0}(\xi)}{n!} \right\|_{\theta, K-\xi+\alpha_n B_1} < \infty.$$

Using $(**)$ and Theorem 3.3, it follows that $f \in H_\theta(U)$. By $(*)$, we have that $p(f-f_\lambda) \leqslant \varepsilon$ if $\lambda \geqslant \lambda_0$. Thus $f_\lambda \to f$ in $(H_\theta(U), \tau_\theta)$ and the theorem is proved.

We now characterize the bounded subsets of $(H_\theta(U), \tau_\theta)$, extending Proposition 1, Section 12 of [5] when θ is the current holomorphy type.

THEOREM 4.2. *Let $\mathfrak{X}$ be a subset of $(H_\theta(U), \tau_\theta)$. Then, the following conditions are equivalent:*

i) *$\mathfrak{X}$ is bounded in $(H_\theta(U), \tau_\theta)$.*

ii) *For each $\xi \in U$, $K \in \mathscr{K}(\xi, U)$, and $\{\alpha_n\}_{n\in\boldsymbol{N}} \in c_0^+$, there are constants $C \geqslant 0$, $c \geqslant 0$, $c < 1$, such that*

$$\left\| \frac{\hat{d}^n f(\xi)}{n!} \right\|_{\theta, K-\xi+\alpha_n B_1} \leqslant Cc^n,$$

for all $n \in \boldsymbol{N}$ and all $f \in \mathfrak{X}$.

iii) *For each $\xi \in U$, $K \in \mathscr{K}(\xi, U)$, there is a number $t > 0$, and constants $C \geqslant 0$, $c \geqslant 0$, $c < 1$, such that*

$$\left\| \frac{\hat{d}^n f(\xi)}{n!} \right\|_{\theta, K-\xi+tB_1} \leqslant Cc^n,$$

for all $n \in \boldsymbol{N}$ and all $f \in \mathfrak{X}$.

Proof. i) $\Rightarrow$ ii). Let ξ, K, and $\{\alpha_n\}$ be chosen as in ii). For some $r > 1$, $r(K-\xi) \subset U - \xi$. By i), there is a number $M < \infty$ such that

$$\sup_{f\in\mathfrak{X}} \sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(\xi)}{n!} \right\|_{\theta, r(K-\xi)+r\alpha_n B_1} \leqslant M.$$

Taking $C = M$ and $c = 1/r$, ii) follows.

ii) $\Rightarrow$ iii). If iii) were not true, then for some ξ and K as in iii),

$$\limsup_n \left\{ \sup_{f\in\mathfrak{X}} \left\| \frac{\hat{d}^n f(\xi)}{n!} \right\|^{1/n}_{\theta, K-\xi+tB_1} \right\} \geqslant 1,$$

for each $t > 0$. It follows that for each $k = 1, 2, \ldots$, there is an integer $n_k > n_{k-1}$ such that

$$\sup_{f\in\mathfrak{X}} \left\| \frac{\hat{d}^{n_k} f(\xi)}{n_k!} \right\|^{1/n_k}_{\theta, K-\xi+\frac{1}{k}B_1} \geqslant 1 - \frac{1}{k}.$$

Define $\{\alpha_n\}_{n\in\boldsymbol{N}} \in c_0^+$ by

$$\alpha_n = \begin{cases} 1 & \text{if } n < n_1, \\ 1/k & \text{if } n_k \leqslant n < n_{k+1}. \end{cases}$$

Then, ii) fails since

$$\limsup_n \left\{ \sup_{f\in\mathfrak{X}} \left\| \frac{\hat{d}^n f(\xi)}{n!} \right\|^{1/n}_{\theta, K-\xi+\alpha_n B_1} \right\} \geqslant 1.$$

Hence, ii) implies iii).

iii) $\Rightarrow$ i). Let p be a τ_θ-continuous seminorm on $H_\theta(U)$. Suppose that p is $\theta-\xi-K$ ported. By iii), there are numbers $t > 0$, $C \geqslant 0$, and $c \geqslant 0$, $c < 1$, such that

$$\left\| \frac{\hat{d}^n f(\xi)}{n!} \right\|_{\theta, K-\xi+tB_1} \leqslant Cc^n,$$

for all $n \in \boldsymbol{N}$ and $f \in \mathfrak{X}$. Also, there is a constant $c(t) \geqslant 0$ such that

$$p(f) \leqslant c(t) \sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(\xi)}{n!} \right\|_{\theta, K-\xi+tB_1},$$

for all $f \in H_\theta(U)$. Hence, for all $f \in \mathfrak{X}$,

$$p(f) \leqslant c(t) \sum_{n=0}^{\infty} Cc^n = c(t) C \frac{1}{1-c},$$

which shows that i) is true. This completes the proof of the theorem.

Remark. When $U = E$, it is easy to see that the constant c in ii) and iii) of Theorem 4.2 can be made arbitrarily small.

By Propositions 3.9 and 3.12, and by Proposition 1, Section 14 of [5], the following is clear.

Corollary 4.3. *Let θ be the current holomorphy type. A subset of $H(U)$ is τ_ω-bounded if and only if one of the conditions in Theorem 4.2 holds.*

The following result is the analogue of Proposition 2, Section 12 of [5]. We recall that the $\tau_{\infty\theta}$-topology on $\mathscr{H}_\theta(U)$ is the locally convex topology generated by the family of seminorms $\{p_{K,m}: K\subseteq U \text{ is compact and } m\in N\}$, where $p_{K,m}(f) = \sup_{x\in K}\|\hat{d}^m f(x)\|_\theta$.

Corollary 4.4. *Let $\mathfrak{X}$ be a bounded subset of $(H_\theta(U), \tau_\theta)$. Then τ_θ and $\tau_{\infty\theta}$ induce the same uniform structure and the same topology on $\mathfrak{X}$.*

Proof. Without loss of generality, $0\in\mathfrak{X}$. By Proposition 3.9, $\tau_\theta \geqslant \tau_{\infty\theta}$. Conversely, let p be a τ_θ-continuous seminorm on $H_\theta(U)$; suppose that p is $\theta-\xi-K$ ported. By iii) of Theorem 4.2, there is a number $t>0$, and constants $C\geqslant 0$, $c\geqslant 0$, $c<1$, such that

$$\left\|\frac{\hat{d}^n f(\xi)}{n!}\right\|_{\theta, K-\xi+tB_1} \leqslant Cc^n,$$

for all $f\in\mathfrak{X}$. Since p is $\theta-\xi-K$ ported, there is a constant $d(t)\geqslant 0$ such that

$$p(f)\leqslant d(t)\sum_{n=0}^{\infty}\left\|\frac{\hat{d}^n f(\xi)}{n!}\right\|_{\theta, K-\xi+tB_1}$$

for all $f\in H_\theta(U)$. Choose $n_0\in N$ such that $d(t)C\sum_{n=n_0}^{\infty} c^n\leqslant\frac{1}{2}$ and define the seminorm q on $\mathscr{H}_\theta(U)$ by

$$q(f) = d(t)\sum_{n=0}^{n_0-1}\left\|\frac{\hat{d}^n f(\xi)}{n!}\right\|_{\theta, K-\xi+tB_1}.$$

q is clearly $\tau_{\infty\theta}$-continuous. Further, if $f\in\mathfrak{X}$ and $q(f)\leqslant\frac{1}{2}$, then $p(f)\leqslant 1$, which proves the corollary.

In fact, Corollary 3.13 and the proof of Corollary 4.4 show that the following is true.

Corollary 4.5. *Let θ be the current holomorphy type and U a ξ-equilibrated open set for some $\xi\in E$. If $\mathfrak{X}$ is a bounded subset of $(H(U), \tau_\omega)$, then the topology on $\mathfrak{X}$ induced by τ_ω is metrizable.*

From Corollary 4.4, one can also obtain a characterization of relatively compact subsets of $(H_\theta(U), \tau_\theta)$, similar to that given in [5]. We recall that a subset $\mathfrak{X}$ of $H_\theta(U)$ is said to be relatively compact at a point $\xi\in U$ if for each $m\in N$, the set $\{d^m f(\xi): f\in\mathfrak{X}\}$ is relatively compact in $\mathscr{P}_\theta({}^mE)$.

Proposition 4.6. *A set $\mathfrak{X}$ in $H_\theta(U)$ is relatively compact for τ_θ if and only if $\mathfrak{X}$ is τ_θ-bounded and relatively compact at each point $\xi\in U$. If U is connected, $\mathfrak{X}$ is relatively compact for τ_θ if $\mathfrak{X}$ is τ_θ-bounded and relatively compact at some point $\xi\in U$.*

The proof of Proposition 4.6 follows from Corollary 4.4 and Propositions 1 and 2, Section 13 of [5].

We consider now the question of the convergence of Taylor series in $(H_\theta(U), \tau_\theta)$. We show below that for all α-holomorphy types θ satisfying a very natural condition, the Taylor series at ξ of every function $f\in H_\theta(U)$ converges to f in τ_θ, where U is an open, convex, ξ-equilibrated subset of E for some $\xi\in E$. We let $\tau_{k,f,\xi}\in H_\theta(U)$ be the kth partial sum of the Taylor series of f at ξ; that is, $\tau_{k,f,\xi}(x) = \sum_{m=0}^{k}\frac{\hat{d}^m f(\xi)}{m!}(x-\xi)$.

The condition necessary for convergence is the following:

(*) For each $f\in H_\theta(U)$, for any equivalent norm on E with unit ball V, and for any $\xi\in U$, if $\limsup_{n\to\infty}\left\|\frac{\hat{d}^n f(\xi)}{n!}\right\|_{\theta,V}^{1/n} < d$, then there are $D>0$ and $\varrho>0$ such that
$\left\|\frac{\hat{d}^n f(x)}{n!}\right\|_{\theta,V}\leqslant Dd^n$ for all $x\in B_\varrho(\xi)$.

In fact, Condition (*) requires little more than that for a given $f\in H_\theta(U)$, the function assigning to each $\xi\in U$ the radius of normal convergence of the Taylor series of f at ξ, relative to holomorphy type θ, is upper-semicontinuous. Using Cauchy's Inequality, it is not difficult to show that the current and compact holomorphy types satisfy Condition (*) (c.f. [3]).

Lemma 4.7. *The nuclear holomorphy type satisfies Condition* (*).

Proof. Let $f\in H_N(U)$ and let V, ξ and d be as in Condition (*); suppose that $\limsup_{n\to\infty}\left\|\frac{\hat{d}^n f(\xi)}{n!}\right\|_{N,V}^{1/n} < c < d$. It follows that for some $C\geqslant 0$, $\left\|\frac{\hat{d}^n f(\xi)}{n!}\right\|_{N,V}\leqslant Cc^n$ for all $n\in N$. Choose $\varrho>0$ so small that $B_\varrho(\xi)\subset U$ and $\frac{c}{1-\varrho c}<d$. By Lemma 7 and Proposition 6 of [4], we have that $\left\|\frac{\hat{d}^h f(x)}{h!}\right\|_{N,V}\leqslant\frac{C}{1-\varrho c}\left(\frac{c}{1-\varrho c}\right)$ for all $h\in N$ and $x\in B_\varrho(\xi)$, proving the lemma.

Proposition 4.8. *Let U be an open, convex, ξ-equilibrated set, for some $\xi\in E$. Let θ be an α-holomorphy type satisfying Condition* (*). *Then, for each $f\in H_\theta(U)$, the Taylor series of f at ξ converges to f in the topology τ_θ.*

The proof of Proposition 4.8 follows from the next two lemmas.

Lemma 4.9. *Let U be an open, convex, ξ-equilibrated set. Let $\eta\in U$ and* *let $K \in \mathscr{K}(\eta, U)$. Let L be the compact, ξ-equilibrated set $\{x: x = t\eta + (1-t)\xi,$ $t \in \boldsymbol{C},\ |t| \leqslant 1\}$. Then, for each $x \in L$, $K - \eta + x \subseteq U$.*

LEMMA 4.10. *Let θ be an a-holomorphy type satisfying Condition (*). Let $f \in H_\theta(U)$ and let U, ξ, K, η, and L be as in Lemma 4.9. Then for some $\varepsilon > 0$, $\gamma > 0$, $\gamma < 1$, $C \geqslant 0$, $c \geqslant 0$, $c < 1$, and some neighborhood $V \subset U$ of L we have*

$$\sup_{x \in V} \left\| \frac{1}{m!} \hat{d}^m (f - \tau_{k,f,\xi})(x) \right\|_{\theta, K-\eta+\varepsilon B_1} \leqslant C c^m \gamma^k.$$

for all $m, k \in \boldsymbol{N}$.

Proof. By Lemma 4.9, $K - \eta + x$ is a compact, convex, x-equilibrated subset of U for each $x \in L$. It follows from Definition 3.1 that for some $\varepsilon_x > 0$, there are $C_x \geqslant 0$, $c_x \geqslant 0$, $c_x < 1$ such that $\left\| \frac{\hat{d}^m f(x)}{m!} \right\|_{\theta, K-\eta+\varepsilon_x B_1} \leqslant$ $\leqslant C_x c_x^m$ for all $m \in \boldsymbol{N}$. By Condition (*) and the compactness of L, we get a finite open cover of L $\{U_i: i = 1, \ldots, n\}$ and positive constants $\{\varepsilon_i: i = 1, \ldots, n\}$, $\{D_i: i = 1, \ldots, n\}$, and $\{d_i: i = 1, \ldots, n\}$ with each $d_i < 1$ such that if $y \in U_i$, then $\left\| \frac{\hat{d}^m f(y)}{m!} \right\|_{\theta, K-\eta+\varepsilon_i B_1} \leqslant D_i d_i^m$ for each $m \in \boldsymbol{N}$. Let $\varepsilon = \min_{1 \leqslant i \leqslant n} \varepsilon_i$, $D = \max_{1 \leqslant i \leqslant n} D_i$, and $d = \max_{1 \leqslant i \leqslant n} d_i$, and let W be any open ξ-equilibrated subset of $\bigcup_{i=1}^{n} U_i$ containing L. Then, for all $y \in W$ and all $m \in \boldsymbol{N}$, $\left\| \frac{\hat{d}^m f(y)}{m!} \right\|_{\theta, K-\eta+\varepsilon B_1} \leqslant D d^m$. Choose a number $\varrho > 1$ such that

i) $\varrho d < 1$.

ii) For some open subset $V \subseteq W$ such that $L \subseteq V$, we have that if $t \in \boldsymbol{C}$, $|t| \leqslant \varrho$, and $x \in V$, then $(1-t)\xi + tx \in W$.

By Lemma 1, Section 6 of [5], we get

$$\sup_{x \in V} \left\| \frac{1}{m!} \hat{d}^m (f - \tau_{k,f,\xi})(x) \right\|_{\theta, K-\xi+\varepsilon B_1} \leqslant \frac{D(\varrho d)^m}{\varrho^k(\varrho - 1)},$$

for all $m, k \in \boldsymbol{N}$. Setting $C = \dfrac{D}{\varrho - 1}$, $\gamma = \dfrac{1}{\varrho}$, and $c = \varrho d$, the lemma is proved.

Proof of Proposition 4.8. Let p be a τ_θ-continuous seminorm on $H_\theta(U)$; suppose that p is $\theta - \eta - K$ ported. Thus, for each $\delta > 0$, there is $c(\delta) > 0$ such that

$$p(g) \leqslant c(\delta) \sum_{m=0}^{\infty} \left\| \frac{\hat{d}^m g(\eta)}{m!} \right\|_{\theta, K-\eta+\delta B_1},$$

for all $g \in H_\theta(U)$. Choose $\delta = \varepsilon$ as in Lemma 4.10. Using Lemma 4.10,

$$p(f - \tau_{k,f,\xi}) \leqslant c(\varepsilon) \sum_{m=0}^{\infty} \left\| \frac{1}{m!} \hat{d}^m (f - \tau_{k,f,\xi})(\eta) \right\|_{\theta, K-\eta+\varepsilon B_1}$$

$$\leqslant c(\varepsilon) \sum_{m=0}^{\infty} C c^m \gamma^k.$$

Since $c < 1$, $c(\varepsilon) \sum_{m=0}^{\infty} C c^m < \infty$, and so $p(f - \tau_{k,f,\xi}) \to 0$ as $k \to \infty$, since $\gamma < 1$. This proves the proposition.

References

[1] R. M. Aron, *Holomorphic Functions on Balanced Subsets of a Banach Space*, Bull. Amer. Math. Soc. (1972).

[2] S. Dineen, *Holomorphic functions on a Banach space*, Bull. Amer. Math. Soc. July (1970), pp. 883-886.

[3] — *Holomorphy types on a Banach space*, Studia Math., 39 (1971), pp. 241-288.

[4] C. Gupta, *Malgrange theorem for nuclearly entire functions of bounded type on a Banach space*, Thesis, University of Rochester, 1966; Notas de Matemática, Instituto de Matematica Pura e Aplicada, Rio de Janeiro, 37, 1-50, 1968; *Convolution Operators and Holomorphic Mappings on a Banach Space*, Université de Sherbrooke.

[5] L. Nachbin, *Topology on Spaces of Holomorphic Mappings*, 1968.

[6] — *Convolution Operators in Spaces of Nuclearly Entire Functions on a Banach Space*, from Functional Analysis and Related Fields, edited by Felix E. Browder, 1970.

[7] — and C. Gupta, *Malgrange theorem for nuclearly entire functions on a Banach space*, to appear.

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