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On power series in the differentiation operator

by

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Abstract. Let s be the differentiation operator in the Mikusiński operational calculus and let $(a_n|a = 0, 1, 2, \ldots)$ be a sequence of complex numbers an infinite number of which are non-zero. It is shown that the infinite series $S = \sum_{n=0}^{\infty} a_n s^n$ is convergent in the field of operators if and only if the Denjoy-Carleman class $C_I\{1/|a_n|\}$ is not quasi-analytic. In particular if $M_0 = 1$, $M_n > 0$, and M_n is a log convex sequence ($M_n^2 \leqslant M_{n-1} M_{n+1}$ for each $n = 1, 2, 3, \ldots$), then $S = \sum_{n=0}^{\infty} \frac{s^n}{M_n}$ is convergent in the field of operators if and only if $\sum_{n=0}^{\infty} M_n/M_{n+1} < \infty$.

Using a reasonable notion of support it is shown that if $S = \sum_{n=0}^{\infty} a_n s^n$ is convergent in the field of operators then S has support equal to $\{0\}$.

1. Introduction. We shall give necessary and sufficient conditions for convergence in the field of Mikusiński operators of power series in the differentiation operator s,

$$S = \sum_{n=0}^{\infty} a_n s^n \qquad a_n \text{ is a complex number for } n = 0, 1, 2, \ldots$$

Our terminology and notation for Mikusiński operators shall be as in Mikusiński's book [4].

In § 2 the classes $C_I\{M_n\}$ of Carleman and Mandelbrojt are discussed. We state the important theorems (Theorems 2.4 and 2.6) of Carleman [2] and Mandelbrojt [3] who characterized in different manners the quasi-analytic classes $C_I\{M_n\}$ in terms of the sequences (M_n). We show some of the properties of these classes in § 2. A rather special property is proved in Corollary 2.8 which is needed in the proof of the important Corollary 3.6.

In § 3 it is shown (our principal theorem) that S is convergent in the field of operators if and only if the class $C_I\{1/|a_n|\}$ is not quasi-analytic. Application of the theorems of Carleman and Mandelbrojt then yield criteria in terms of the coefficients a_n which are necessary and sufficient for S to be convergent (Corollaries 3.3, 3.4). It is shown that if S is a convergent series then (using the terminology of [1]) the support of S is the single point zero.

§ 4 proves a uniqueness theorem. Namely, if $S = \sum_{n\geqslant 0}^{\infty} \alpha_n s^n$ is convergent in the field of operators then $S = 0$ implies $\alpha_n = 0$ for every n.

2. Quasi-analytic classes. Let $(M_n | n = 0, 1, 2, \ldots)$ be a sequence of positive real numbers. By $C_I\{M_n\}$ we mean the class of all infinitely differentiable functions φ such that there are constants $\beta_\varphi > 0$ and B_φ depending on φ and

$$\max_{x\in I} |\varphi^{(n)}(x)| \leqslant \beta_\varphi B_\varphi{}^n M_n \quad \text{each } n = 0, 1, 2, \ldots \tag{1}$$

We shall allow M_n to be infinite so long as infinitely many M_n are finite. We always suppose

$$M_0 = 1, \quad 0 < M_{n-1} \leqslant \infty \text{ for each } n = 0, 1, 2, \ldots, \quad M_n < \infty \text{ for infinitely many } n. \tag{2}$$

DEFINITION 2.1. A sequence (M_n) is said to be *logarithmically convex* if

$$M_n^2 \leqslant M_{n-1} M_{n+1} \quad \text{for each } n = 0, 1, 2, \ldots$$

LEMMA 2.2. *$C_I\{M_n\}$ is a vector space under pointwise addition of functions. Let J be an interval, a and b real numbers, such that $x \in J$ implies $ax + b \in I$. Then for φ in $C_I\{M_n\}$ we have $\psi(x) = \varphi(ax+b) \in C_J\{M_n\}$. If (M_n) is a logarithmic convex sequence then $C_I\{M_n\}$ is an algebra under pointwise multiplication of functions.*

Proof. That $C_I\{M_n\}$ is a vector space and has the stated property under affine transformations follows from (1). If moreover (M_n) is logarithmically convex and $M_0 = 1$ then

$$M_k M_{n-k} \leqslant M_0 M_n = M_n \quad 0 \leqslant k \leqslant n.$$

Thus if φ and ψ are in $C_I\{M_n\}$ then

$$|D^n \varphi(x)\psi(x)| \leqslant \beta_\varphi \beta_\psi \sum_{k=0}^{n} \binom{n}{k} B_\varphi{}^k B_\psi{}^{n-k} M_k M_{n-k} \leqslant \beta_\varphi \beta_\psi (B_\varphi + B_\psi)^n M_n \tag{3}$$

for each $n = 0, 1, 2, \ldots$, and the lemma follows from (3).

DEFINITION 2.3. $C_I\{M_n\}$ is said to be quasi-analytic if $\varphi \in C_I\{M_n\}$, $x_0 \in I$, and

$$\varphi^{(n)}(x_0) = 0 \quad \text{for each } n = 0, 1, 2, \ldots$$

implies $\varphi(x) \equiv 0$ on I.

We are particularly interested in those $C_I\{M_n\}$ which are *not* quasi-analytic.

If (M_n) is logarithmically convex there is a simple characterization due to Mandelbrojt of the quasi-analytic classes in terms of the sequence (M_n). In the general case, where the M_n are not necessarily logarithmically convex, necessary and sufficient conditions in order that $C_I\{M_n\}$ be quasi-analytic were first given by Carleman [2].

THEOREM 2.4 (CARLEMAN). *Let*

$$\mu_n = (M_n)^{1/n} \quad \text{for each } n = 0, 1, 2, \ldots$$

and

$$\mu_n^* = \min_{k\geqslant 0} \mu_{n+k} \quad \text{for each } n = 0, 1, 2, \ldots.$$

Then $C_I\{M_n\}$ is not quasi-analytic if and only if

$$\sum_{n=0}^{\infty} \frac{1}{\mu_n^*} < \infty.$$

When the sequence (M_n) is not log convex it is still possible to associate with (M_n) a log convex sequence $(M_n{}^c)$. This process is described in detail by Mandelbrojt [3]. He calls the sequence $(M_n{}^c)$ the *convex regularized sequence of (M_n) regularized by means of logarithms.*

DEFINITION 2.5. Let $(M_n | n = 0, 1, 2, \ldots)$ satisfy (2) and suppose that

$$\lim_{n\to\infty} \frac{\log M_n}{n} = \infty. \tag{4}$$

Let U be the convex hull in R^2 of the set $\{(n, \log M_n)\colon n = 0, 1, 2, \ldots\}$.

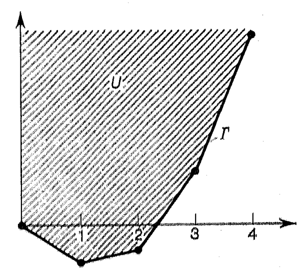

Fig. 1

The boundary of U consists of the union of $\{(0, x_2)\colon x_2 > 0\}$ with a convex polygonal curve Γ. The sequence $(M_n{}^c | n = 0, 1, 2, \ldots)$ is uniquely defined by the condition

$$(n, \log M_n{}^c) \in \Gamma \quad \text{for each } n = 0, 1, 2, \ldots$$

Some of the most important properties of the sequence (M_n^c) are

P_1: $(M_n^c | n = 0, 1, 2, \ldots)$ is a log convex sequence.

P_2: If M_n is log convex to begin with, then $M_n^c = M_n$ for each $n = 0, 1, 2, \ldots$

P_3: If (M_n) and $(\bar{M}_n)$ are two sequences such that $M_n \leqslant \bar{M}_n$ for such $n = 0, 1, 2, \ldots$ then $M_n^c \leqslant \bar{M}_n^c$ for each n.

P_4: $M_n^c \leqslant M_n$ for each $n = 0, 1, 2, \ldots$

In Theorem XII of ([3], p. 78) Mandelbrojt proves the following:

THEOREM 2.6 (MANDELBROJT). *Suppose I is a compact interval and that (M_n) satisfies* (2). *In order that $C_I\{M_n\}$ not be quasi-analytic it is necessary that* (4) *holds. If* (4) *holds then $C_I\{M_n\}$ is not quasi-analytic if and only if*

$$\sum_{n=0}^{\infty} \frac{M_n^c}{M_{n+1}^c} < \infty. \tag{5}$$

In particular we get the following corollaries.

COROLLARY 2.7. *If (M_n) is a logarithmically convex sequence then $C_I\{M_n\}$ is not quasi-analytic if and only if*

$$\sum_{n=0}^{\infty} \frac{M_n}{M_{n+1}} < \infty. \tag{6}$$

Proof. If (M_n) is logarithmically convex and (6) holds then (4) necessarily holds. Thus for a logarithmically convex sequence (5) alone (and thus (6) alone) is necessary and sufficient in order that $C_I\{M_n\}$ not be quasi-analytic.

COROLLARY 2.8. *Suppose $C_I\{M_n\}$ is not quasi-analytic. Then there exists a logarithmically convex sequence $\bar{M}_n^c$ such $C_I\{\bar{M}_n^c\} \subset C_I\{M_n\}$, $C_I\{\bar{M}_n^c\}$ is not quasi-analytic, and for every $B > 0$*

$$\sum_{n=0}^{\infty} \frac{B^n \bar{M}_n^c}{M_n} < \infty. \tag{7}$$

Proof. Let $\varrho_n = M_n^c / M_{n+1}^c$ for each $n = 1, 2, \ldots$ Since $C_I\{M_n\}$ is not quasi-analytic $\sum_{n=0}^{\infty} \varrho_n < \infty$ and since (M_n^c) is logarithmically

convex, $\varrho_n \downarrow 0$. Pick two sequences of integers (n_i) and (m_i) such that

$$n_i < m_i \leqslant n_{i+1} \quad \text{for each } i = 0, 1, 2, \ldots,$$

$$\sum_{n \geqslant n_i} \varrho_n < 10^{-i} \quad \text{for each } i = 0, 1, 2, \ldots$$

and

$$\varrho_{m_i} \leqslant \varrho_{n_i}/2 \quad \text{for each } i = 0, 1, 2, \ldots$$

We will define a new sequence $(\bar{\varrho}_n)$ by

$$\bar{\varrho}_n = 2^i \varrho_n \quad \text{when } m_i \leqslant n < n_{i+1}$$

and

$$\bar{\varrho}_n = 2^i \varrho_{m_i} \quad \text{when } n_i \leqslant n < m_i.$$

Then

$$\bar{\varrho}_n \downarrow 0 \tag{8}$$

and

$$\sum_{n=0}^{\infty} \bar{\varrho}_n \leqslant \sum_{n<n_0} \bar{\varrho}_n + \sum_{i \geqslant 0} \Big(\sum_{n \geqslant n_i} \bar{\varrho}_n \Big) \leqslant \sum_{n<n_0} \bar{\varrho}_n + \sum_{i \geqslant 0} \Big(\frac{2}{10} \Big)^i < \infty. \tag{9}$$

Let $(\bar{M}_n^c)$ be given by

$$\bar{M}_n^c = \Big[\prod_{j=0}^{n} \bar{\varrho}_j \Big]^{-1}.$$

Since

$$\frac{\bar{M}_n^c}{\bar{M}_{n+1}^c} = \bar{\varrho}_{n+1} \quad \text{for } n = 0, 1, 2, \ldots$$

$(\bar{M}_n^c)$ is logarithmically convex in view of (8), and $C_I\{\bar{M}_n^c\}$ is not quasi-analytic in view of (9).

Moreover, for each i let $C_i = \prod_{n<n_i} (\bar{\varrho}_n / \varrho_n)$ and $\bar{C}_i = C_i 2^{in_i}$. Then

$$\frac{\bar{M}_n^c}{M_n^c} = C_i \prod_{j \geqslant n_i}^{n} (\varrho_j / \bar{\varrho}_j) < C_i (2^{-i})^{n-n_i} < \bar{C}_i 2^{-in} \quad \text{for } n > n_i.$$

Since $M_n^c \leqslant M_n$ we have

$$\frac{\bar{M}_n^c}{M_n} \leqslant \frac{\bar{M}_n^c}{M_n^c} \leqslant \bar{C}_i (2^{-i})^n \quad \text{when } n > n_i$$

which proves (7).

Corollary 2.9. *Suppose $C_I\{M_n\}$ is not quasi-analytic. If $I' \subset I = [a, b]$ there is a nontrivial, nonnegative function $\varphi \in C_I\{M_n\}$ with support in I'.*

Proof. By Corollary 2.8 $C_I\{M_n\} \supset C_I\{\overline{M}_n^c\}$ where $(\overline{M}_n^c)$ is logarithmically convex and $C_I\{\overline{M}_n^c\}$ is also not quasi-analytic. Let ψ be a nontrivial function in $C_I\{\overline{M}_n^c\}$ such that $\psi^{(n)}(x_0) = 0$ for all $n = 0, 1, 2, \ldots$ Now ψ does not vanish identically on at least one of the two intervals $[a, x_0]$, $[x_0, b]$. Suppose ψ is nontrivial on $[x_0, b]$ and that x_1 is the largest value of x such that ψ vanishes identically on $[x_0, x]$. Let $\psi^*(x) = \psi(x)$ for $x \in [x_1, b]$ and $\psi^*(x) = 0$ on $[a, x_1]$. If $b - x_1 > \varepsilon > 0$ then

$$\theta(x) = \psi^*(x)\psi\big(-(x-x_1)+x_1+\varepsilon\big)$$

has support on $[x_1, x_1+\varepsilon]$, is nontrivial, and since $(\overline{M}_n^c)$ is logarithmically convex (3) shows that $\theta \in C_I\{\overline{M}_n^c\}$. Taking ε sufficiently small and letting φ be a translate of θ yields a function which satisfies the requirements of the corollary.

3. The convergence of power series in s. Because of the property of $C_I\{M_n\}$ under affine transformations (Lemma 2.2) if I is a compact interval and $C_I\{M_n\}$ is not quasi-analytic then $C_{I'}\{M_n\}$ is not quasi-analytic for any compact interval I'. Thus all compact intervals are equivalent with regard to whether a sequence (M_n) defines a quasi-analytic class on I. The convergence of the power series

$$S = \sum_{n=0}^{\infty} a_n s^n, \qquad n = 0, 1, \ldots$$

can be characterized in terms of the quasi-analyticity of $C_I\{1/|a_n|\}$.

Theorem 3.1. *S is convergent in $\mathscr{M}$ if and only if $C_I\{1/|a_n|\}$ is not quasi-analytic.*

Proof. If S is convergent there is an infinitely differentiable function φ in $\mathscr{C}$ such that

$$\sum_{n=0}^{\infty} a_n \varphi^{(n)} \tag{10}$$

is almost uniformly convergent. Let I be a compact interval which contains the support number of φ. Since (10) is uniformly convergent on I the terms $a_n\varphi^{(n)}$ are uniformly bounded on I and thus for some $\beta_\varphi > 0$

$$\operatorname*{Max}_I |\varphi^{(n)}(t)| \leqslant \beta_\varphi/|a_n|, \qquad n = 0, 1, 2, \ldots$$

which proves that $C_I\{1/|a_n|\}$ is not quasi-analytic.

Conversely, if $C_I\{1/|a_n|\}$ is not quasi-analytic then (Corollary 2.9) there is a nontrivial function $\varphi \in C_I\{1/|a_n|\}$ with support in the interior of I. If B_φ is a constant such that (1) holds and $\psi(x) = \varphi(x/2B_\varphi)$ for $x/2B_\varphi$ in I, $\psi(x) = 0$ otherwise, then

$$\operatorname*{Sup}_R |a_n\psi^{(n)}(x)| = \operatorname*{Sup}_I |a_n\varphi^{(n)}(x)/(2B_\varphi)^n| \leqslant \beta_\varphi/2^n$$

for each $n = 0, 1, 2, \ldots$. Thus $\sum_{n=0}^{\infty} a_n\psi^{(n)}$ is uniformly convergent on R and S is convergent in $\mathscr{M}$.

Corollary 3.2. *Let $M_n = 1/|a_n|$. A necessary condition that S be convergent in $\mathscr{M}$ is that (4) holds. If (4) holds a necessary and sufficient condition that S be convergent is that*

$$\sum_{n=0}^{\infty} \frac{M_n^c}{M_{n+1}^c} < \infty.$$

Proof. Theorem 3.1 and Theorem 2.6.

For logarithmically convex sequences this criterion of convergence is particularly simple.

Corollary 3.3. *Suppose (M_n) is such that $M_n^2 \leqslant M_{n+1}M_{n-1}$ for each $n = 0, 1, 2, \ldots$ Then*

$$S = \sum_{n=0}^{\infty} \frac{s^n}{M_n}$$

is convergent in $\mathscr{M}$ if and only if

$$\sum_{n=0}^{\infty} \frac{M_n}{M_{n+1}} < \infty.$$

From Carleman's Theorem we have another characterization.

Corollary 3.4. *Let*

$$\nu_n = \operatorname*{Max}_{h \geqslant 0} |a_{n+h}|^{\frac{1}{n+h}}.$$

Then S is convergent if and only if

$$\sum_{n=0}^{\infty} \nu_n < \infty.$$

The following definition is from [1].

Definition 3.5. An operator $a \in \mathscr{M}$ has support equal to a single point $\{x_0\}$ if for each $\delta > 0$ and each $\varepsilon > 0$ there is a φ with support in $(-\varepsilon, \varepsilon)$ and a ψ with support in $(x_0-\delta, x_0+\delta)$, $\varphi(x) \geqslant 0$, $\int_{-\infty}^{\infty}\varphi\,dx = 1$, and $a\varphi = \psi$.

Corollary 3.6. *If S is convergent then* $\operatorname{supp} S = \{0\}$.

Proof. If S is convergent $C_{[-1,1]}\{1/|a_n|\}$ is not quasi-analytic. Let $\overline{M}_n^c$ be as in Corollary 2.8. By Corollary 2.9, for any $\varepsilon > 0$, $\varepsilon_0 > 0$, $C_{[-1,1]}\{\overline{M}_n^c\}$ contains a nontrivial positive function with support in $(-\varepsilon_0, \varepsilon_0)$ and by (7)

$$S\varphi = \sum_{n=0}^{\infty} a_n \varphi^{(n)} \tag{11}$$

is uniformly convergent and from (11) we see $\operatorname{supp} S\varphi \subset [-\varepsilon_0, \varepsilon_0]$. Taking $\varepsilon_0 < \operatorname{Min}[\varepsilon, \delta]$ completes the proof.

4. The uniqueness theorem. We shall prove the following theorem.

Theorem 4.1. *If*

$$S = \sum_{n=0}^{\infty} a_n s^n$$

is convergent in $\mathcal{M}$ and $S = 0$ then $a_n = 0$ for $n = 0, 1, 2, \ldots$

Proof. Let $\varphi \in \mathscr{C}^\infty$ be a convergence factor for S. Let $\lambda_0 \in (0, 1)$ and define $f(x, \lambda)$

$$f(x, \lambda) = \sum_{n=0}^{\infty} a_n \lambda^n \varphi^{(n)}(x). \tag{12}$$

If $T > 0$ then Dirichlet's test for uniform convergence shows that (12) is uniformly convergent on D where

$$D = \{x\colon x \leqslant T\} \times \{\lambda\colon 0 \leqslant \lambda \leqslant \lambda_0,\ 0 \leqslant \lambda_0 < 1\}.$$

In particular for each $\lambda \in [0, 1]$ the series (12) is almost uniformly convergent. Let $\varphi_\lambda(x) = \varphi(\lambda x)$ when $0 < \lambda \leqslant \lambda_0$. For each such λ

$$\sum_{n=0}^{\infty} a_n \lambda^n \varphi^{(n)}(\lambda x) = \sum_{n=0}^{\infty} a_n \varphi_\lambda^{(n)}(x) = \lim_{N\to\infty} \Big(\sum_{n=0}^{N} a_n s^n\Big) \varphi_\lambda$$

is almost uniformly convergent and by the uniqueness of sequential limits in $\mathcal{M}$ the right hand side of this equation must be zero. Thus the left hand side is also zero and (12) is zero for every $\lambda \in (0, 1)$ and every x.

For each fixed x_0, $f(x_0, \lambda)$ is analytic on $|\lambda| < 1$ and since it vanishes identically we have

$$a_n \varphi^{(n)}(x_0) = 0$$

for every $n = 0, 1, 2, \ldots$ and every x_0. Since no $\varphi^{(n)}$ vanishes identically $a_n = 0$ for $n = 0, 1, 2, \ldots$

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