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Maps which preserve equality of distance

by

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Abstract. If $f\colon X \to Y$ is a continuous surjection, $f(0) = 0$, and $\|fx-fy\|$ depends functionally on $\|x-y\|$, then f is linear.

A theorem due to Mazur and Ulam ([2], p. 166 and [4]) states that every isometry of a normed real vector space onto a normed real vector space is linear up to translation. Charzyński [3] and Rolewicz [6] have shown, respectively, that surjective isometries of finite-dimensional F-spaces and of locally bounded spaces with concave norm are also linear.

The present paper extends the result of Mazur and Ulam in a different direction. The spaces remain normed real vector spaces, but we replace isometries by the more general notion of equality of distance preserving maps, maps with the property that the distance between image points depends functionally on the distance between domain points.

We prove in Section 1 that every continuous equality of distance preserving map from a normed real vector space onto a normed real vector space is affine-linear. This result generalizes the Mazur–Ulam theorem and yields a characterization of the similarity group of a space which does not presuppose linearity. In Section 2 the continuity hypothesis of Section 1 is shown to be a consequence of surjectivity when the domain has dimension $\geqslant 2$.

Schoenberg [7] and von Neumann and Schoenberg [5] investigated and classified all continuous equality of distance preserving maps from one separable or finite-dimensional Hilbert space into another. Corollary 2.3 in Section 2 shows that their continuity assumption is also redundant when the domain has dimension $\geqslant 2$.

1. Let R_0^+ denote the set of non-negative real numbers. Let X and Y be normed real vector spaces of dimension $\geqslant 1$, the norms in each space being denoted by the symbol $\|\ \|$.

DEFINITION 1.1. A map $f\colon X \to Y$ *preserves equality of distance* iff there exists a function $p\colon R_0^+ \to R_0^+$ such that for each x and y in X $\|fx-fy\| = p(\|x-y\|)$. The function p is called the *gauge function* for f.

A map f which preserves equality of distance may be characterized equivalently by the requirement that whenever x, y, z and w are in X and $\|x-y\| = \|z-w\|$, then $\|fx-fy\| = \|fz-fw\|$.

Mazur and Ulam proved that all surjective isometries — that is, all surjective equality of distance preserving maps with gauge function the identity function — are affine-linear. The proof of the result here depends on the Mazur–Ulam technique. That technique is implicit in the following theorem in metric space theory, similar to one stated by Aronszajn [1].

THEOREM 1.2. *Let (M, d) be a bounded metric space. Suppose there exist an element m in M, a surjective isometry $g\colon M \to M$ and a constant $K > 1$ such that for all x in M $d(gx, x) \geqslant Kd(m, x)$. Then every surjective isometry $h\colon M \to M$ fixes m.*

Proof. Since metric space isometries are injective, h^{-1} and g^{-1} exist and h, g, h^{-1} and g^{-1} are bijective isometries of M together with arbitrary (finite) compositions of them.

Define a sequence of isometries $g_n\colon M \to M$ and elements m_n in M indexed by the integers $n \geqslant 1$.

Let

$$\begin{aligned} g_1 &= g, & m_1 &= m, \\ g_2 &= hgh^{-1}, & m_2 &= hm, \\ g_{n+1} &= g_{n-1}g_n(g_{n-1})^{-1}, & m_{n+1} &= g_{n-1}m_n, \qquad n \geqslant 2. \end{aligned}$$

Each g_n is a bijective, invertible isometry of M and a straight-forward induction yields:

$$d(g_n x, x) \geqslant Kd(m_n, x) \qquad x \in M,\ n \geqslant 1. \tag{1.21}$$

If we let $x = m_{n+1}$ in 1.21, we obtain $d(m_{n+2}, m_{n+1}) = d(g_n m_{n+1}, m_{n+1}) \geqslant Kd(m_n, m_{n+1}) = Kd(m_{n+1}, m_n)$, and by another induction $d(m_{n+2}, m_{n+1}) \geqslant K^n d(m_2, m_1)$ for all $n \geqslant 1$.

Since M is a bounded metric space, there exists a positive number $N \geqslant d(m_{n+2}, m_{n+1})$ for all n. Then $N/K^n \geqslant d(m_2, m_1)$ for all n and since $K > 1$, $d(m_2, m_1) = 0$. Therefore, $hm = m_2 = m_1 = m$, and h fixes m. ■

THEOREM 1.3. *Let $f\colon X \to Y$ with $f(0) = 0$ be a continuous surjective map which preserves equality of distance. Then:*

(i) *f is linear*

and

(ii) *$f = \lambda T$ where λ is a non-zero real number and T is an isometry of X onto Y.*

Proof. Fix any x in X.

Let $M = \{y\colon y \text{ is in } Y \text{ and } \|y\| = \|2f(x)-y\| \leqslant 2\|fx\|\}$, let $m = fx$ and define $g\colon M \to M$ by $gy = 2f(x)-y$. Then M, m and g have the following properties:

(1.31) *M is a bounded metric space with $d(y_1, y_2) = \|y_1-y_2\|$.*

(1.32) *$m = fx$ is in M.*

(1.33) *g is an isometry from M onto M.*

$g(M) = M$ since g is its own inverse.

(1.34) *$d(gy, y) \geqslant Kd(m, y)$ for all y in M with $K = 2$.*

This follows from $d(gy, y) = \|gy-y\| = \|(2f(x)-y)-y\| = 2\|f(x)-y\| = 2d(m, y)$.

By Theorem 1.2 and Properties 1.31 thru 1.34, we conclude that every surjective isometry of M fixes m. We therefore define an appropriate isometry.

Fix x_0 in $f^{-1}(2f(x))$. Define $h\colon M \to M$ by $hy = f(x_0 - f^{-1}(y))$.

h has the following properties:

(1.35) *h is well-defined.*

If $f(x_1) = f(x_2) = y$, then $\|f(x_0-x_1)-f(x_0-x_2)\| = p(\|(x_0-x_1) - (x_0-x_2)\|) = p(\|x_2-x_1\|) = \|f(x_2)-f(x_1)\| = \|y-y\| = 0$. Here p is the gauge function for f.

(1.36) *h is an isometry.*

If $f(x_1) = y_1$ and $f(x_2) = y_2$, $\|h(y_1)-h(y_2)\| = \|f(x_0-x_1)-f(x_0-x_2)\| = p(\|x_2-x_1\|) = \|f(x_2)-f(x_1)\| = \|y_2-y_1\|$.

(1.37) *h is an isometry from M into M.*

With $f(x_1) = y_1 \in M$, $\|2f(x)-h(y_1)\| = \|2f(x)-f(x_0-x_1)\| = \|f(x_0) - f(x_0-x_1)\| = p(\|x_1-0\|) = \|f(x_1)-f(0)\| = \|y_1-0\| = \|y_1\| = \|2f(x)-y_1\| = \|f(x_0)-f(x_1)\| = p(\|x_0-x_1\|) = p(\|(x_0-x_1)-0\|) = \|f(x_0-x_1)-f(0)\| = \|h(y_1)-0\| = \|h(y_1)\|$, and so $h(M) \subseteq M$.

(1.38) *h is a surjective isometry.*

$h(M) = M$ since h is its own inverse.

By Theorem 1.2 h fixes m, and $fx = m = hm = f(x_0-f^{-1}(m)) = f(x_0-f^{-1}(fx)) = f(x_0-x)$. Consequently, $0 = \|fx-f(x_0-x)\| = p(\|x - (x_0-x)\|) = p(\|2x-x_0\|) = \|f(2x)-f(x_0)\| = \|f(2x)-2f(x)\|$, and we conclude:

(1.39) *$f(2x) = 2f(x)$ for all x in X.*

Now, for fixed y in X, define $f_y(x) = f(x+y)-fy$ for all x in X. $f_y\colon X \to Y$ inherits continuity and surjectivity from f and $f_y(0) = 0$. Further, f_y preserves equality of distance since $\|f_y(x_1)-f_y(x_2)\| = \|f(x_1+y)-f(x_2+y)\| = p(\|(x_1+y)-(x_2+y)\|) = p(\|x_1-x_2\|)$. (In fact, f_y has the same gauge function as f.) It follows that 1.39 applies to f_y as well as f.

Let x and y be arbitrary elements in X. Then $f((x-y)+y)-fy = f_y(x-y) = 2f_y((x-y)/2) = 2[f((x-y)/2+y)-fy]$. Equating the end terms and simplifying, we have: $fx+fy = 2f((x+y)/2) = f(x+y)$. f is consequently additive. Since f is continuous, f must then be linear and (i) of Theorem 1.3 is established.

Let u be a unit vector in X. For a in R_0^+, $p(a) = p(\|au-0\|) = \|f(au)-f(0)\| = \|af(u)\| = a\|fu-f(0)\| = ap(\|u-0\|) = ap(1)$. Let $\lambda = p(1)$. $\lambda \neq 0$ since if $\lambda = 0$, $p \equiv 0$ and f cannot be surjective. Let $T = f/\lambda$. $T\colon X \to Y$ is surjective and $\|Tx-Ty\| = \|(fx)/\lambda-(fy)/\lambda\| = (1/\lambda)\|fx-fy\| = (1/\lambda)p(\|x-y\|) = (1/\lambda)\|x-y\|p(1) = \|x-y\|$ for all x and y in X. This establishes (ii) of Theorem 1.3. ■

Remark 1.4. Curiously, the additivity of f in Theorem 1.3 depends neither on the continuity of f nor on the norm in the domain space X. If $f\colon X \to Y$ with $f(0) = 0$, if $f(X)$ is an additive subgroup of Y, and if there exists a function $P\colon X \to R_0^+$ such that $\|fx-fy\| = P(x-y)$ for all x and y in X, then f is additive. The argument in Theorem 1.3 applies to this case with minor changes.

2. We now attempt to eliminate the continuity hypothesis in Theorem 1.3.

LEMMA 2.1. *Given $\varepsilon > 0$ and a normed real vector space X with dimension $X \geqslant 2$, then for each x in X with $\|x\| \leqslant 2\varepsilon$ there exist x_1 and x_2 in X such that $\|x_1\| = \|x_2\| = \varepsilon$ and $x = x_1+x_2$.*

Proof. If $x = 0$, choose u in X with $\|u\| = 1$. Letting $x_1 = \varepsilon u = -x_2$, we have our conclusion. Assume then that $x \neq 0$.

Consider $S_\varepsilon = \{y\colon y \text{ is in } X \text{ and } \|y\| = \varepsilon\}$. If v and w are independent vectors in S_ε, then

$$\alpha_{v,w}(t) = \frac{\varepsilon[(\cos t)v+(\sin t)w]}{\|(\cos t)v+(\sin t)w\|}$$

is a path in S_ε from v to w. If v and w are dependent vectors in S_ε, then $w = \pm v$. Using dimension $X \geqslant 2$, choose z in S_ε such that z is independent of v (and hence of w also). Then

$$\beta_{v,w}(t) = \begin{cases} \alpha_{v,z}(t), & t \leqslant \pi/2, \\ \alpha_{z,w}(t-(\pi/2)), & t \geqslant \pi/2 \end{cases}$$

is a path in S_ε from v to w. We conclude that S_ε is path-connected.

Now define $\mu\colon S_\varepsilon \to R_0^+$ by $\mu(y) = \|x-y\|$ for y in S_ε. Then μ is a continuous function on S_ε. Since $\pm\varepsilon(x/\|x\|)$ is in S_ε, we have:

$$\mu(\varepsilon(x/\|x\|)) = \|x-\varepsilon(x/\|x\|)\| = |\|x\|-\varepsilon| \leqslant \varepsilon < \|x\|+\varepsilon = \|x-(-\varepsilon)(x/\|x\|)\| = \mu((-\varepsilon)(x/\|x\|)).$$

By the path-connectedness of S_ε there exists an element x_1 in S_ε such that the intermediate value $\varepsilon = \mu(x_1)$.

Let $x_2 = x-x_1$. Then $x = x_1+x_2$, $\|x_1\| = \varepsilon$ since x_1 is in S_ε, and $\|x_2\| = \|x-x_1\| = \mu(x_1) = \varepsilon$. ■

LEMMA 2.2. *Let $f\colon X \to Y$ preserve equality of distance and suppose dimension $X \geqslant 2$. If for every $\varepsilon > 0$ there exist x and y in X such that $x \neq y$ and $\|fx-fy\| < \varepsilon$, then f is uniformly continuous.*

Proof. Given $\varepsilon > 0$, choose x and y in X such that $x \neq y$ and $\|fx-fy\| < \varepsilon/3$. Let $\delta = 2\|x-y\| > 0$.

If v and w are in X and $\|v-w\| \leqslant \delta$, by Lemma 2.1 there exist x_1 and x_2 in X with $\|x_1\| = \|x_2\| = \delta/2$ and $v-w = x_1+x_2$. Then $\|fv-fw\| = p(\|v-w\|) = p(\|x_1-(-x_2)\|) = \|f(x_1)-f(-x_2)\| \leqslant \|f(x_1)-f(0)\|+\|f(0)-f(-x_2)\| = p(\|x_1-0\|)+p(\|0-(-x_2)\|) = 2p(\delta/2) = 2p(\|x-y\|) = 2\|fx-fy\| < 2(\varepsilon/3) < \varepsilon$.

Thus if $\|v-w\| \leqslant \delta$, $\|fv-fw\| < \varepsilon$ and so f is uniformly continuous. ■

COROLLARY 2.3. *Let $f\colon X \to Y$ preserve equality of distance with dimension $X \geqslant 2$ and Y separable. Then f is uniformly continuous.*

Proof. Assume $\varepsilon > 0$ is given such that for all x and y in X with $x \neq y$ $\|fx-fy\| \geqslant \varepsilon$. Let Σ be a countable dense subset of Y. For each x in X choose s_x in Σ such that $\|fx-s_x\| < \varepsilon/2$. Then $x \to s_x$ is an injective map from X into the countable set Σ. Consequently, X is countable or finite. Since X is a real vector space of positive dimension, this is impossible. Lemma 2.2 now applies to yield uniform continuity of f. ■

THEOREM 2.4. *Let $f\colon X \to Y$ with $f(0) = 0$ be a surjective map which preserves equality of distance. Then the conclusions of Theorem 1.3 hold provided dimension $X \geqslant 2$.*

Proof. Continuity of f is the only additional hypothesis needed for the conclusions of Theorem 1.3 to hold. But since f is surjective, f is uniformly continuous by Lemma 2.2. ■

EXAMPLE 2.5. Let $\{1\} \cup \{e_a\colon a \text{ is in } A\}$ be a Hamel base for the set R of real numbers relative to the field Q of rational numbers. (Here A is some suitable index set.) Define $f\colon R \to R$ by setting $f(1) = 1$, $f(e_a) = 2e_a$ for a in A, and by extending Q-linearly to all of R. Then:

(2.51) *f is a bijection with $f(0) = 0$.*

This follows from the fact that f is Q-linear and takes the original Hamel base onto $\{1\} \cup \{2e_a\colon a \text{ is in } A\}$, which is also a Hamel base for R relative to Q.

(2.52) *f is non-linear.*

Suppose $f(\lambda x) = \lambda fx$ for all x in R. Since λ is a real number, $\lambda = q+\Sigma q_a e_a$ for some q and q_a's in Q. Hence, $q+\Sigma q_a e_a = \lambda = \lambda f(1) = f(\lambda) = f(q+\Sigma q_a e_a) = q+\Sigma 2q_a e_a$. This implies that $\Sigma q_a e_a = 0$ and $\lambda = q$.

So λ is rational and f is not R-linear.

(2.53) *f preserves equality of distance.*

Any additive map f from R into a normed real vector space Y preserves equality of distance. For, define $p(t) = \|f(t)\|$ for t in R_0^+. Then for all x and y in R, $\|fx-fy\| = \|f(x-y\| = \|\pm f(|x-y|)\| = \|f(|x-y|)\| = p(|x-y|)$.

Properties 2.51, 2.52 and 2.53 of Example 2.5 show that Theorem 2.4 fails if X is permitted to be one-dimensional.

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On the conjugates of some function spaces

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Abstract. For $p < 1$ and the underlying measure space non atomic, $L(p, \infty)^* = \{0\}$. Results are also given in the atomic case.

I. Introduction. The function spaces $L(p, q)$ form a two parameter family which incorporates the familiar L^p spaces ($L^p = L(p, p)$) as well as other important function spaces. The family $L(p, q)$ is a convenient setting for interpolation theorems for operators, and so is of interest for problems in harmonic analysis.

The dual spaces $L(p, q)^*$ have been studied, and in many cases characterised. (See [1], [2]). This note considers the previously untreated case when $0 < p < 1$ and $q = \infty$.

Throughout this note (X, Σ, μ) is a σ-finite measure space with $0 \leqslant \mu$.

Definition 1. For each measurable f we define

$$f_*(y) = \mu\{x \mid |f(x)| > y\}.$$

Cconfining ourselves to those f such that $f_*(y) < \infty$ for some $y > 0$ define.

Definition 2.

$$f^*(t) = \inf\{y \mid f_*(y) \leqslant t\}.$$

Definition 3. For $0 < p < \infty$, $0 < q < \infty$

$$\|f\|_{p,q}^* = \left[\int_0^\infty [t^{1/p} f^*(t)]^q \frac{dt}{t}\right]^{1/q}$$

and for $0 < p \leqslant \infty$

$$\|f\|_{p,\infty}^* = \sup_{t>0} t^{1/p} f^*(t)$$

Define also $L(p, q) = \{f \mid \|f\|_{p,q}^* < \infty\}$.

A detailed discussion of $L(p, q)$ spaces may be found in [2].