STUDIA MATHEMATICA, T. XLV. (1973)

Function algebras on the interval and circle

by

DAVID M. WELLS (Pittsburgh, Penn.)

Abstract. A function algebra whose maximal ideal space is the interval I is local. A function algebra whose maximal ideal space is the circle T is either local or antisymmetric. A strongly regular function algebra on T must be $C(T)$.

Throughout this paper A will denote a function algebra on a compact Hausdorff space X, with maximal ideal space M_A and Shilov boundary S_A. If K is a closed subset of X, the algebra of restrictions to K of functions in A will be denoted by $A|_K$, and its uniform closure in $C(K)$ will be denoted by A_K. The unit interval and unit circle will be denoted by I and T respectively. We will need the definitions listed below. For additional background, see [1] or [3].

Say that a function f in $C(X)$ *belongs locally* to A if there are open sets $U_1, \ldots, U_n$ covering X and functions $g_1, \ldots, g_n$ in A such that the restriction of f to U_i agrees with g_i for each i. A is said to be a *local algebra* if it contains every function belonging locally to A. A is called *approximately normal* if for any two disjoint closed subsets K and L of X and any $r > 0$, there is a function h in A with $|h| < r$ on K and $|h-1| < r$ on L.

At present, there are no known examples of function algebras other than $C(I)$ or $C(T)$ with $M_A = I$ or T. In [4] it is shown that any such algebra must be approximately normal. Our main result concerns functions belonging locally to such algebras. We will make use of the following fact from [4]: if A is approximately normal on X, $f \in C(X)$ and there are open sets U_1, U_2 covering X and functions g_1, g_2 in A with $f|_{U_i} = g_i|_{U_i}$ then $f \in A$.

THEOREM. *If A is a function algebra whose maximal ideal space is I, then A is a local algebra.*

Let $g \in C(I)$ and suppose there is an open covering of I by sets $U_1, \ldots, U_n$, such that $g|_{U_i} = g_i|_{U_i}$ for some $g_i \in A$, $i = 1, \ldots, n$. We may assume each U_i is a subinterval of I, $U_i \cup U_{i+1} \neq \emptyset$ for $i = 1, \ldots, n-1$, and $\{U_1, \ldots, U_n\}$ has no proper subcover. Since A is approximately normal on I, we can find a function $f_1 \in A$ agreeing with g_1 on U_1 and with g_2 on $U_2 \cup \ldots \cup U_n$. Then f_1 agrees with g on $U_1 \cup U_2$. Continuing inductively, we can find functions $f_i \in A$ agreeing with f_{i-1} on $U_1 \cup \ldots \cup U_i$, and with g_{i+1} on $U_{i+1} \cup \ldots \cup U_n$, for $i = 2, \ldots, n-1$. Since f_i agrees with g on $U_1 \cup \ldots \cup U_{i+1}$, $f_{n-1} = g$.

COROLLARY. *If A is a function algebra whose maximal ideal space is T, then A is either antisymmetric or local.*

Suppose $g \in C(T)$ is locally in A. Let K be an arbitrary maximal set of antisymmetry for A. If $K = T$, then A is antisymmetric. Otherwise we need to show $g \in A$, and by a result in [2] it is enough to show $g|_K \in A_K$. Since $K \neq T$ we can find an arc J containing K. It is shown in [4] that J must be A-convex, so A_J is a function algebra whose maximal ideal space is the interval J. Now $g|_J$ is locally in A_J, and A_J is a local algebra. Hence $g|_J \in A_J$, and in particular, $g|_K \in A_K$.

A function algebra A on X is said to be analytic if any function in A vanishing on a non-empty open set in X must vanish identically on X. It is known that an analytic function algebra on X must be an integral domain, and that a function algebra on X which is an integral domain must be antisymmetric on X. It is easy to see that an analytic function algebra is local whenever X is connected. We will be able to show that if $M_A = T$, A is local as long as it is an integral domain.

LEMMA. *Let A be approximately normal on X. Suppose A is an integral domain. If Z is the zero-set of any function $f \in A$, then $X - Z^0$ is connected.*

If $X - Z^0$ is not connected, it is the union of two disjoint non-empty compact sets K and L. It is easy to show that K and L have non-empty interiors in X, so that $f|_K$ and $f|_L$ are not identically zero. Using approximate normality, we can obtain functions g and h in A such that $g|_{X-K} = 0 = h|_{X-L}$, g agrees with f on K and h agrees with f on L. But then $gh = 0$, a contradiction since A is an integral domain.

THEOREM. *Let A be approximately normal on T. If A is an integral domain, then A is local.*

Suppose $f \in C(T)$ is locally in A but not in A. Choose a cover $\mathscr{U} = \{U_1, \ldots, U_n\}$ of T and functions $g_1, \ldots, g_n \in A$ such that $f|_{U_i} = g_i$. Assume each U_i is an arc, U has no proper subcover and $f|_{U_i \cup U_j} \notin A|_{U_i \cup U_j}$ if $U_i \cap U_j \neq \emptyset$. Let $W_i = U_i$ together with the interiors of the zero-sets of $g_i - g_{i-1}$ and $g_i - g_{i+1}$, $i = 1, \ldots, n$, where we understand $n+1 = 1$. By the above lemma, each W_i is an arc, so if $\mathscr{V} = \{V_1, \ldots, V_m\}$ is a minimal subcover of $\{W_1, \ldots, W_n\}$, then $\mathscr{V}$ has all the properties attributed above to $\mathscr{U}$. We may assume $m = n$ and f agrees with g_i on V_i, so the interior of the zero-set of $g_i - g_{i+1}$ is exactly $V_i \cap V_{i+1}$. Let $f_i = (g_i - g_{i+1})(g_i - g_{i+1})$. If Z_i is the zero-set of f_i, then Z_i^0 contains $(V_i \cap V_{i-1}) \cup (V_i \cap V_{i+1})$ but $Z_i^0 \subseteq V_i$ by the construction of $\mathscr{V}$. Hence $Z_i^0 = V_i$, but then $f_1 f_2 \ldots f_n = 0$, a contradiction.

A function algebra A on X is called *strongly regular* if any function in A vanishing at a point $x \in X$ can be uniformly approximated in A by functions vanishing in a neighborhood of x, the neighborhood depending on the function. In [5] it is shown that any strongly regular function algebra is normal, and that a strongly regular function algebra on I must be $C(I)$. We can prove the corresponding result for T.

THEOREM. *If A is a strongly regular function algebra on T, then $A = C(T)$.*

It is enough to show that for any $r > 0$ and any two disjoint closed arcs K and L of T, there is a function $g \in A$ with $|g| < r$ on K, $|g-1| < r$ on L and $\|g\|$ bounded by some constant independent of r. The Shilov boundary of A must be T, so we may assume the endpoints of K are peak points for A, say x and y. Choose a neighborhood U of $\{x, y\}$ with $\overline{U} \cap L = \emptyset$. We can find a function $f \in A$ such that $\|f\| < 2$, $f(x) = f(y) = 0$ and $|f-1| < r$ outside U. Choose functions h, $k \in A$ such that $\|h-f\| < r$, $\|k-f\| < r$ and h and k vanish in neighborhoods U and W of x and y respectively. Now hk vanishes on $V \cup W$, is close to 1 on L, and $\|hk\| < 5$ for sufficiently small values of r. Using normality, we can now obtain a function $g \in A$ vanishing on K and agreeing with hk on $T - K$.

References

[1] A. Browder, *Introduction of Function Algebras*, 1969.
[2] I. Glicksberg, *Measures orthogonal to algebras and sets of antisymmetry*, Trans. Amer. Math. Soc. 105 (1962), pp. 414–435.
[3] G. Leibowitz, *Lectures on Complex Function Algebras*, 1970.
[4] D. Wilken, *Approximate normality and function algebras on the interval and circle*, pp. 98–111, Proc. Internat. Sympos. on Function Algebras, Tulane Univ., 1965–1966.
[5] — *A Note on Strongly Regular Function Algebras*, Can. J. Math. 21 (1969).

UNIVERSITY OF PITTSBURGH

Received December 10, 1971 (451)