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Nonsymmetric group algebras

by

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Abstract. The principle result of this paper states that the $\mathscr{L}^1$-algebra of a connected reductive Lie group with parabolic rank $\geqslant 1$ is not symmetric. Also, using a relationship that is derived for the transfinite diameter of the spectrum of any element in a Banach *-algebra, it is shown that the group algebra of $\mathrm{PGL}(2, Q_p)$ is not symmetric.

A complex Banach *-algebra $\mathscr{U}$ is said to be *symmetric* (or by some authors, *completely symmetric*) if xx^* is quasi-regular for each x in $\mathscr{U}$. The problem of determining which locally compact groups, G, have symmetric group algebras, $\mathscr{L}^1(G)$, has received the attention of several authors (cf. [1]–[4], [11]–[16]). The present work continues this investigation.

Section one is devoted to arbitrary Banach *-algebras and of particular importance is the relationship between the transfinite diameter of the spectrum of x and the norm of certain polynomials in x. In section two, this relationship is interpreted in the group algebra and applied for the group $\mathrm{PGL}(2, Q_p)$.

The extent to which the occurrence of free nonabelian subsemigroups of G effect symmetry of $\mathscr{L}^1(G)$ is discussed in section three.

In section four it is shown that connected, noncompact semisimple Lie groups have nonsymmetric group algebras.

§1. Given a Banach *-algebra $\mathscr{U}$, $\mathscr{U}_e$ will denote either $\mathscr{U}$, if $\mathscr{U}$ has an identity, or the algebra obtained by adjoining an identity to $\mathscr{U}$. $P(\mathscr{U}_e)$ will denote the set of all linear functionals f defined on $\mathscr{U}_e$ such that $f(e) = 1$ and $f(xx^*) \geqslant 0$ for each x in $\mathscr{U}_e$. For each x in $\mathscr{U}$, $\nu(x)$ and $\sigma(x)$ equal, respectively, the spectral radius and the spectrum of x.

DEFINITION 1.1. Given a Banach *-algebra $\mathscr{U}$, $\mathscr{S}(\mathscr{U})$ is defined to be the set of all x in $\mathscr{U}$ such that

$$\sigma(x) \subset \{f(x) \mid f \in P(\mathscr{U}_e)\}.$$

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PROPOSITION 1.2. *A Banach $*$-algebra $\mathcal{U}$ is symmetric if and only if $\mathcal{S}(\mathcal{U}) = \mathcal{U}$.*

Proof. Assume that $\mathcal{U}$ is symmetric. It is well known that is equivalent to assuming that $\mathcal{U}_e$ is symmetric. We will show that under this assumption, $\sigma(x) \subset \{f(x) | f \in P(\mathcal{U}_e)\}$ for each x in $\mathcal{U}_e$.

Let x be in $\mathcal{U}_e$ and let $\mathcal{M}$ be a maximal commuting $*$-subalgebra of $\mathcal{U}_e$ containing xx^*. Since $\mathcal{M}$ is a closed $*$-subalgebra of $\mathcal{U}_e$, $\mathcal{M}$ is also symmetric. If xx^* is singular there is a nonzero, continuous, multiplicative, linear functional f_0 on $\mathcal{M}$ such that $f_0(xx^*) = 0$. Since $\mathcal{M}$ is symmetric, $f_0(y^*) = \overline{f_0(y)}$ for each y in $\mathcal{M}$. Hence, since f_0 is multiplicative, $f_0(yy^*) \geqslant 0$ for each y in $\mathcal{M}$, and thus $f_0 \in P(\mathcal{M})$. Because $\mathcal{U}_e$ is symmetric, each f in $P(\mathcal{M})$ can be extended to an element $\bar{f}$ of $P(\mathcal{U}_e)$ (cf. Naimark [18]). Therefore, if xx^* is singular, there is an $f(=\bar{f}_0)$ in $P(\mathcal{U}_e)$ such that $f(xx^*) = 0$. A similar statement holds if x^*x is singular.

Now, if $x \in \mathcal{U}_e$ and x has no right inverse, xx^* is singular; otherwise, if $(xx^*)y = e$, x^*y would be a right inverse for x. Hence, there is an f in $P(\mathcal{U}_e)$ such that $f(xx^*) = 0$. But then $|f(x)|^2 \leqslant f(xx^*) = 0$. Therefore, $f(x) = 0$.

A similar argument shows that if x has no left inverse, then for some f in $P(\mathcal{U}_e)$, $f(x) = 0$. Therefore, if $\alpha \in \sigma(x)$, there is an element f of $P(\mathcal{U}_e)$ such that $f(x - \alpha e) = f(x) - \alpha = 0$. Consequently, $\sigma(x) \subset \{f(x) | f \in P(\mathcal{U}_e)\}$ for each x in $\mathcal{U}_e$.

The converse is immediate from the definition of symmetry.

For each compact subset A of the complex plane, and for each continuous complex valued function f defined on A, we set

$$\|f\|_A = \sup\{|f(t)| \; t \in A\}.$$

For each positive integer n, there exist a unique monic polynomial of degree n, denoted p_n, such that $\|p_n\|_A \leqslant \|q_n\|_A$ for each monic polynomial q_n of degree n. p_n is called the *n-th Tchebycheff polynomial.* A fundamental theorem for Tchebycheff polynomials states that

$$\lim_n \|p_n\|_A^{1/n}$$

exist. This limit, which we denote by $\tau(A)$, is called the *transfinite diameter of A*. We will use the fact that if A is the closed interval $[s, t]$, then $\tau(A) = (t-s)/4$. (A discussion of Tchebycheff polynomials can be found in [5].)

Given a Banach algebra $\mathcal{U}$ and an x in $\mathcal{U}$, let $C[x]$ be the algebra of polynomials in x (with identity), and let $C_n[x]$ denote the set of all such monic polynomials of degree n. We set

$$\tau_x = \liminf_n (\inf\{\|p(x)\|^{1/n} | p(x) \in C_n[x]\}).$$

PROPOSITION 1.3. (1) *Let $\mathcal{U}$ be a Banach algebra. For each x in $\mathcal{U}$, $\tau_x = \tau(\sigma(x))$.*

Proof. We first show that $\tau_x \leqslant \tau\sigma((x))$. Let p_n be the nth Tchebycheff polynomial for $\sigma(x)$. Then

$$\lim_n \|p_n\|_{\sigma(x)}^{1/n} = \tau(\sigma(x)).$$

We also have

$$\|p_n\|_{\sigma(x)} = \sup_{t \in \sigma(x)} |p_n(t)| = \nu(p_n(x)).$$

But

$$\nu(p_n(x)) = \lim_m \|p_n(x)^m\|^{1/m}.$$

Hence, for each fixed $n \geqslant 1$,

$$\lim_m \|p_n(x)^m\|^{1/nm} = \nu(p_n(x))^{1/n},$$

and so

$$\lim_n \lim_m \|p_n(x)^m\|^{1/nm} = \tau(\sigma(x)).$$

Therefore, there is a sequence (n_k, m_k) of $Z \times Z$ such that

$$\lim_k \|p_{n_k}(x)^{m_k}\|^{1/n_k m_k} = \tau(\sigma(x)).$$

Since $p_{n_k}(t)^{m_k}$ is a monic polynomial of degree $n_k m_k$, $\tau_x \leqslant \tau(\sigma(x))$.

Suppose now that $q_{n_k}(x) \in C_{n_k}[x]$ for $k = 1, 2, \ldots$, and that

$$\tau_x = \lim_k \|q_{n_k}(x)\|^{1/n_k}.$$

Let $0 \neq \alpha \in \sigma(x)$. Then, since $\sigma_{C[x]}(x) \cup \{0\} \supset \sigma(x)$, there is a continuous homomorphism ξ_α defined on $C[x]$ such that $\xi_\alpha(x) = \alpha$. Thus

$$|q_{n_k}(\alpha)| = |\xi_\alpha(q_{n_k}(x))| \leqslant \|\xi_\alpha\| \; \|q_{n_k}(x)\|.$$

Hence, if we set $q'_{n_k+1}(t) = tq_{n_k}(t)$, $\|q'_{n_k+1}\|_{\sigma(x)} \leqslant \|\xi_\alpha\| \; \|x\| \; \|q_{n_k}(x)\|$.

Now, if p_n is the nth Tchebycheff polynomial for $\sigma(x)$,

$$\|p_{n_k+1}\|_{\sigma(x)} \leqslant \|q'_{n_k+1}\|_{\sigma(x)}$$

for each $k = 1, 2, \ldots$ Therefore

$$\tau(\sigma(x)) = \lim_n \|p_n\|_{\sigma(x)}^{1/n} \leqslant \lim_k \|q'_{n_k+1}\|_{\sigma(x)}^{1/n_k+1} \leqslant \lim_k (\|\xi_\alpha\| \; \|x\| \; \|q_{n_k}(x)\|)^{1/n_k+1} = \tau_x.$$

COROLLARY 1.4. *Let $\mathcal{U}$ be a Banach $*$-algebra and let $x = x^*$ be in $\mathcal{S}(\mathcal{U})$. There exist a sequence of monic polynomials q'_n of degree n such that*

$$\liminf \|q'_n(x)\|^{1/n} \leqslant \nu(x)/2.$$

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(1) It was recently learned that this proposition was also proved by P. Halmos, *Capacity in Banach algebras*, Indiana Math. J. 20 (1971), p. 855.

Proof. Since $x = x^*$, $f(x)$ is real for each f in $P(\mathscr{U}_e)$. Hence, since $x \in \mathscr{S}(\mathscr{U})$, $\sigma(x) \subset [-\nu(x), \nu(x)]$, and thus, $\tau(\sigma(x)) \leqslant \nu(x)/2$. Since $\tau_x = \tau(\sigma(x))$ the desired sequence exist.

The following proposition has a proof very similar in spirit to that of Proposition 1.3.

PROPOSITION 1.5. *Let $\mathscr{U}$ be a Banach $*$-algebra, $x = x^*$ be an element of $\mathscr{U}$, and $C[x]$ be the ring of polynomials in x (with identity). If there is a $\delta > 0$ such that*

$$\Big\| \sum \alpha_n x^n \Big\| \geqslant \delta \sum |\alpha_n|$$

for each $\sum \alpha_n x^n$ in $C[x]$, then $x \notin \mathscr{S}(\mathscr{U})$.

Proof. Suppose $x \in \mathscr{S}(\mathscr{U})$. Then $\sigma(x)$ is real and hence $\sigma_{\mathscr{U}_e}(x)$ is also real. It follows from the spectral permanence theorem (cf. e.g. [10]) that if $\mathfrak{A}$ is the closure in $\mathscr{U}_e$ of $C[x]$, $\sigma_{\mathfrak{A}}(x)$ is real. We will show that under our assumptions on x, this is not the case.

For each γ in C with $|\gamma| \leqslant 1$ and each $y = \alpha e + \sum \alpha_n x^n$ in $C[x]$, define

$$\xi_\gamma(y) = \alpha + \sum \alpha_n \gamma^n.$$

Clearly ξ_γ is linear on $C[x]$, and if $y = \alpha e + \sum \alpha_n x^n$ is in $C[x]$,

$$|\xi_\gamma(y)| = \Big|\alpha + \sum \alpha_n \gamma^n\Big| \leqslant |\alpha| + \sum |\alpha_n| \leqslant \|y\|/\delta.$$

Therefore, ξ_γ is continuous on $C[x]$. We will show that it is also a homomorphism.

Let $\sum \alpha_n x^n$ and $\sum \beta_m x^m$ be in $C[x]$ and suppose

$$\sum \lambda_p x^p = \Big(\sum \alpha_n x^n\Big)\Big(\sum \beta_m x^m\Big).$$

Then

$$\sum_p \Big(\lambda_p - \sum_{n+m=p} \alpha_n \beta_m\Big) x^p = 0.$$

Hence

$$0 = \Big\|\sum_p \Big(\lambda_p - \sum_{n+m=p} \alpha_n \beta_m\Big) x^p\Big\| \geqslant \delta \sum_p \Big|\lambda_p - \sum_{n+m=p} \alpha_n \beta_m\Big|.$$

Consequently,

$$\lambda_p = \sum_{n+m=p} \alpha_n \beta_m$$

for each p.

It now follows that

$$\xi_\gamma\Big[\Big(\sum \alpha_n x^n\Big)\Big(\sum \beta_m x^m\Big)\Big] = \Big(\sum \alpha_n \gamma^n\Big)\Big(\sum \beta_n \gamma^m\Big) = \Big[\xi_\gamma\Big(\sum \alpha_n x^n\Big)\Big]\Big[\xi_\gamma\Big(\sum \beta_m x^m\Big)\Big].$$

Therefore ξ_γ is a continuous homomorphism on $C[x]$ for each $|\gamma| \leqslant 1$.

By continuously extending each ξ_γ to $\mathfrak{A}$, we can conclude that $\sigma_{\mathfrak{A}}(x) \supset \{\gamma \mid |\gamma| \leqslant 1\}$.

§2. Let G be a locally compact group with left Haar measure λ. (We will write dt for $d\lambda(t)$, etc.) $\mathscr{L}^1(G)$ (or $l^1(G)$ if G is discrete), the space of absolutely integrable complex valued functions on G, is a Banach $*$-algebra with multiplication and involution defined for λ — a.a. t in G by

$$x * y(t) = \int x(s) y(s^{-1}t)\, dt$$

and

$$x^*(t) = \overline{x(t^{-1})}\, \Delta(t^{-1})$$

for all x and y in $\mathscr{L}^1(G)$. ($\Delta(\cdot)$ denotes the modular function of G.) $\mathfrak{A}(G)$ will denote either $l^1(G)$ or the algebra obtained by adjoining an identity to $\mathscr{L}^1(G)$. We will apply the results of §1 to $\mathfrak{A}(G)$.

The following notation will be used: G denotes a locally compact group. For $A \subset G$, $A^{-1} = \{a^{-1} \mid a \in A\}$, ${}^cA = \{g \in G \mid g \notin A\}$, and if A is finite, $|A|$ is the cardinality of A. For n a positive integer,

$$A^n = \{a_1 a_2 \ldots a_n \mid a_i \in A,\ 1 \leqslant i \leqslant n\}$$

and for $n \geqslant 2$, ${}^nA = A^n \cap {}^c\big(\bigcup_{i=1}^{n-1} A^i\big)$.

For x in $\mathscr{L}^1(G)$, $N(x) = \operatorname{ess\,supp}(x)$. We write $\mathscr{S}(G)$ for $\mathscr{S}(\mathscr{L}^1(G))$ and $P(G)$ for $P(\mathfrak{A}(G))$.

One can easily verify

LEMMA 2.1. *Let x and y be in $\mathscr{L}^1(G)$. Then*

(i) $N(x*y) \subset N(x)N(y)$,

(ii) $N(x^*) = N(x)^{-1}$,

(iii) *if* $N(x) \cup N(y) = \emptyset$, $\|x+y\| = \|x\| + \|y\|$.

PROPOSITION 2.2. *Let x be in $\mathscr{L}^1(G)$ and let $G_x(n) = {}^c\big(\bigcup_{i=1}^{n-1} N(x^i)\big)$ for each $n \geqslant 2$. Then*

(i) $\liminf_n \{\int_{G_x(n)} |x^n(t)|\, dt\}^{1/n} \leqslant \tau(\sigma(x))$.

In particular, if $x = x^*$ and $x \in \mathscr{S}(G)$,

(ii) $\liminf_n \{\int_{G_x(n)} |x^n(t)|\, dt\}^{1/n} \leqslant \nu(x)/2$.

Proof. By Corollary 1.4 there exist a sequence of monic polynomials q_n of degree n such that $\liminf_n \|q_n(x)\|^{1/n} \leqslant \tau(\sigma(x))$. Let $q_n(t) = t^n + tq_n'(t) + c_n$. Then the degree of $tq_p'(t)$ is $n-1$ and

$$N(xq_n'(x)) \subset \bigcup_{i=1}^{n-1} N(x)^i.$$

Hence

$$\|q_n(x)\| = \|x^n + xq_n'(x)\| + |c_n| \geqslant \|x^n + xq_n'(x)\|$$
$$= \int_G |x^n(t) + xq_n'(x)(t)|\,dt \geqslant \int_{G_x(n)} |x^n(t)|\,dt.$$

This proves (i). The modification for (ii) merely takes into account the fact that if $x = x^* \in \mathscr{S}(G)$, then $\tau(\sigma(x)) \leqslant \nu(x)/2$.

In the following example we use Proposition 2.2 to obtain nonsymmetry of the group algebra of PGL $(2, Q_p)$.

Let Q_p be a p-adic completion of the rationals, $\mathscr{O}$ the valuation ring of Q_p, $\mathscr{P}$ the maximal (principle) ideal of $\mathscr{O}$ and τ a generator of $\mathscr{P}$ in $\mathscr{O}$. $\mathscr{O}$ is compact and open in Q_p and if q denotes the cardinality of $\mathscr{O}/\mathscr{P}$ then $1 < q < \infty$.

Let $\mathrm{GL}(2, Q_p)$ (resp. $\mathrm{GL}(2, \mathscr{O})$) denote the group of non-singular 2×2 matrices with coefficients in Q_p (resp. $\mathscr{O}$), let Z denote the center of $\mathrm{GL}(2, Q_p)$, and let G $(= \mathrm{PGL}(2, Q_p)) = \mathrm{GL}(2, Q_p)/Z$. $\mathrm{GL}(2, \mathscr{O})$ is a compact open subgroup of $\mathrm{GL}(2, Q_p)$, and hence, so also is its image K in G. Let g' be the matrix $\begin{pmatrix}\tau & 0\\ 0 & 1\end{pmatrix}$ in $\mathrm{GL}(2, Q_p)$ and g the image of g' in G. Normalize the Haar measure λ on G so that $\lambda(K) = 1$. One has the following facts (cf. [17] or [19]):

(i) $G = \bigcup_{n=0}^{\infty} Kg^nK$,

(ii) $Kg^nK = Kg^mK$ *if and only if* $n = \pm m$,

(iii) $\lambda(Kg^nK) = q^{n-1}(q+1)$.

Let x_n denote the characteristic function of Kg^nK. If $x = \sum_{i=0}^{n} x(g^i)x_i$ and $y = \sum_{j=0}^{m} y(g^j)x_j$, one can show that $x*y = \sum_{i=0}^{n+m} x*y(g^i)x_i$. Furthermore if $x(g^n) = 1 = y(g^m)$, then

$$x*y(g^{n+m}) = x_n * x_m(g^{n+m})$$
$$= \int x_n(t)\,x_m(t^{-1}g^{n+m})\,dt$$
$$= \lambda(Kg^nK \cap g^{n+m}Kg^mK)$$
$$\geqslant \lambda(K) = 1.$$

Thus, $(x_1)^n = x_n + \sum_{i=0}^{n-1} (x_1)^n(g^i)x_i$, and so $G_{x_1}(n) = N(x_1^n) \cap^c \Big(\bigcup_{i=1}^{n-1} N(x_1^i)\Big) \subset Kg^nK$. Therefore

$$\lim_n \Big\{\int_{G_{x_1}(n)} |(x_1)^n(t)|\,dt\Big\}^{1/n} \geqslant \lim_n \Big\{\int_{Kg^nK} x_n(t)\,dt\Big\}^{1/n}$$
$$= \lim_n \lambda(Kg^nK)^{1/n} = q.$$

Now, since $(KgK)^{-1} = KgK$ and since G is unimodular, $x_1 = x_1^*$. Thus, if $x_1 \in \mathscr{S}(G)$, $\tau(\sigma(x_1)) \leqslant \nu(x_1)/2 = q + 1/2$. This contradiction of Proposition 2.2 shows that $x_1 \notin \mathscr{S}(G)$.

Recent results by Bruhat and Tits indicate that the method illustrated in this example can be successfully applied to a large class of reductive algebraic groups defined over a locally compact field with discrete valuation. The details, however, have not yet been worked out.

In [14], we proved that if each finite subset $e \in A = A^{-1}$ of G satisfies

$$\liminf_n |A^n \cap^c (A^{n-1})|^{1/n} \leqslant 1,$$

then $\mathscr{S}(G)$ $(= \mathscr{S}(l^1(G)))$ contains all x of $l^1(G)$ for which $|N(x)| < \infty$. The following corollary of Proposition 2.2 (ii) contains a partial converse to this theorem.

Corollary 2.3. *For each finite subset A of G let x_A be the characteristic function of A. For each such A,*

$$\tau(\sigma(x_A)) \geqslant \liminf_n |{}^nA|^{1/n}.$$

If $e \in A = A^{-1}$ and $x_A \in \mathscr{S}(G)$ then

$$\nu(x_A)/2 \geqslant \liminf_n |A^n \cap^c (A^{n-1})|^{1/n}.$$

Proof. First observe that for each t in A^n, $x_A^n(t) \geqslant 1$. We also have

$$N(x^n) \cap^c \Big(\bigcup_{i=1}^{n-1} N(x^i)\Big) = A^n \cap^c \Big(\bigcup_{i=1}^{n-1} A^i\Big) = {}^nA.$$

Applying Proposition 2.2 (i) we have

$$\tau(\sigma(x_A)) \geqslant \liminf_n \Big[\int_{G_x(n)} |x^n(t)|\,dt\Big]^{1/n}$$
$$= \liminf_n \Big[\sum_{t \in N(x^n)\cap G_x(n)} |x^n(t)|\Big]^{1/n}$$
$$\geqslant \liminf_n \Big[\sum_{t \in {}^nA} 1\Big]^{1/n}$$
$$= \liminf_n |{}^nA|^{1/n}.$$

For the second statement, we note that since $e \in A = A^{-1}$, $A^n \supset A^{n-1}$ for $n \geqslant 2$. Hence ${}^nA = A^n \cap^c (A^{n-1})$. Also, since $A = A^{-1}$, $x_A = x_A^*$. Thus if $x_A \in \mathscr{S}(G)$, $\tau(\sigma(x_A)) \leqslant \nu(x_A)/2$.

§3. In this section we discuss the relationship between symmetry of $\mathfrak{A}(G)$ and the occurrence of free semigroups on two generators in G.

Let a and b be elements of G. We denote by $[a, b]$ the subsemigroup of G generated by a and b, and we say $[a, b]$ is free if $a[a, b] \cap b[a, b] = \emptyset$. In [15], we have shown that if G is discrete and contains a free semigroup

$[a, b]$ then $\mathscr{S}(G)$ does not contain all elements of $l^1(G)$ which have finite support. In particular, $l^1(G)$ is not symmetric. In attempting to extend this result to nondiscrete groups, it is obvious that a topological requirement must be added for $[a, b]$. For example, $\mathrm{SO}(3)$ ($= \mathrm{SO}(3, R)$) contains a free nonabelian group but $\mathfrak{A}(\mathrm{SO}(3))$ is symmetric since $\mathrm{SO}(3)$ is compact (cf. van Dijk [3]). Here the free subgroup is not closed. If one requires that $[a, b]$ be free and closed in G, this still is not sufficient to imply nonsymmetry of $\mathfrak{A}(G)$. (In [16], we have shown that $\mathrm{SO}(3)\times Z$ contains such a subsemigroup and that $\mathfrak{A}(\mathrm{SO}(3)\times Z)$ is symmetric.)

A subsemigroup S of G is said to be uniformly discrete if G has a neighborhood of the identity, U, such that $sU \cap tU = \emptyset$ for s, t in S, $s \neq t$. (In general this is stronger than requiring that S be discrete in G.) In [16], we studied groups containing free, uniformly discrete semigroups on two generators, and conjectured that such groups have nonsymmetric group algebras. (Examples of these groups include certain solvable non-nilpotent groups such as the "$ax+b$" group and all almost connected nonamenable groups. This latter category includes all reductive algebraic groups with split rank $\geqslant 1$.) A proof of this conjecture has not been found. The weakest known condition on $[a, b]$ that is sufficient to imply nonsymmetry of $\mathfrak{A}(G)$ is given in

PROPOSITION 3.1. *Let a and b be elements of G such that $[a, b]$ is free. For s in $[a, b]$ and $s = s_1 s_2 \dots s_n$, where $s_i \in \{a, b\}$ for $1 \leqslant i \leqslant n$, let $U^s = s_1 U s_2 U \dots s_n U$ for any $U \subset G$. If G contains a compact neighborhood of the identity, U, such that $U^s \cap U^t = \emptyset$ for s, t in $[a, b]$, $s \neq t$ then there exist x in $\mathscr{L}^1(G)$ with $N(x)$ compact such that $x \notin \mathscr{S}(G)$.*

We begin by proving the following lemmas. For x in $\mathscr{L}^1(G)$, t in G, we denote by ${}_tx$ the element of $\mathscr{L}^1(G)$ defined by ${}_tx(s) = x(t^{-1}s)$ for λ — a.a. s in G.

LEMMA 3.2. *Suppose x is a normal element of $\mathscr{L}^1(G)$ and that x, ${}_tx$ are in $\mathscr{S}(G)$. Then $\nu({}_tx) \leqslant \nu(x)$.*

Proof. First note that for any x in $\mathscr{L}^1(G)$ and t in G.

$$\begin{aligned} x^* * x(s) &= \int x^*(r)x(r^{-1}s)\,dr \\ &= \int \overline{x(r^{-1})}\,\Delta(r^{-1})x(r^{-1}s)\,dr \\ &= \int \overline{x(t^{-1}r)}\,x(t^{-1}rs)\,dr \\ &= \int \overline{{}_tx(r)}\,{}_tx(rs)\,dr \\ &= \int \overline{{}_tx(r^{-1})}\,\Delta(r^{-1})\,{}_tx(r^{-1}s)\,dr \\ &= ({}_tx)^* * ({}_tx)(s). \end{aligned}$$

Since ${}_tx \in \mathscr{S}(G)$, there is an f in $P(G)$ such that $\nu({}_tx) = |f({}_tx)|$. But then

$$|f({}_tx)|^2 \leqslant f\big(({}_tx)^* * ({}_tx)\big) = f(x^* x) \leqslant \nu(x^* x).$$

Since x is normal,

$$\nu(x^* x) \leqslant \nu(x^*)\nu(x) = \nu(x)^2.$$

Thus,

$$\nu({}_tx) = |f({}_tx)| \leqslant \nu(x).$$

LEMMA 3.3. *Let a, b, and U be as in Proposition 3.1, and let $s_1, s_2, \dots, s_n$ be distinct elements of $[a, b]$. Suppose that for $1 \leqslant i \leqslant n$, $x_i \in \mathscr{L}^1(G)$ such that $x_i(t) \geqslant 0$ for λ — a.a. t and that $N(x_i) \subset s_i U$. If $x = \sum_{i=1}^n a_i x_i$, $a_i \in C$, then*

$$\nu(x) = \|x\| = \sum_{i=1}^n |a_i|\ \|x_i\|.$$

Proof. The second equality follows immediately from the fact that $s_i U \subset U^{s_i}$ and that $U^{s_i} \cap U^{s_j} = \emptyset$ for $i \neq j$.

For p a positive integer, let Ω_p be the set of all maps of $\{1, 2, \dots, p\}$ into $\{1, 2, \dots, n\}$. Then

$$x^p = \sum_{\omega \in \Omega_p} a_{\omega(1)} \dots a_{\omega(p)} x_{\omega(1)} \dots x_{\omega(p)}.$$

Since $x_i(t) \geqslant 0$, $\|x_{\omega(1)} \dots x_{\omega(p)}\| = \|x_{\omega(1)}\| \dots \|x_{\omega(p)}\|$ for each ω in Ω_p. Also, if $\omega \neq \omega'$,

$$s_{\omega(1)} U \dots s_{\omega(p)} U \cap s_{\omega'(1)} U \dots s_{\omega'(p)} U = \emptyset.$$

Hence

$$\|x^p\| = \sum_{\omega \in \Omega_p} \prod_{i=1}^p |a_{\omega(i)}|\ \|x_{\omega(i)}\| = \Big[\sum_{i=1}^n |a_i|\ \|x_i\|\Big]^p.$$

Therefore

$$\nu(x) = \lim_p \|x^p\|^{1/p} = \sum_{i=1}^n |a_i|\ \|x_i\|.$$

Proof. (Proposition 3.1): Let $V = V^{-1}$ be a compact neighborhood of the identity sufficiently small so that

$$V \cup aVa^{-1} \cup a^2 V a^{-1} V a^{-1} \subset U.$$

Let x be the normalized characteristic function of aV and set $x_1 = {}_{ba^2}x$, $x_2 = {}_{ba^2}(x^*)$, $x_3 = {}_{ba^2}(x^2)$, $x_4 = {}_{ba^2}(x^*)^2$ and $x_5 = {}_{ba^2}(xx^* + x^*x)$. Finally, let $y = x_1 + x_2 + i(x_3 + x_4 + x_5)$.

Now, $N(x_i) \subset ba^{n_i}U$ for $1 \leqslant i \leqslant 5$ and $n_i \geqslant 0$. Also, $n_i \neq n_j$ if $i \neq j$. Therefore, by Lemma 3.3, $\nu(y) = \|y\| = \sum_{i=1}^5 \|x_i\| = 6$.

Let $z = x+x^*+i(x+x^*)^2$. Then $y = {}_{ba^2}z$. z is normal, and hence by Lemma 3.1, $\nu(y) \leqslant \nu(z)$ if $y, z \in \mathcal{S}(G)$. But if $z \in \mathcal{S}(G)$, $\nu(z) = |f(z)|$ for some f in $P(G)$. Since $x+x^*$ is hermitian, $f(x+x^*)$ and $f([x+x^*]^2)$ are real. Thus,

$$\begin{aligned} |f(z)| &= |f(x+x^*)+if([x+x^*]^2)| \\ &= [f(x+x^*)^2+f([x+x^*]^2)^2]^{1/2} \\ &\leqslant [\nu(x+x^*)^2+\nu(x+x^*)^4]^{1/2} \\ &= 2\sqrt{5}. \end{aligned}$$

This contradiction shows that not both y and z are in $\mathcal{S}(G)$.

§4. We will show in this section that the algebra of spherical functions on a connected semisimple Lie group with finite center is not symmetric. From this it follows that the group algebra of a connected reductive Lie group with noncompact adjoint group is not symmetric. The proof, which parallels that originally given by Naimark [18] to show that the group algebra of $SL(2, C)$ is not symmetric, is basically a collection of recent results from representation theory.

Let G be a connected semisimple Lie group with finite center. G has a maximal compact subgroup K and if $\mathcal{L}^{\Delta}(G)$ denotes the subspace of all functions in $L^1(G)$ which are invariant under both right and left translation by elements of K then $\mathcal{L}^{\Delta}(G)$ is a commutative Banach $*$-algebra. Furthermore, the maximal ideal space of $\mathcal{L}^{\Delta}(G)$ can be identified with the set of all bounded continuous functions $\mathcal{U}$ defined as G such that

$$\int_K \varphi(xky)\,dk = \varphi(x)\varphi(y) \tag{4.1}$$

for all x, y in G (dk denotes the normalized Haar measure on K.) (For these results see Helgason [8].)

Let $G = KAN$ be the Iwasawa decomposition of G. Let $\mathfrak{A}$ be the Lie algebra of A, $\mathfrak{A}^*$ the dual space of $\mathfrak{A}$ and $\mathfrak{A}_c^*$ the complexification of $\mathfrak{A}^*$. For a fixed Weyl chamber in $\mathfrak{A}$ let Δ_+ denote the corresponding set of positive roots and let

$$\varrho = 1/2 \sum_{\alpha\in\Delta_+} m_\alpha \alpha$$

where m_α is the dimension of the root space of α. Define H: $G \to \mathfrak{A}$ by requiring that $x \in K \exp H(x) N$ for each x in G.

Harish-Chandra [7] has shown that if φ is continuous, $\varphi \not\equiv 0$, and if for all x, y in G.

$$\int_K \varphi(xky)\,dk = \varphi(x)\varphi(y),$$

then there is a λ in $\mathfrak{A}_c^*$ such that

$$\varphi(x) = \varphi_\lambda(x) = \int_K e^{(i\lambda-\varrho)H(xk)}\,dk$$

for all x in G.

Let $g \to Ad_G(\cdot)$ denote the adjoint representation of G and let W (the Weyl group) denote group of linear transformations in $Ad_G(K)$ that leave $\mathfrak{A}$ invariant. For each s in W, also denote by s the linear map of $\mathfrak{A}_c^*$ defined by $s\lambda(H) = \lambda(s^{-1}H)$ for each $\lambda \in \mathfrak{A}^*$, $H \in \mathfrak{A}$. Denote by C_ϱ the convex hull of $\{s\varrho \mid s \in W\}$. Helgason and Johnson [9] have shown that for $\lambda = \xi + i\eta$, $\xi, \eta \in \mathfrak{A}^*$, φ_λ is bounded if and only if $\eta \in C_\varrho$. Combining these results we have

Lemma 4.2. *The maximal ideal space of $\mathcal{L}^{\Delta}(G)$ can be identified with* $\{\varphi_\lambda \mid \lambda \in \mathfrak{A} + iC_\varrho\}$.

For each f in $\mathcal{L}^1(G)$ let

$$f_K(x) = \int_K\int_K f(k_1 x k_2)\,dk_1\,dk_2.$$

Clearly $f_K \in \mathcal{L}^{\Delta}(G)$ for each f in $\mathcal{L}^1(G)$. If $\lambda \in \mathfrak{A} + iC_\varrho$, $\varphi_\lambda \in \mathcal{L}^\infty(G)$ and

$$\begin{aligned} \langle\varphi_\lambda, f_K\rangle &= \int_G f_K(x)\varphi_\lambda(x)\,dx \\ &= \int_G\int_K\int_K f(k_1 x k_2)\varphi_\lambda(x)\,dk_1\,dk_2\,dx \\ &= \int_G\int_K\int_K f(x)\varphi_\lambda(k_1^{-1} x k_2^{-1})\,dk_1\,dk_2\,dx_2 \\ &= \int_G f(x)\varphi_\lambda(x)\,dx \\ &= \langle\varphi_\lambda, f\rangle. \end{aligned}$$

(Here we have made use of the fact that G is unimodular, being that it is semisimple, and that φ_λ satisfies 4.1.)

Recall the Iwasawa decomposition $G = KAN$. Define $\mathfrak{k}$: $G \to K$ by requiring that $x \in \mathfrak{k}(x)AN$ for each x in G. It is known (cf. Harish-Chandra [6]) that for any continuous function φ defined on K, and any x in G

$$\int_K \varphi(k)\,dk = \int_K \varphi(\mathfrak{k}(xk))e^{-2\varrho(H(xk))}\,dk.$$

Using this, and the fact that $H(x^{-1}\mathfrak{k}(xk)) = -H(xk)$, we have

$$\begin{aligned} \varphi_\lambda(x^{-1}) &= \int_K e^{(i\lambda-\varrho)H(x^{-1}k)}\,dk \\ &= \int_K e^{(i\lambda-\varrho)H(x^{-1}\mathfrak{k}(xk))}e^{-2\varrho H(xk)}\,dk \end{aligned}$$

$$= \int_K e^{(-i\lambda-\varrho)H(xk)}\,dk$$

$$= \varphi_{-\lambda}(x).$$

Also, if for $\lambda = \xi + i\eta$, ξ, $\eta \in \mathfrak{A}^*$ we write $\bar{\lambda} = \xi - i\eta$, we have

$$\operatorname{conj}(\varphi_\lambda(x)) = \int_K \operatorname{conj}(e^{(i\lambda-\varrho)H(xk)})\,dk$$

$$= \int_K e^{(i\bar{\lambda}-\varrho)H(xk)}\,dk$$

$$= \varphi_{\bar{\lambda}}(x).$$

We can now write

$$\operatorname{conj}\langle\varphi_\lambda, f^*\rangle = \operatorname{conj}\Big(\int_G \varphi_\lambda(x)\operatorname{conj}\big(f(x^{-1})\big)\,dx\Big)$$

$$= \int_G \operatorname{conj}\big(\varphi_\lambda(x^{-1})\big) f(x)\,dx$$

$$= \int_G \varphi_{-\bar{\lambda}}(x) f(x)\,dx$$

$$= \langle\varphi_{-\bar{\lambda}}, f\rangle.$$

Observing that for all f in $\mathscr{L}^1(G)$,

$$\operatorname{conj}\langle\varphi_\lambda, (f^*)_K\rangle = \langle\varphi_\lambda, f_K\rangle$$

if and only if

$$\operatorname{conj}\langle\varphi_\lambda, f^*\rangle = \langle\varphi_\lambda, f\rangle$$

if and only if

$$\langle\varphi_{-\bar{\lambda}}, f\rangle = \langle\varphi_\lambda, f\rangle$$

we have

LEMMA 4.3. *φ_λ is hermitian if and only if $\varphi_\lambda = \varphi_{-\bar{\lambda}}$.*

In [7], it is shown that $\varphi_\lambda = \varphi_\delta$ if and only if $\delta \in \{s\lambda \mid s \in W\}$. Therefore, in particular, φ_λ is not hermitian if $\lambda = \eta + i\eta$ where $0 \neq \eta \in C_\varrho$. Hence we have proved

PROPOSITION 4.4. *If G is a connected, noncompact, semisimple Lie group with finite center then $\mathscr{L}^d(G)$ is not symmetric.*

PROPOSITION 4.5. *If G is a connected reductive Lie group with noncompact semisimple component then $\mathfrak{A}(G)$ is not symmetric.*

Proof. We first make the following observation: Let π be a continuous homomorphism of H_1 onto H_2 with kernel H_0. Then $\mathscr{L}^1(H_2)$ can be identified with $\mathscr{L}^1(H_1/H_0)$. For proper choice of Haar measure we have that the mapping

$$f(x) \to \int_{H_0} f(xh_0)\,dh_0$$

is an isometric isomorphism of $\mathscr{L}^1(H_1)$ onto $\mathscr{L}^1(H_1/H_0)$. Therefore, $\mathscr{L}^1(H_1)$ is not symmetric if $\mathscr{L}^1(H_2)$ is not symmetric.

It only remains to observe that with our assumptions on G, $Ad_G(G)$ is a connected, noncompact semisimple Lie group with trivial center. Hence, since $g \to Ad_G(g)$ is a homomorphism of G onto $Ad_G(G)$, we can conclude that $\mathscr{L}^1(G)$ is not symmetric.

References

[1] D. W. Bailey, *On symmetry in certain group algebra*, Pacific J. Math. 24 (1968), pp. 413–419.

[2] R. A. Bonic, *Symmetry in group algebras of discrete groups*, Pacific J. Math. 11 (1961), pp. 73–94.

[3] G. van Dijk, *On symmetry of group algebras of motion groups*, Math. Ann. 179 (1969), pp. 219–226.

[4] I. M. Gelfand, u. M. A. Neumark, Unitäre Darstellungen der Klassischen Gruppen, Berlin, 1957.

[5] G. M. Goluzin, *Geometric Theory of Functions of a Complex Variable*, Translations of Math. Monographs, Amer. Math. Soc.

[6] Harish-Chandra, *Representations of a semisimple Lie group on a Banach space I*, Trans. Amer. Math. Soc. 75 (1953), pp. 185–243.

[7] — *Spherical functions on a semisimple Lie group I, II* Amer. J. Math. 80 (1958), pp. 241–310.

[8] S. Helgason, *Differential Geometry and Symmetric Spaces*, New York, 1962.

[9] — and K. Johnson, *The bounded spherical functions on symmetric spaces*, Adv. in Math. 3 (1969), pp. 586–593.

[10] K. Hoffman, *Fundamentals of Banach Algebras*, Monografias Mathemáticas Da Universidade Do Paraná (1962).pp. 227–287.

[11] A. Hulanicki, *On the spectral radius of hermitian elements in group algebras*, Pacific J. Math. 18 (1966), pp. 277-287.

[12] — *On symmetry of group algebras of discrete nilpotent groups*, Studia Math. 35 (1970), pp. 207–219.

[13] — *On the spectral radius in group algebras*, Studia Math. 34 (1970), pp. 209–214.

[14] J. W. Jenkins, *Symmetry and nonsymmetry in the group algebra of discrete groups*, Pacific J. Math. 32 (1970), pp. 131–145.

[15] — *On the spectral radius of elements in a group algebra*, Ill. J. Math. 15(1971), pp. 551–554.

[16] — *Free semigroups and unitary group representations*, Studia Math., 43 (1972), pp. 27–39.

[17] F. I. Mautner, *Spherical functions over p-adic fields I*, Amer. J. Math. 80 (1958), pp. 441–457.

[18] M. A. Naimark, *Normed Rings*, Groningen 1960.

[19] A. J. Silberger, PGL_2 *over the p-adics*, Lecture Notes in Math. (166), 1970.

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