

The class of all possible limit distributions of sequences  $c_n |S_n|$ , where  $c_n > 0$  and  $S_n = X_1 + X_2 + \dots + X_n$  ( $n = 1, 2, \dots$ )  $X_k$  being independent random  $r$ -vectors with spherical symmetric distribution coincides with the class of all probability distributions  $P$  on  $[0, \infty)$  whose integral transform (26) is of the form

$$\Phi_P(t) = \exp \int_0^\infty \int_x^\infty \frac{F\left(\frac{r}{2}\right) \left(\frac{2}{tu}\right)^{\frac{r}{2}-1} J_{\frac{r}{2}-1}(tu) - 1}{u} du \frac{m(dx)}{\log(1+x^2)},$$

where  $m$  is a finite Borel measure on  $[0, \infty)$ .

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#### On generalized variations (II)

by

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**Abstract.** A  $\varphi$ -function is a non-decreasing function, continuous for  $u > 0$ ,  $\varphi(u) = 0$  only for  $u = 0$  and  $\lim_{u \rightarrow \infty} \varphi(u) = \infty$ . For a function  $x$  with domain  $[a, b]$  put

$$V_\varphi(x) = \sup \sum_{v=1}^n \varphi(|x(t_v) - x(t_{v-1})|),$$

supremum is taken over all partitions of  $[a, b]$ ,  $\mathcal{V}^{*\varphi}$  denotes the class of all functions  $x$  defined on  $[a, b]$  for which  $x(a) = 0$  and  $V_\varphi(\lambda x) < \infty$  for certain  $\lambda > 0$ , and  $\mathcal{C}\mathcal{V}^{*\varphi}$  denotes the class of all functions continuous on  $[a, b]$  belonging to  $\mathcal{V}^{*\varphi}$ . Among all  $\varphi$ -functions the log-convex  $\varphi$ -functions are distinguished i.e. ones satisfying the condition

$$\varphi(u^a v^b) \leq a\varphi(u) + b\varphi(v) \quad \text{for } u, v > 0, a, b \geq 0, a + b = 1.$$

There are presented two proofs of L. C. Young's Theorem that if  $\varphi$  and  $\varphi^*$  are log-convex  $\varphi$ -functions satisfying the following L. C. Young's condition

$$(*) \quad \sum_{v=1}^{\infty} \varphi_{-1}(1/v) \varphi_{-1}^*(1/v) < \infty$$

where  $\varphi_{-1}$  and  $\varphi_{-1}^*$  are the inverse functions to  $\varphi$  and  $\varphi^*$  respectively then the integral  $\int_a^b x(t) dy(t)$  for functions  $x \in \mathcal{C}\mathcal{V}^{*\varphi}$  and  $y \in \mathcal{V}^{*\varphi^*}$  exists in the sense of Riemann-Stieltjes.

Estimations of this integral with the use of series in  $(*)$  are given. On the same assumptions is proved the theorem on passing to the limit under the sign of RS-integral, in particular — the analogue of Helly's theorem. It is shown also that if  $\varphi$  and  $\varphi^*$  are convex  $\varphi$ -functions satisfying the certain conditions for which L. C. Young's condition  $(*)$  does not hold then there are functions  $x \in \mathcal{C}\mathcal{V}^{*\varphi}$  and  $y \in \mathcal{C}\mathcal{V}^{*\varphi^*}$  such that their RS-integral does not exist. These results proved for scalar functions are generalized for functions with values in Banach spaces.

**0. Introduction.** The present paper can be regarded as a second part of paper [9] which, under the same title, appeared in Studia Math. in 1959 (results of [9] were earlier announced in [8]). In the present paper the notations essentially differ from those employed in [9] i.e. in all places where in [9] and other papers dealing with the theory of Orlicz spaces symbols  $M, N$  etc. were used we now write  $\varphi, \psi, \dots$ . The purpose

of this paper is to give existence proofs of Riemann-Stieltjes integrals  $\int_a^b x dy$  where both  $x$  and  $y$  are functions of finite generalized variations. Fundamental papers in this field are due to L. C. Young who obtained sufficient conditions for the existence of the integral  $\int_a^b x dy$  in form of convergence of a certain series, which is in our paper called the L. C. Young's series. This series plays also an important role in our investigations; in general, our paper develops and explains the ideas presented in L. C. Young's papers [11] and [12]. The last section of the paper briefly outlines the problem of the existence of the Riemann-Stieltjes integral when one of the functions is a scalar function and the other is a vector function with values in a Banach space.

**1.  $\varphi$ -functions and L. C. Young's series.** We call a  $\varphi$ -function every real, nondecreasing and continuous for  $u \geq 0$  function equal 0 only at the point  $u = 0$  and tending to  $\infty$  when  $u \rightarrow \infty$ . Among all  $\varphi$ -functions certain classes of functions are of a special importance: the so called *log-convex  $\varphi$ -functions*, i.e. those which satisfy the inequality

$$\varphi(u^\alpha v^\beta) \leq \alpha \varphi(u) + \beta \varphi(v) \quad \text{for } u, v > 0 \text{ and } \alpha, \beta \geq 0, \alpha + \beta = 1,$$

and convex  $\varphi$ -functions i.e. those for which the inequality

$$\varphi(\alpha u + \beta v) \leq \alpha \varphi(u) + \beta \varphi(v) \quad \text{for } u, v \geq 0 \text{ and } \alpha, \beta \geq 0, \alpha + \beta = 1$$

holds. Clearly, a  $\varphi$ -function  $\varphi$  is log-convex if and only if it can be represented in the form  $\varphi(u) = \Phi(\lg u)$  for  $u > 0$ , where  $\Phi$  is a convex function on the whole real axis. From this we easily deduce that a log-convex  $\varphi$ -function is strictly increasing for  $u \geq 0$ . Log-convex  $\varphi$ -functions were considered in [4], where they were used in generalizations of Hardy spaces  $H^p$ ,  $p > 0$ . Convex  $\varphi$ -functions are a particular case of log-convex  $\varphi$ -functions and their applications are well known in the literature; they were extensively dealt with in the monograph [3]. The following conditions imposed on convex  $\varphi$ -functions are essential for those applications:

$$(0_1) \quad \lim_{u \rightarrow 0+} u^{-1} \varphi(u) = 0 \quad \text{and} \quad (\infty_1) \quad \lim_{u \rightarrow \infty} u^{-1} \varphi(u) = \infty.$$

Under conditions  $(0_1)$  and  $(\infty_1)$  we can define by the formula

$$\varphi^*(v) = \sup \{uv - \varphi(u) : u \geq 0\} \quad (v \geq 0)$$

a *complementary function* for  $\varphi$  which is also a convex  $\varphi$ -function, satisfies conditions  $(0_1)$  and  $(\infty_1)$  and moreover  $(\varphi^*)^* = \varphi$ . The functions  $\varphi$  and  $\varphi^*$  satisfy the inequality

$$v \leq \varphi_{-1}(v) \varphi_{-1}^*(v) \leq 2v \quad \text{for } v \geq 0 \text{ (see [3])},$$

where  $\varphi_{-1}$  and  $\varphi_{-1}^*$  are the inverse functions to  $\varphi$ , resp.  $\varphi^*$ . Indeed, the definition of  $\varphi^*$  implies the Young inequality  $uv \leq \varphi(u) + \varphi^*(v)$  for  $u, v \geq 0$  whence we get  $\varphi_{-1}(v) \cdot \varphi_{-1}^*(v) \leq v + v = 2v$  for  $v \geq 0$ . The other part of the above inequality is obvious for  $v = 0$ . Excluding this case we observe that for  $0 < t < u$  it holds  $\frac{u}{t} \varphi(t) \leq \varphi(u)$  and so we have for  $t > 0$

$$\varphi^*\left(\frac{\varphi(t)}{t}\right) = \sup \left\{ \frac{u}{t} \varphi(t) - \varphi(u) : 0 \leq u \leq t \right\} \leq \sup \left\{ \frac{u}{t} \varphi(t) : 0 \leq u \leq t \right\} = \varphi(t);$$

now, putting  $t = \varphi_{-1}(v)$ , we get  $v \leq \varphi_{-1}(v) \varphi_{-1}^*(v)$  for  $v > 0$ .

Generally, for a  $\varphi$ -function  $\varphi$  we define the indices:

$$s_\varphi = \lim_{a \rightarrow \infty} \frac{\lg \bar{l}_\varphi(a)}{\lg a} \quad \text{and} \quad \sigma_\varphi = \lim_{a \rightarrow \infty} \frac{\lg \bar{l}_\varphi(a)}{\lg a},$$

where

$$\bar{l}_\varphi(a) = \liminf_{u \rightarrow 0+} \frac{\varphi(au)}{\varphi(u)} \quad \text{and} \quad \bar{l}_\varphi(a) = \limsup_{u \rightarrow 0+} \frac{\varphi(au)}{\varphi(u)}, \quad ([5], [6], [7]).$$

These indices are closely connected with widely used conditions  $(\Delta_2)$ ,  $(V_2)$  and  $(V_2)$ . Namely  $\varphi$  satisfies condition  $(\Delta_2)$ :  $\varphi(2u) \leq c \varphi(u)$  for  $0 \leq u \leq u_0$  with some constants  $c > 1$  and  $u_0 > 0$  if and only if  $\sigma_\varphi < \infty$ ,  $\varphi$  satisfies condition  $(V_2)$ :  $2\varphi(u) \leq \varphi(du)$  for  $0 \leq u \leq u_0$  with some constants  $d > 1$ ,  $u_0 > 0$  if and only if  $s_\varphi > 0$ , and finally  $\varphi$  satisfies condition  $(V_2)$ :  $2\varphi(u) \leq d^{-1} \varphi(du)$  for  $0 \leq u \leq u_0$  with some constants  $d > 1$  and  $u_0 > 0$  ( $\varphi(u)/u$  satisfies  $(V_2)$ ) if and only if  $s_\varphi > 1$ .

If  $\varphi$  is a convex  $\varphi$ -function then  $1 \leq s_\varphi \leq \infty$ , and if it besides satisfies  $(0_1)$  and  $(\infty_1)$ , then

$$\frac{1}{s_\varphi} + \frac{1}{\sigma_{\varphi^*}} = 1 \quad \text{and} \quad \frac{1}{\sigma_\varphi} + \frac{1}{s_{\varphi^*}} = 1 \quad ([5]).$$

**1.1.** Let  $\varphi$  and  $\varphi^-$  be strictly increasing  $\varphi$ -functions. The series

$$(*) \quad \sum_{v=1}^{\infty} \varphi_{-1}\left(\frac{1}{v}\right) \varphi_{-1}^-\left(\frac{1}{v}\right),$$

where  $\varphi_{-1}$  and  $\varphi_{-1}^-$  are the inverse functions respectively to  $\varphi$  and  $\varphi^-$ , we shall from now on call the *L. C. Young's series for functions  $\varphi$  and  $\varphi^-$* . This is due to the fact that this series was first used in L. C. Young's paper [12].

In this section we shall give some lemmas concerning L. C. Young's series.

**1.11.** If the L. C. Young's series for strictly increasing  $\varphi$ -functions  $\varphi$  and  $\varphi^-$  is convergent, then for arbitrary numbers  $A, B \geq 0$  also the series  $\sum_{\nu=0}^{\infty} \varphi_{-1}(A/\nu) \varphi_{-1}^-(B/\nu)$  is convergent ([12]).

Proof. Let  $m$  be a positive integer such that  $A \leq m$  and  $B \leq m$ . Then for  $km \leq \nu < (k+1)m$  we have  $\frac{A}{\nu} \leq \frac{m}{km} = \frac{1}{k}$  and  $\frac{B}{\nu} \leq \frac{1}{k}$ . Hence we get

$$\sum_{\nu=1}^{\infty} \varphi_{-1}\left(\frac{A}{\nu}\right) \varphi_{-1}^-\left(\frac{B}{\nu}\right) \leq \sum_{\nu=1}^{m-1} \varphi_{-1}\left(\frac{A}{\nu}\right) \varphi_{-1}^-\left(\frac{B}{\nu}\right) + m \sum_{k=1}^{\infty} \varphi_{-1}\left(\frac{1}{k}\right) \varphi_{-1}^-\left(\frac{1}{k}\right) < \infty.$$

**1.12.** The L. C. Young's series for strictly increasing  $\varphi$ -functions  $\varphi$  and  $\varphi^-$  is convergent if and only if for any positive integer  $m \geq 2$  the series  $\sum_{k=0}^{\infty} m^k \varphi_{-1}(1/m^k) \varphi_{-1}^-(1/m^k)$  is convergent. Moreover, for arbitrary numbers  $A, B \geq 0$  the following inequality holds:

$$\begin{aligned} \frac{m-1}{m} \sum_{k=0}^{\infty} m^k \varphi_{-1}\left(\frac{A}{m^k}\right) \varphi_{-1}^-\left(\frac{B}{m^k}\right) &\leq \sum_{\nu=1}^{\infty} \varphi_{-1}\left(\frac{A}{\nu}\right) \varphi_{-1}^-\left(\frac{B}{\nu}\right) \\ &\leq (m-1) \sum_{k=0}^{\infty} m^k \varphi_{-1}\left(\frac{A}{m^k}\right) \varphi_{-1}^-\left(\frac{B}{m^k}\right). \end{aligned}$$

Proof. Since for  $m^k \leq \nu < m^{k+1}$  we have  $\frac{1}{m^{k+1}} < \frac{1}{\nu} \leq \frac{1}{m^k}$ , hence

$$\begin{aligned} \frac{m-1}{m} \sum_{k=0}^{\infty} m^k \varphi_{-1}\left(\frac{A}{m^k}\right) \varphi_{-1}^-\left(\frac{B}{m^k}\right) \\ \leq \varphi_{-1}(A) \varphi_{-1}^-(B) + \sum_{k=0}^{\infty} (m^{k+1} - m^k) \varphi_{-1}\left(\frac{A}{m^{k+1}}\right) \varphi_{-1}^-\left(\frac{B}{m^{k+1}}\right) \\ \leq \varphi_{-1}(A) \varphi_{-1}^-(B) + \sum_{k=0}^{\infty} \sum_{\nu=m^{k+1}}^{m^{k+1}-1} \varphi_{-1}\left(\frac{A}{\nu}\right) \varphi_{-1}^-\left(\frac{B}{\nu}\right) = \sum_{\nu=1}^{\infty} \varphi_{-1}\left(\frac{A}{\nu}\right) \varphi_{-1}^-\left(\frac{B}{\nu}\right) \\ \leq \sum_{k=0}^{\infty} (m^{k+1} - m^k) \varphi_{-1}\left(\frac{A}{m^k}\right) \varphi_{-1}^-\left(\frac{B}{m^k}\right) = (m-1) \sum_{k=0}^{\infty} m^k \varphi_{-1}\left(\frac{A}{m^k}\right) \varphi_{-1}^-\left(\frac{B}{m^k}\right). \end{aligned}$$

**1.13.** If the L. C. Young's series for strictly increasing  $\varphi$ -functions  $\varphi$  and  $\varphi^-$  is convergent, then there exists a convex  $\varphi$ -function  $\psi$  satisfying condition  $(0_1)$ , such that the L. C. Young's series for  $\varphi$ -functions  $\varphi^-$  and  $\varphi^+$ , where  $\varphi^+(u) = \psi(\varphi(u))$  and  $\varphi^-(u) = \psi(\varphi^-(u))$  for  $u \geq 0$ , is convergent (see [12], (5.3)).

Proof. From the hypothesis it follows, in view of 1.11, that  $\sum_{\nu=1}^{\infty} \varphi_{-1}(3/\nu) \varphi_{-1}^-(3/\nu) < \infty$ . Hence there exists a sequence of positive integers  $\{k_n\}$  such that for  $n = 1, 2, \dots$  we have

$$k_{n+1} > \frac{n+1}{n} k_n \quad \text{and} \quad \sum_{\nu=k_n}^{\infty} \varphi_{-1}\left(\frac{3}{\nu}\right) \varphi_{-1}^-\left(\frac{3}{\nu}\right) \leq \frac{1}{n^3}.$$

Let us put

$$\psi(u) = \int_0^u p(t) dt, \quad \text{where } p(t) = \begin{cases} \frac{1}{n} & \text{for } \frac{1}{k_{n+1}} < t \leq \frac{1}{k_n} \quad (n = 1, 2, \dots), \\ 1+t & \text{for } \frac{1}{k_1} < t. \end{cases}$$

Since the function  $p$  is positive and nondecreasing for  $t > 0$  and tends to 0 when  $t \rightarrow 0+$ , so  $\psi$  is a convex  $\varphi$ -function satisfying condition  $(0_1)$ . Let us observe that for  $nk_n \leq \nu < n k_{n+1}$  the following inequalities hold:

$$\psi\left(\frac{3n}{\nu}\right) \geq \int_{n/\nu}^{2n/\nu} p(t) dt \geq p\left(\frac{n}{\nu}\right) \cdot \frac{n}{\nu} = \frac{1}{n} \cdot \frac{n}{\nu} = \frac{1}{\nu},$$

since  $\frac{1}{k_{n+1}} < \frac{n}{\nu} \leq \frac{1}{k_n}$ , and for  $nk_{n+1} \leq \nu < (n+1)k_{n+1}$

$$\psi\left(\frac{3n}{\nu}\right) \geq \int_{(n+1)/\nu}^{(2n+1)/\nu} p(t) dt \geq p\left(\frac{n+1}{\nu}\right) \cdot \frac{n}{\nu} = \frac{1}{n} \cdot \frac{n}{\nu} = \frac{1}{\nu},$$

since  $\frac{1}{k_{n+1}} < \frac{n+1}{\nu} \leq \frac{n+1}{nk_{n+1}} < \frac{1}{k_n}$ .

Thus, for  $nk_n \leq \nu < (n+1)k_{n+1}$  we get the inequality  $\psi_{-1}\left(\frac{1}{\nu}\right) \leq \left(\frac{3n}{\nu}\right)$ .

Moreover, for  $n(k_n + s) \leq \nu < n(k_n + s + 1)$  it is true that  $\frac{3n}{\nu} \leq \frac{3}{k_n + s}$ .

From these inequalities we obtain

$$\begin{aligned} \sum_{\nu=1}^{\infty} \varphi_{-1} \left( \psi_{-1} \left( \frac{1}{\nu} \right) \right) \varphi_{-1} \left( \psi_{-1} \left( \frac{1}{\nu} \right) \right) &= \sum_{\nu=1}^{k_1-1} \dots + \sum_{\nu=nk_n}^{\infty} \sum_{\nu=nk_n}^{(n+1)k_{n+1}-1} \dots \\ &\leq \sum_{\nu=1}^{k_1-1} \varphi_{-1} \left( \psi_{-1} \left( \frac{1}{\nu} \right) \right) \varphi_{-1} \left( \psi_{-1} \left( \frac{1}{\nu} \right) \right) + \sum_{n=1}^{\infty} \sum_{\nu=nk_n}^{(n+1)k_{n+1}-1} \varphi_{-1} \left( \frac{3n}{\nu} \right) \varphi_{-1} \left( \frac{3n}{\nu} \right) \\ &\leq \sum_{\nu=1}^{k_1-1} \varphi_{-1} \left( \psi_{-1} \left( \frac{1}{\nu} \right) \right) \varphi_{-1} \left( \psi_{-1} \left( \frac{1}{\nu} \right) \right) + \sum_{n=1}^{\infty} n \sum_{s=0}^{\infty} \varphi_{-1} \left( \frac{3}{k_n+s} \right) \varphi_{-1} \left( \frac{3}{k_n+s} \right) \\ &\leq \sum_{\nu=1}^{k_1-1} \varphi_{-1} \left( \psi_{-1} \left( \frac{1}{\nu} \right) \right) \varphi_{-1} \left( \psi_{-1} \left( \frac{1}{\nu} \right) \right) + \sum_{n=1}^{\infty} n \frac{1}{n^3} < \infty. \end{aligned}$$

**1.14.** *L. C. Young's series for strictly increasing  $\varphi$ -functions  $\varphi$  and  $\varphi_{-1}$  is convergent if  $\frac{1}{\sigma_{\varphi}} + \frac{1}{\sigma_{\varphi^{-1}}} > 1$  and is divergent if  $\frac{1}{\sigma_{\varphi}} + \frac{1}{\sigma_{\varphi^{-1}}} < 1$ . This theorem remains true when the indices are equal to 0 or  $\infty$  if we adopt the conventions  $\frac{1}{0} = \infty$  and  $\frac{1}{\infty} = 0$ .*

*Proof.* Let  $\frac{1}{\sigma_{\varphi}} + \frac{1}{\sigma_{\varphi^{-1}}} > 1$ . We take  $\sigma > \sigma_{\varphi}$  and  $\sigma^{-1} > \sigma_{\varphi^{-1}}$  such that still  $\frac{1}{\sigma} + \frac{1}{\sigma^{-1}} > 1$  holds. We know that  $\sigma_{\varphi} = \lim_{a \rightarrow \infty} \lg \bar{l}_{\varphi}(a) / \lg a < \sigma$  and  $\sigma_{\varphi^{-1}} = \lim_{a \rightarrow \infty} \lg \bar{l}_{\varphi^{-1}}(a) / \lg a < \sigma^{-1}$  and so we deduce that there exists an integer  $m \geq 2$  such that  $\lg \bar{l}_{\varphi}(m^{1/\sigma}) / \lg m^{1/\sigma} < \sigma$  and  $\lg \bar{l}_{\varphi^{-1}}(m^{1/\sigma^{-1}}) / \lg m^{1/\sigma^{-1}} < \sigma^{-1}$ . Hence we get

$$\bar{l}_{\varphi}(m^{1/\sigma}) = \limsup_{u \rightarrow 0+} \varphi(m^{1/\sigma} u) / \varphi(u) < m$$

and

$$\bar{l}_{\varphi^{-1}}(m^{1/\sigma^{-1}}) = \limsup_{u \rightarrow 0+} \varphi^{-1}(m^{1/\sigma^{-1}} u) / \varphi^{-1}(u) < m.$$

It follows now that there exists a  $u_0 > 0$  such that for  $0 < u \leq u_0$  we have

$$\varphi(m^{1/\sigma} u) \leq m \varphi(u) \quad \text{and} \quad \varphi^{-1}(m^{1/\sigma^{-1}} u) \leq m \varphi^{-1}(u).$$

From these we get for  $0 < v \leq A = \inf\{m \varphi(u_0), m \varphi^{-1}(u_0)\}$

$$\varphi_{-1}(v/m) \leq m^{-1/\sigma} \varphi_{-1}(v) \quad \text{and} \quad \varphi_{-1}(v/m) \leq m^{-1/\sigma^{-1}} \varphi_{-1}(v).$$

Now, the induction yields

$$\varphi_{-1} \left( \frac{A}{m^k} \right) \leq \frac{1}{m^{k/\sigma}} \varphi_{-1}(A) \quad \text{and} \quad \varphi_{-1} \left( \frac{A}{m^k} \right) \leq \frac{1}{m^{k/\sigma^{-1}}} \varphi_{-1}(A) \quad \text{for } k = 0, 1, \dots$$

We take a positive integer  $r$  such that  $Ar \geq 1$ . Since  $\frac{Ar}{\nu} \leq \frac{A}{m^k}$  for  $m^k r \leq \nu < m^{k+1} r$ , we finally get

$$\begin{aligned} \sum_{\nu=1}^{\infty} \varphi_{-1} \left( \frac{1}{\nu} \right) \varphi_{-1} \left( \frac{1}{\nu} \right) &\leq \sum_{\nu=1}^{r-1} \varphi_{-1} \left( \frac{1}{\nu} \right) \varphi_{-1} \left( \frac{1}{\nu} \right) + \sum_{\nu=r}^{\infty} \varphi_{-1} \left( \frac{Ar}{\nu} \right) \varphi_{-1} \left( \frac{Ar}{\nu} \right) \\ &\leq \sum_{\nu=1}^{r-1} \varphi_{-1} \left( \frac{1}{\nu} \right) \varphi_{-1} \left( \frac{1}{\nu} \right) + \sum_{k=0}^{\infty} m^{k+1} r \varphi_{-1} \left( \frac{A}{m^k} \right) \varphi_{-1} \left( \frac{A}{m^k} \right) \\ &\leq \sum_{\nu=1}^{r-1} \varphi_{-1} \left( \frac{1}{\nu} \right) \varphi_{-1} \left( \frac{1}{\nu} \right) + \sum_{k=0}^{\infty} m^{k+1} r \frac{1}{m^{k/\sigma}} \frac{1}{m^{k/\sigma^{-1}}} \varphi_{-1}(A) \varphi_{-1}(A) < \infty. \end{aligned}$$

Now let  $\frac{1}{\sigma_{\varphi}} + \frac{1}{\sigma_{\varphi^{-1}}} < 1$ . We take  $0 < s < \sigma_{\varphi}$  and  $0 < s^{-1} < \sigma_{\varphi^{-1}}$  such that

$\frac{1}{s} + \frac{1}{s^{-1}} < 1$ . Since  $\sigma_{\varphi} = \lim_{a \rightarrow \infty} \lg \bar{l}_{\varphi}(a) / \lg a > s$  and  $\sigma_{\varphi^{-1}} = \lim_{a \rightarrow \infty} \lg \bar{l}_{\varphi^{-1}}(a) / \lg a > s^{-1}$ , where  $\bar{l}_{\varphi}(a) = \liminf_{u \rightarrow 0+} \varphi(au) / \varphi(u)$  and  $\bar{l}_{\varphi^{-1}}(a) = \liminf_{u \rightarrow 0+} \varphi^{-1}(au) / \varphi^{-1}(u)$ , similarly as before we obtain the inequalities

$$\varphi_{-1} \left( \frac{A}{m^k} \right) \geq \frac{1}{m^{k/s}} \varphi_{-1}(A) \quad \text{and} \quad \varphi_{-1} \left( \frac{A}{m^k} \right) \geq \frac{1}{m^{k/s^{-1}}} \varphi_{-1}(A) \quad \text{for } k = 0, 1, \dots$$

where  $m$  is an integer not less than 2 and  $A$  is a certain positive constant not larger than 1. For  $m^k \leq \nu < m^{k+1}$  we have  $\frac{1}{\nu} > \frac{1}{m^{k+1}} \geq \frac{A}{m^{k+1}}$ ; hence we get

$$\begin{aligned} \sum_{\nu=1}^{\infty} \varphi_{-1} \left( \frac{1}{\nu} \right) \varphi_{-1} \left( \frac{1}{\nu} \right) &\geq \sum_{k=0}^{\infty} m^k \varphi_{-1} \left( \frac{A}{m^{k+1}} \right) \varphi_{-1} \left( \frac{A}{m^{k+1}} \right) \\ &\geq \sum_{k=0}^{\infty} m^k \frac{1}{m^{(k+1)/s}} \frac{1}{m^{(k+1)/s^{-1}}} \varphi_{-1}(A) \varphi_{-1}(A) = \infty. \end{aligned}$$

To complete the proof we need only to consider the case  $\sigma_{\varphi} < 1$  and  $\sigma_{\varphi^{-1}} = \infty$ . Taking in this case  $\sigma_{\varphi} < \sigma < 1$  and applying a procedure analogous to that of the first part of this proof but only to the  $\varphi$ -function  $\varphi$ , we obtain the inequality

$$\varphi_{-1} \left( \frac{A}{m^k} \right) \leq \frac{1}{m^{k/\sigma}} \varphi_{-1}(A) \quad \text{for } k = 0, 1, \dots,$$

where  $m$  is some integer  $\geq 2$  and  $A$  is some positive constant. Further, similarly as before, we take a positive integer  $r$  such that  $Ar \geq 1$  and conclude that

$$\begin{aligned} \sum_{\nu=1}^{\infty} \varphi_{-1}\left(\frac{1}{\nu}\right) \varphi_{-1}^*\left(\frac{1}{\nu}\right) &\leq \sum_{\nu=1}^{r-1} \varphi_{-1}\left(\frac{1}{\nu}\right) \varphi_{-1}^*\left(\frac{1}{\nu}\right) + \sum_{\nu=r}^{\infty} \varphi_{-1}\left(\frac{Ar}{\nu}\right) \varphi_{-1}^*(1) \\ &\leq \sum_{\nu=1}^{r-1} \varphi_{-1}\left(\frac{1}{\nu}\right) \varphi_{-1}^*\left(\frac{1}{\nu}\right) + \sum_{k=0}^{\infty} m^{k+1} r \varphi_{-1}\left(\frac{A}{m^k}\right) \varphi_{-1}^*(1) \\ &\leq \sum_{\nu=1}^{r-1} \varphi_{-1}\left(\frac{1}{\nu}\right) \varphi_{-1}^*\left(\frac{1}{\nu}\right) + \sum_{k=1}^{\infty} m^{k+1} r \frac{1}{m^{k/\sigma}} \varphi_{-1}(A) \varphi_{-1}^*(1) < \infty. \end{aligned}$$

Theorem 1.14 does not decide on convergence of L. C. Young's series when  $\frac{1}{\sigma_{\varphi}} + \frac{1}{\sigma_{\varphi^*}} \leq 1$  and  $\frac{1}{s_{\varphi}} + \frac{1}{s_{\varphi^*}} \geq 1$ .

Let  $\varphi$  be a convex  $\varphi$ -function satisfying conditions  $(0_1)$  and  $(\infty_1)$  and let  $\varphi^*$  be its complementary  $\varphi$ -function. Then we have  $\frac{1}{\sigma_{\varphi}} + \frac{1}{\sigma_{\varphi^*}} \leq 1$  and  $\frac{1}{s_{\varphi}} + \frac{1}{s_{\varphi^*}} \geq 1$  and, in view of the inequality  $v \leq \varphi_{-1}(v) \varphi_{-1}^*(v)$  for  $v \geq 0$ , we get

$$\sum_{\nu=1}^{\infty} \varphi_{-1}\left(\frac{1}{\nu}\right) \varphi_{-1}^*\left(\frac{1}{\nu}\right) \geq \sum_{\nu=1}^{\infty} \frac{1}{\nu} = \infty.$$

**1.15.** Let  $\varphi$  be a convex  $\varphi$ -function satisfying conditions  $(0_1)$  and  $(\infty_1)$  and let  $\varphi^-$  be a strictly increasing  $\varphi$ -function. If L. C. Young's series for functions  $\varphi$  and  $\varphi^-$  is convergent then, for every  $\lambda > 0$ ,

$$\lim_{u \rightarrow 0+} \frac{\varphi^*(\lambda u)}{\varphi^-(u)} = 0.$$

**Proof.** The relation in question occurs if and only if for every  $\lambda > 0$  and for every  $0 < \varepsilon \leq 1$  there exists a  $u_0 > 0$  such that  $\varphi^*(\lambda u) \leq \varepsilon \varphi^-(u)$  for  $0 < u \leq u_0$ . This is equivalent to the following inequality:

$$\lambda \varphi_{-1}(v) \leq \varphi_{-1}^*(\varepsilon v) \quad \text{for } 0 < v \leq v_0 = \varphi^-(u_0).$$

Now suppose that the equality in question is not true. Then there exist numbers  $\lambda > 0$  and  $0 < \varepsilon \leq 1$  and a sequence  $\{v_n\}$  of positive numbers tending to 0 such that

$$\lambda \varphi_{-1}(v_n) > \varphi_{-1}^*(\varepsilon v_n) \quad \text{for } n = 1, 2, \dots$$

Since  $\{v_n\}$  is a sequence of positive numbers tending to 0 we can take its subsequence  $\{v_{n_k}\}$  such that

$$\frac{1}{2^{n_k+1}} < v_{n_k} \leq \frac{1}{2^{n_k}} \quad \text{for } k = 1, 2, \dots$$

where  $\{v_k\}$  is an increasing sequence of positive integers. Now, in virtue of the inequality  $v \leq \varphi_{-1}(v) \varphi_{-1}^*(v)$  for  $v \geq 0$ , we get

$$\begin{aligned} \sum_{\nu=1}^{\infty} \varphi_{-1}\left(\frac{1}{\nu}\right) \varphi_{-1}^*\left(\frac{1}{\nu}\right) &\geq \frac{1}{2} \sum_{\nu=1}^{\infty} 2^{\nu} \varphi_{-1}\left(\frac{1}{2^{\nu}}\right) \varphi_{-1}^*\left(\frac{1}{2^{\nu}}\right) \\ &\geq \frac{1}{2} \sum_{k=1}^{\infty} 2^{n_k} \varphi_{-1}\left(\frac{1}{2^{n_k}}\right) \varphi_{-1}^*\left(\frac{1}{2^{n_k}}\right) \\ &\geq \frac{1}{2} \sum_{k=1}^{\infty} 2^{n_k} \varphi_{-1}\left(\frac{1}{2^{n_k}}\right) \varphi_{-1}^*(v_{n_k}) \\ &\geq \frac{1}{2\lambda} \sum_{k=1}^{\infty} 2^{n_k} \varphi_{-1}\left(\frac{1}{2^{n_k}}\right) \varphi_{-1}^*(\varepsilon v_{n_k}) \\ &\geq \frac{1}{2\lambda} \sum_{k=1}^{\infty} 2^{n_k} \varphi_{-1}\left(\frac{\varepsilon}{2^{n_k+1}}\right) \varphi_{-1}^*\left(\frac{\varepsilon}{2^{n_k+1}}\right) \\ &\geq \frac{1}{2\lambda} \sum_{k=1}^{\infty} 2^{n_k} \frac{\varepsilon}{2^{n_k+1}} = \infty, \end{aligned}$$

what contradicts our assumption.

**2. Functions of finite  $\varphi$ -variation.** Let  $x$  be a real- or complex-valued function defined on a closed interval  $[a, b]$ . For a  $\varphi$ -function  $\varphi$  and for a partition  $\pi$  of  $[a, b]$  by points  $a = t_0 < t_1 < \dots < t_n = b$  we define

$$V_{\varphi}(x; \pi; a, b) = \sum_{\nu=1}^n \varphi(|x(t_{\nu}) - x(t_{\nu-1})|).$$

Further, we define

$$V_{\varphi}(x; a, b) = \sup_{\pi} V_{\varphi}(x; \pi; a, b).$$

In the sequel, where no confusion may occur, we shall simply write  $V_{\varphi}(x; \pi)$  and  $V_{\varphi}(x)$  instead of  $V_{\varphi}(x; \pi; a, b)$  and, respectively,  $V_{\varphi}(x; a, b)$ . The quantity  $V_{\varphi}(x) = V_{\varphi}(x; a, b)$  is called the  $\varphi$ -variation of a function  $x$  on the interval  $[a, b]$  (see [9]).

By  $\mathcal{V}^{\varphi}$  we shall denote in this paper the class of all functions  $x$  of finite  $\varphi$ -variation on  $[a, b]$  satisfying the normalizing condition  $x(a) = 0$ ,

and by  $\mathcal{V}^{*\varphi}$  the class of all functions  $x$  such that for some constant  $\lambda > 0$  (depending, in general, on  $x$ )  $\lambda x \in \mathcal{V}^\varphi$  holds. Further,  $\mathcal{E}\mathcal{V}^\varphi$  will denote the class of all continuous functions on  $[a, b]$  belonging to  $\mathcal{V}^\varphi$ , and  $\mathcal{E}\mathcal{V}^{*\varphi}$  the class of all continuous functions on  $[a, b]$  belonging to  $\mathcal{V}^{*\varphi}$ . The class  $\mathcal{V}^{*\varphi}$  is a vector subspace of the space consisting of all functions defined on  $[a, b]$  and satisfying the condition  $x(a) = 0$ , and the class  $\mathcal{E}\mathcal{V}^{*\varphi}$  is a linear subspace of the space  $\mathcal{V}^{*\varphi}$ .

**2.01.** If  $x(a) = 0$  then

$$\varphi(\bar{d}(x)) \leq V_\varphi(x) \quad \text{where} \quad \bar{d}(x) = \sup\{|x(t)| : a \leq t \leq b\}.$$

**2.02.** If  $\varphi$  and  $\varphi^*$  are two  $\varphi$ -functions such that the function  $\chi(u) = \varphi^*(u)/\varphi(u)$  for  $u > 0$  and  $\chi(0) = 0$  is nondecreasing for  $u \geq 0$ , then  $V_{\varphi^*}(x) \leq \chi(2\bar{d}(x)) V_\varphi(x)$  where  $\bar{d}(x)$  is defined as above.

**2.03.** If  $\{x_n\}$  is a sequence of functions convergent everywhere on the interval  $[a, b]$  to a function  $x$ , then  $V_\varphi(x) \leq \liminf_{n \rightarrow \infty} V_\varphi(x_n)$ .

**2.11.** If  $a < c < b$  then

$$\begin{aligned} V_\varphi(x; a, c) + V_\varphi(x; c, b) &\leq V_\varphi(x; a, b) \leq V_\varphi(x; a, c) + V_\varphi(x; c, b) + \\ &+ \sup\{\varphi(|x(t_1) - x(t_2)|) - \varphi(|x(t_1) - x(c)|) - \varphi(|x(c) - x(t_2)|) : \\ &a \leq t_1 \leq c \leq t_2 \leq b\}. \end{aligned}$$

Simple proofs of 2.01, 2.02, 2.03 and 2.11 are omitted. Immediately from this lemma it follows that for a function  $x$  of finite  $\varphi$ -variation on  $[a, b]$  the function  $v(t) = V_\varphi(x; a, t)$  is nondecreasing on  $[a, b]$ .

Another corollary to 2.11 is this:

**2.12.** Let  $x$  be a function of finite  $\varphi$ -variation on  $[a, b]$ . The function  $v(t) = V_\varphi(x; a, t)$  is constant on a certain subinterval  $[c, d]$  of  $[a, b]$  if and only if the function  $x$  is constant on this subinterval.

Proof. Clearly, if  $x$  is constant on the subinterval  $[c, d]$  of  $[a, b]$  then also  $v(t)$  is constant on this subinterval. Conversely, if the function  $v(t) = V_\varphi(x; a, t)$  is constant on the subinterval  $[c, d]$  then, in virtue of 2.11, we have  $V_\varphi(x; t, d) = 0$  for  $c \leq t \leq d$  and from the inequality  $\varphi(|x(t) - x(d)|) \leq V_\varphi(x; t, d)$  we deduce that  $x(t) = x(d)$  for  $c \leq t \leq d$ .

**2.13.** Let  $x$  be a function of finite  $\varphi$ -variation on  $[a, b]$ . The equality

$$\lim_{h \rightarrow 0+} V_\varphi(x; t_0, t_0 + h) = 0$$

is true for  $a \leq t_0 < b$  if and only if the function  $x$  is right-continuous at the point  $t_0$ . Similarly, for  $a < t_0 \leq b$

$$\lim_{h \rightarrow 0+} V_\varphi(x; t_0 - h, t_0) = 0$$

holds if and only if the function  $x$  is left-continuous at the point  $t_0$ .

Proof. The proof is essentially the same as for the functions of finite (ordinary) variation. We shall prove only the first part of the theorem, the second, being analogous, is omitted. If  $V_\varphi(x; t_0, t_0 + h) \rightarrow 0$  when  $h \rightarrow 0+$  then by the inequality

$$\varphi(|x(t_0 + h) - x(t_0)|) \leq V_\varphi(x; t_0, t_0 + h)$$

we get  $x(t_0 + h) \rightarrow x(t_0)$  as  $h \rightarrow 0+$ .

Conversely, assume that  $x$  is right-continuous at  $t_0$  and  $V_\varphi(x; t_0 + h, t_0) \rightarrow \varepsilon > 0$  when  $h \rightarrow 0+$ . Then we have  $V_\varphi(x; t_0, t) \geq \varepsilon$  for every  $t_0 < t \leq b$  and  $\varphi(|x(t) - x(t_0)|) < \frac{\varepsilon}{3}$  for every  $t_0 < t \leq t_1$  for some  $t_0 < t_1 \leq b$ .

But  $V_\varphi(x; t_0, t_1) \geq \varepsilon$  implies that there exists a partition of  $[t_0, t_1]$ :  $t_0 = t_{1,0} < t_{1,1} < \dots < t_{1,n_1} = t_1$  such that

$$\sum_{r=1}^{n_1} \varphi(|x(t_{1,r}) - x(t_{1,r-1})|) > \frac{2}{3} \varepsilon.$$

Since here  $\varphi(|x(t_{1,1}) - x(t_{1,0})|) < \frac{\varepsilon}{3}$ , so we get

$$\sum_{r=2}^{n_1} \varphi(|x(t_{1,r}) - x(t_{1,r-1})|) > \frac{\varepsilon}{3}.$$

Replacing in the above procedure  $t_1$  by  $t_{1,1}$  we conclude that there exists a partition of  $[t_0, t_{1,1}]$ :  $t_0 = t_{2,0} < t_{2,1} < \dots < t_{2,n_2} = t_{1,1}$  such that

$$\sum_{r=2}^{n_2} \varphi(|x(t_{2,r}) - x(t_{2,r-1})|) > \frac{\varepsilon}{3}.$$

Continuing this we see that for the  $k$ th step there exists a partition of  $[t_0, t_{k-1,1}]$ :  $t_0 = t_{k,0} < t_{k,1} < \dots < t_{k,n_k} = t_{k-1,1}$  such that

$$\sum_{r=2}^{n_k} \varphi(|x(t_{k,r}) - x(t_{k,r-1})|) > \frac{\varepsilon}{3}.$$

Hence we get

$$V_\varphi(x; a, b) \geq V_\varphi(x; t_0, t_1) \geq \sum_{k=1}^r \sum_{r=2}^{n_k} \varphi(|x(t_{k,r}) - x(t_{k,r-1})|) > r \frac{\varepsilon}{3}$$

for  $r = 1, 2, \dots$  and thus  $V_\varphi(x; a, b) = \infty$ , contrary to our assumption.

**2.14.** Let  $x$  be a function of finite  $\varphi$ -variation on  $[a, b]$ . The function  $v(t) = V_\varphi(x; a, t)$  is continuous (left continuous, right continuous) at a point  $t_0$  if and only if the function  $x$  is continuous (left continuous, right continuous) at this point.



**Proof.** We shall prove this only for right-continuous functions, since the other proofs are analogous.

If  $v(t)$  is right continuous at a point  $t_0$ ,  $a \leq t_0 < b$ , then by the inequality

$$0 \leq V_\varphi(x; t_0, t_0+h) \leq V_\varphi(x; a, t_0+h) - V(x; a, t_0) \quad \text{for } 0 < h \leq b - t_0$$

which is a consequence of 2.11 we get  $V_\varphi(x; t_0, t_0+h) \rightarrow 0$  when  $h \rightarrow 0+$ . Now, in view of 2.13, we see that  $x$  is right continuous at  $t_0$ . Conversely, let  $x$  be right continuous at  $t_0$  where  $a \leq t_0 < b$ . Then, by 2.13, for every

$\varepsilon > 0$  there exists  $0 < \delta_1 \leq b - t_0$  such that  $V_\varphi(x; t_0, t_0+h) < \frac{\varepsilon}{2}$  for

$0 < h \leq \delta_1$ . Further, let  $u_0 = \sup\{|x(t_1) - x(t_2)| : a \leq t_1 \leq t_2 \leq b\}$ . Since the function  $\varphi$  is continuous on  $[0, u_0]$  and hence is uniformly continuous on this interval, there exists  $\eta > 0$  such that  $|\varphi(u_1) - \varphi(u_2)| < \frac{\varepsilon}{2}$

for  $|u_1 - u_2| < \eta$  and  $u_1, u_2 \in [0, u_0]$ . The function  $x$  is right-continuous at  $t_0$ , thus there exists  $0 < \delta_2 \leq b - t_0$  such that  $|x(t) - x(t_0)| < \eta$  for  $t, t_0 < t \leq t_0 + \delta_2$ . Then we also have

$$|x(t) - x(\tau)| - |x(\tau) - x(t_0)| \leq |x(t) - x(t_0)| < \eta$$

for  $t_0 < t \leq t_0 + \delta_2$  and  $a \leq \tau \leq t_0$ . From this we get

$$\sup\{\varphi(|x(t) - x(\tau)|) - \varphi(|x(\tau) - x(t_0)|) : a \leq \tau \leq t_0 \leq t \leq t_0 + h\} < \frac{\varepsilon}{2}.$$

Now, for  $0 < h \leq \delta = \inf\{\delta_1, \delta_2\}$  we have, in view of 2.11,

$$0 \leq V_\varphi(x; a, t_0+h) - V_\varphi(x; a, t_0) \leq V_\varphi(x; t_0, t_0+h) + \sup\{\varphi(|x(t) - x(\tau)|) - \varphi(|x(\tau) - x(t_0)|) : a \leq \tau \leq t_0 \leq t \leq t_0+h\} < \varepsilon,$$

and this completes the proof.

**2.2.** Let  $x$  be a function defined on the interval  $[a, b]$  and let  $\pi$  be a partition of  $[a, b]$  by points  $a = t_0 < t_1 < \dots < t_n = b$ . In the remainder of this paper we shall denote by  $x_\pi^0$  the step function defined by the formula:

$$x_\pi^0(t) = \begin{cases} x(t_0) & \text{for } t = t_0, \\ x(t_\nu) & \text{for } t_{\nu-1} < t \leq t_\nu, \nu = 1, 2, \dots, n, \end{cases}$$

where  $\theta$  denotes a set of points  $\theta_\nu$  satisfying inequalities  $t_{\nu-1} \leq \theta_\nu \leq t_\nu$  for  $\nu = 1, 2, \dots, n$ . By  $x_\pi$  we shall denote the polygonal function defined by:

$$x_\pi(t) = \begin{cases} x(t_0) & \text{for } t = t_0, \\ \frac{t_\nu - t}{t_\nu - t_{\nu-1}} x(t_{\nu-1}) + \frac{t - t_{\nu-1}}{t_\nu - t_{\nu-1}} x(t_\nu) & \text{for } t_{\nu-1} < t \leq t_\nu, \nu = 1, 2, \dots, n. \end{cases}$$

$$\mathbf{2.21.} \quad V_\varphi(x_\pi^0) \leq V_\varphi(x).$$

Easy proof of this lemma is omitted.

$$\mathbf{2.22.} \quad \text{If } \varphi \text{ is a convex } \varphi\text{-function then } V_\varphi(x_\pi) \leq V_\varphi(x).$$

**Proof.** Let  $\pi$  be a partition of  $[a, b]$ :  $a = t_0 < t_1 < \dots < t_n = b$ , and let  $\pi'$  be another partition of this interval:  $a = \tau_0 < \tau_1 < \dots < \tau_m = b$ . Let in the interval  $(t_{\nu-1}, t_\nu)$  lie more than one point of the partition  $\pi'$  and let, further,  $\tau_{\mu_1-1} \leq t_{\nu-1} < \tau_{\mu_1} < \dots < \tau_{\mu_2} < t_\nu \leq \tau_{\mu_2+1}$ . Since then

$$x_\pi(\tau_\mu) = \frac{\tau_{\mu_2} - \tau_\mu}{\tau_{\mu_2} - t_{\nu-1}} x_\pi(t_{\nu-1}) + \frac{\tau_\mu - t_{\nu-1}}{\tau_{\mu_2} - t_{\nu-1}} x_\pi(\tau_{\mu_2}) \quad \text{for } \mu = \mu_1, \dots, \mu_2,$$

so we get

$$\begin{aligned} & \sum_{\mu=\mu_1}^{\mu_2} \varphi(|x_\pi(\tau_\mu) - x_\pi(\tau_{\mu-1})|) \\ &= \varphi\left(\frac{\tau_{\mu_2} - \tau_{\mu_1}}{\tau_{\mu_2} - t_{\nu-1}} (x_\pi(t_{\nu-1}) - x_\pi(\tau_{\mu_1-1})) + \frac{\tau_{\mu_1} - t_{\nu-1}}{\tau_{\mu_2} - t_{\nu-1}} (x_\pi(\tau_{\mu_2}) - x_\pi(\tau_{\mu_1-1}))\right) + \\ & \quad + \sum_{\mu=\mu_1+1}^{\mu_2} \varphi\left(\frac{\tau_\mu - \tau_{\mu-1}}{\tau_{\mu_2} - t_{\nu-1}} (x_\pi(\tau_{\mu_2}) - x_\pi(t_{\nu-1}))\right) \\ &\leq \frac{\tau_{\mu_2} - \tau_{\mu_1}}{\tau_{\mu_2} - t_{\nu-1}} \varphi(|x_\pi(t_{\nu-1}) - x_\pi(\tau_{\mu_1-1})|) + \frac{\tau_{\mu_1} - t_{\nu-1}}{\tau_{\mu_2} - t_{\nu-1}} \varphi(|x_\pi(\tau_{\mu_2}) - x_\pi(\tau_{\mu_1-1})|) + \\ & \quad + \sum_{\mu=\mu_1+1}^{\mu_2} \frac{\tau_\mu - \tau_{\mu-1}}{\tau_{\mu_2} - t_{\nu-1}} \varphi(|x_\pi(\tau_{\mu_2}) - x_\pi(t_{\nu-1})|) \\ &= \frac{\tau_{\mu_2} - \tau_{\mu_1}}{\tau_{\mu_2} - t_{\nu-1}} (\varphi(|x_\pi(t_{\nu-1}) - x_\pi(\tau_{\mu_1-1})|) + \varphi(|x_\pi(\tau_{\mu_2}) - x_\pi(t_{\nu-1})|)) + \\ & \quad + \frac{\tau_{\mu_1} - t_{\nu-1}}{\tau_{\mu_2} - t_{\nu-1}} \varphi(|x_\pi(\tau_{\mu_2}) - x_\pi(\tau_{\mu_1-1})|) \\ &\leq \sup\{\varphi(|x_\pi(t_{\nu-1}) - x_\pi(\tau_{\mu_1-1})|) + \varphi(|x_\pi(\tau_{\mu_2}) - x_\pi(t_{\nu-1})|), \\ & \quad \varphi(|x_\pi(\tau_{\mu_2}) - x_\pi(\tau_{\mu_1-1})|)\}. \end{aligned}$$

Now we conclude that there exists a partition  $\pi''$  of  $[a, b]$  such that to every interval  $(t_{\nu-1}, t_\nu)$  there belongs at most one point of this partition, and such that  $V_\varphi(x_\pi; \pi') \leq V_\varphi(x_\pi; \pi'')$ .

Let now  $\pi' = \pi''$  and let to the interval  $(t_{\nu-1}, t_\nu)$  belong one point  $\tau_\mu$ . Then we have

$$x_\pi(\tau_\mu) = \frac{t_\nu - \tau_\mu}{t_\nu - t_{\nu-1}} x_\pi(t_{\nu-1}) + \frac{\tau_\mu - t_{\nu-1}}{t_\nu - t_{\nu-1}} x_\pi(t_\nu),$$

whence we get

$$\begin{aligned}
 & \varphi(|x_\pi(\tau_\mu) - x_\pi(\tau_{\mu-1})|) + \varphi(|x_\pi(\tau_{\mu+1}) - x_\pi(\tau_\mu)|) \\
 &= \varphi\left(\left|\frac{t_\nu - \tau_\mu}{t_\nu - t_{\nu-1}}(x_\pi(t_{\nu-1}) - x_\pi(\tau_{\mu-1})) + \frac{\tau_\mu - t_{\nu-1}}{t_\nu - t_{\nu-1}}(x_\pi(t_\nu) - x_\pi(\tau_{\mu-1}))\right|\right) + \\
 &+ \varphi\left(\left|\frac{t_\nu - \tau_\mu}{t_\nu - t_{\nu-1}}(x_\pi(\tau_{\mu+1}) - x_\pi(t_{\nu-1})) + \frac{\tau_\mu - t_{\nu-1}}{t_\nu - t_{\nu-1}}(x_\pi(\tau_{\mu+1}) - x_\pi(t_\nu))\right|\right) \\
 &\leq \frac{t_\nu - \tau_\mu}{t_\nu - t_{\nu-1}}\left(\varphi(|x_\pi(t_{\nu-1}) - x_\pi(\tau_{\mu-1})|) + \varphi(|x_\pi(\tau_{\mu+1}) - x_\pi(t_{\nu-1})|)\right) + \\
 &+ \frac{\tau_\mu - t_{\nu-1}}{t_\nu - t_{\nu-1}}\left(\varphi(|x_\pi(t_\nu) - x_\pi(\tau_{\mu-1})|) + \varphi(|x_\pi(\tau_{\mu+1}) - x_\pi(t_\nu)|)\right) \\
 &\leq \sup\{\varphi(|x_\pi(t) - x_\pi(\tau_{\mu-1})|) + \varphi(|x_\pi(\tau_{\mu+1}) - x_\pi(t)|) : t = t_{\nu-1}, t_\nu\}.
 \end{aligned}$$

From the above consideration we conclude that there exists a partition  $\pi'''$  of  $[a, b]$  such that in neither of the intervals  $(t_{\nu-1}, t_\nu)$  lies a point of this partition, and such that  $V_\varphi(x_\pi; \pi'') \leq V_\varphi(x_\pi; \pi''')$ . Partition  $\pi'''$  is clearly a subpartition of  $\pi$ . Since at the point of  $\pi$  the functions  $x$  and  $x_\pi$  take the same values, hence  $V_\varphi(x_\pi; \pi''') = V_\varphi(x; \pi''') \leq V_\varphi(x)$ . Thus we obtain  $V_\varphi(x_\pi; \pi') \leq V_\varphi(x)$ . Since  $\pi'$  is an arbitrary partition of  $[a, b]$ , we get  $V_\varphi(x_\pi) \leq V_\varphi(x)$ .

**2.31.** For any  $\varphi$ -function  $\varphi$  the functional  $V_\varphi(\cdot)$  on the space of functions  $x$  defined on the interval  $[a, b]$  and satisfying condition  $x(a) = 0$  is a modular in the sense of [10]. Hence it follows that on the linear space

$$\overline{\mathcal{V}^{*\varphi}} = \{x \in \mathcal{V}^{*\varphi} : \lim_{\lambda \rightarrow 0+} V_\varphi(\lambda x) = 0\}$$

we may define an  $F$ -norm by

$$\|x\|_\varphi = \inf\{\varepsilon > 0 : V_\varphi(x/\varepsilon) \leq \varepsilon\}, \quad (x \in \mathcal{V}^{*\varphi}).$$

The space  $\overline{\mathcal{V}^{*\varphi}}$  is complete with respect to this norm (see 5.1.1).

**2.32.** If  $\varphi$  satisfies condition  $(V_2)$  (in particular, if  $\varphi(u) = \psi(u^s)$  for  $u \geq 0$ , where  $\psi$  is a convex  $\varphi$ -function and  $s > 0$ ) then  $\overline{\mathcal{V}^{*\varphi}} = \mathcal{V}^{*\varphi}$ .

Proof. By the hypotheses we have  $2\varphi(u) \leq \varphi(du)$  for  $0 \leq u \leq u_0$  where  $d > 1$  and  $u_0 > 0$  are some constants. For arbitrary  $x \in \mathcal{V}^{*\varphi}$ ,  $x \neq 0$  we have by 2.01,  $0 < d(x) < \infty$  and so for  $0 < \lambda \leq u_0/(2d(x))$  and a partition  $\pi$  of  $[a, b]$ :  $a = t_0 < t_1 < \dots < t_n = b$  we obtain

$$2V_\varphi(\lambda x; \pi) = \sum_{\nu=1}^n 2\varphi(\lambda |x(t_\nu) - x(t_{\nu-1})|) \leq \sum_{\nu=1}^n \varphi(d\lambda |x(t_\nu) - x(t_{\nu-1})|) \leq V_\varphi(d\lambda x)$$

and further  $2V_\varphi(\lambda x) \leq V_\varphi(d\lambda x)$ . Hence for arbitrary  $x \in \mathcal{V}^{*\varphi}$  we get

$$2 \lim_{\lambda \rightarrow 0+} V_\varphi(\lambda x) \leq \lim_{\lambda \rightarrow 0+} V_\varphi(d\lambda x) = \lim_{\lambda \rightarrow 0+} V_\varphi(\lambda x)$$

what yields  $\lim_{\lambda \rightarrow 0+} V_\varphi(\lambda x) = 0$ . This implies that  $\mathcal{V}^{*\varphi} \subset \overline{\mathcal{V}^{*\varphi}}$ . Since the other inclusion is obvious, the proof is complete.

**2.33.** Let  $\varphi(u) = \psi(u^s)$  for  $u \geq 0$ , where  $\psi$  is a convex  $\varphi$ -function and  $s$  is a number such that  $0 < s \leq 1$ . Then we may define an  $s$ -homogeneous norm in the space  $\mathcal{V}^{*\varphi}$  by

$$\|x\|_{\varphi^s} = \inf\{\varepsilon > 0 : V_\varphi(x/\varepsilon^{1/s}) \leq 1\} \quad (x \in \mathcal{V}^{*\varphi}).$$

This norm is equivalent to  $\|\cdot\|_\varphi$ ; more precisely

$$\|x\|_\varphi^{1+s} \leq \|x\|_{\varphi^s} \leq \|x\|_\varphi^s \quad \text{for } x \in \mathcal{V}^{*\varphi} \text{ such that } V_\varphi(x) \leq 1.$$

Proof. We shall prove only the above inequality. A  $\varphi$ -function  $\varphi$  satisfies the condition  $\varphi(\varepsilon^{1/s}u) \leq \varepsilon\varphi(u)$  for  $u \geq 0$  and  $0 < \varepsilon \leq 1$ . From this we deduce that

$$V_\varphi(x/\varepsilon) = V_\varphi(\varepsilon^{1/s}x/\varepsilon^{(1+s)/s}) \leq \varepsilon V_\varphi(x/\varepsilon^{(1+s)/s}) \quad \text{for } 0 < \varepsilon \leq 1.$$

Now, for  $x \in \mathcal{V}^{*\varphi}$  such that  $V_\varphi(x) \leq 1$  we get

$$\begin{aligned}
 \|x\|_{\varphi^s}^{1/(1+s)} &= \inf\{0 < \varepsilon \leq 1 : V_\varphi(x/\varepsilon^{(1+s)/s}) \leq 1\} \\
 &\geq \inf\{0 < \varepsilon \leq 1 : V_\varphi(x/\varepsilon) \leq \varepsilon\} = \|x\|_\varphi
 \end{aligned}$$

and

$$\begin{aligned}
 \|x\|_\varphi^s &= \inf\{0 < \varepsilon \leq 1 : V_\varphi(x/\varepsilon^{1/s}) \leq \varepsilon^{1/s}\} \\
 &\geq \inf\{0 < \varepsilon \leq 1 : V_\varphi(x/\varepsilon^{1/s}) \leq 1\} = \|x\|_{\varphi^s}.
 \end{aligned}$$

**2.34.** If  $\varphi$  is a convex  $\varphi$ -function satisfying conditions  $(0_1)$  and  $(\infty_1)$  we may define on  $\mathcal{V}^{*\varphi}$ , besides the homogeneous norm  $\|\cdot\|_{1\varphi}$ , another homogeneous norm

$$\|x\|_{(\varphi)} = \sup \left| \sum_{\nu=1}^n a_\nu (x(t_\nu) - x(t_{\nu-1})) \right|, \quad (x \in \mathcal{V}^{*\varphi}),$$

where the supremum is taken over all partitions  $\pi$  of  $[a, b]$ :  $a = t_0 < t_1 < \dots < t_n = b$  and over all  $n$ -tuples  $(a_1, \dots, a_n)$  such that  $\sum_{\nu=1}^n \varphi^*(|a_\nu|) \leq 1$ . The norms  $\|\cdot\|_{1\varphi}$  and  $\|\cdot\|_{(\varphi)}$  are equivalent, namely

$$\|x\|_{1\varphi} \leq \|x\|_{(\varphi)} \leq 2\|x\|_{1\varphi} \quad \text{for every } x \in \mathcal{V}^{*\varphi} \quad ([9]).$$

**3. Riemann-Stieltjes sums.** Let  $x$  and  $y$  be functions defined on an interval  $[a, b]$ . Further, let  $\pi$  be a partition of  $[a, b]$ :  $a = t_0 < t_1 < \dots$



$< t_n = b$ , and  $\theta$  a system of points  $\theta$ , satisfying inequalities  $t_{v-1} \leq \theta_v \leq t_v$  for  $v = 1, 2, \dots, n$ . Let us denote

$$R_n^0(x, y) = \sum_{v=1}^n x(\theta_v) (y(t_v) - y(t_{v-1})).$$

This sum is called the *Riemann-Stieltjes sum*.

In this section we shall estimate the sums  $R_n^0(x, y)$  using  $\varphi$ -variations of functions  $x$  and  $y$ . This estimation will be obtained by two mutually independent methods.

**3.11.** Let  $a = \{a_v\}$  be a sequence of real or complex numbers and let  $\varphi$  be a  $\varphi$ -function. Let us write

$$\varrho_\varphi(a) = \sum_{v=1}^{\infty} \varphi(|a_v|)$$

and

$$\varrho_\varphi^*(a) = \sup_a \varrho_\varphi(a(a)),$$

where the supremum is taken over all strictly increasing sequences of positive integers  $\alpha = \{\alpha_v\}$ , while  $\alpha(a)$  denotes a sequence whose terms are

$$a_1 = \sum_{\mu=1}^{\alpha_1} a_\mu \quad \text{and} \quad a_v = \sum_{\mu=\alpha_{v-1}+1}^{\alpha_v} a_\mu \quad \text{for} \quad v = 2, 3, \dots$$

Clearly,  $\varrho_\varphi(a) \leq \varrho_\varphi^*(a)$ .

**3.12.** Let  $\varphi$  be a log-convex  $\varphi$ -function and  $a = \{a_v\}$  a sequence of real or complex numbers. Then for every positive integer  $n$  the following inequality is true:

$$\left| \prod_{v=1}^n a_v \right|^{1/n} \leq \varphi_{-1} \left( \frac{1}{n} \varrho_\varphi(a) \right).$$

*Proof.* The inequality in question is obvious when at least one number  $a_v$ ,  $1 \leq v \leq n$ , is equal 0. So we may assume that all numbers  $a_v$ ,  $1 \leq v \leq n$ , are different from 0. Since a log-convex  $\varphi$ -function  $\varphi$  is representable in the form  $\varphi(u) = \Phi(\lg u)$  for  $u > 0$ , where  $\Phi$  is a convex function on the whole real line, we get

$$\begin{aligned} \varphi \left( \left| \prod_{v=1}^n a_v \right|^{1/n} \right) &= \Phi \left( \frac{1}{n} \sum_{v=1}^n \lg |a_v| \right) \leq \frac{1}{n} \sum_{v=1}^n \Phi(\lg |a_v|) \\ &= \frac{1}{n} \sum_{v=1}^n \varphi(|a_v|) \leq \frac{1}{n} \varrho_\varphi(a), \end{aligned}$$

and the desired inequality follows.

**3.13.** Let  $\varphi$  and  $\varphi^\sim$  be two log-convex  $\varphi$ -functions and let  $a = \{a_v\}$  and  $b = \{b_v\}$  be two sequences of real or complex numbers. Then for every positive integer  $n$  there exists a positive integer  $k$  such that  $1 \leq k \leq n$  and

$$|a_k b_k| \leq \varphi_{-1} \left( \frac{1}{n} \varrho_\varphi(a) \right) \varphi_{-1}^\sim \left( \frac{1}{n} \varrho_{\varphi^\sim}(b) \right).$$

*Proof.* It is clear that for every positive integer  $n$  there exists a positive integer  $k$  such that  $1 \leq k \leq n$  and  $|a_k b_k| \leq |a_v b_v|$  for  $v = 1, 2, \dots, n$ . Then, by 3.12, we have

$$|a_k b_k| \leq \left( \prod_{v=1}^n |a_v b_v| \right)^{1/n} = \left| \prod_{v=1}^n a_v \right|^{1/n} \cdot \left| \prod_{v=1}^n b_v \right|^{1/n} \leq \varphi_{-1} \left( \frac{1}{n} \varrho_\varphi(a) \right) \varphi_{-1}^\sim \left( \frac{1}{n} \varrho_{\varphi^\sim}(b) \right).$$

**3.14.** Let  $\varphi$  and  $\varphi^\sim$  be two log-convex  $\varphi$ -functions and let  $a = \{a_v\}$  and  $b = \{b_v\}$  be two sequences of real or complex numbers. Then, for every positive integer  $n$  there holds the inequality

$$\left| \sum_{\mu=1}^n \sum_{v=1}^v a_\mu b_v \right| \leq \varphi_{-1}(\varrho_\varphi^*(a)) \varphi_{-1}^\sim(\varrho_{\varphi^\sim}^*(b)) + \sum_{v=1}^{n-1} \varphi_{-1} \left( \frac{1}{v} \varrho_\varphi^*(a) \right) \varphi_{-1}^\sim \left( \frac{1}{v} \varrho_{\varphi^\sim}^*(b) \right).$$

For  $n = 1$  we assume that the second term on the right-hand side is equal 0.

*Proof.* It is easy to see that this inequality is satisfied for  $n = 1$ . We assume that it is satisfied for  $n-1$  and we shall prove that it then holds also for  $n$ . To this end we consider two sequences  $a' = \{a_{v+1}\}$  and  $b = \{b_v\}$ . In virtue of 3.13 there exists a positive integer  $k$  such that  $1 \leq k \leq n-1$  and

$$|a_{k+1} b_k| \leq \varphi_{-1} \left( \frac{1}{n-1} \varrho_\varphi(a') \right) \varphi_{-1}^\sim \left( \frac{1}{n-1} \varrho_{\varphi^\sim}(b) \right).$$

Since  $\varrho_\varphi(a') \leq \varrho_\varphi(a) \leq \varrho_\varphi^*(a)$  and  $\varrho_{\varphi^\sim}(b) \leq \varrho_{\varphi^\sim}^*(b)$ , we also have

$$|a_{k+1} b_k| \leq \varphi_{-1} \left( \frac{1}{n-1} \varrho_\varphi^*(a) \right) \varphi_{-1}^\sim \left( \frac{1}{n-1} \varrho_{\varphi^\sim}^*(b) \right).$$

Now we define two other sequences  $c = \{c_v\}$  and  $d = \{d_v\}$  in the following fashion:

$$c_v = \begin{cases} a_v & \text{for } v < k, \\ a_k + a_{k+1} & \text{for } v = k, \\ a_{v+1} & \text{for } v > k, \end{cases} \quad \text{and} \quad d_v = \begin{cases} b_v & \text{for } v < k, \\ b_k + b_{k+1} & \text{for } v = k, \\ b_{v+1} & \text{for } v > k. \end{cases}$$

We observe that  $c = \alpha(a)$  and  $d = \alpha(b)$ , where  $\alpha = \{\alpha_v\}$  is the increasing sequence of positive integers whose terms are  $\alpha_v = v$  for  $v = 1, 2, \dots, k-1$

and  $\alpha_\nu = \nu + 1$  for  $\nu = k, k+1, \dots$ . In view of the above we have  $\varrho_\varphi^*(c) \leq \varrho_\varphi^*(a)$  and  $\varrho_{\varphi^\sim}^*(d) \leq \varrho_{\varphi^\sim}^*(b)$ . Further we check that

$$\begin{aligned} \sum_{\nu=1}^{n-1} \sum_{\mu=1}^{\nu} c_{\mu} d_{\nu} &= \sum_{\nu=1}^{k-1} \sum_{\mu=1}^{\nu} a_{\mu} b_{\nu} + \sum_{\mu=1}^{k+1} a_{\mu} (b_k + b_{k+1}) + \sum_{\nu=k+1}^{n-1} \sum_{\mu=1}^{\nu+1} a_{\mu} b_{\nu+1} \\ &= a_{k+1} b_k + \sum_{\nu=1}^n \sum_{\mu=1}^{\nu} a_{\mu} b_{\nu}. \end{aligned}$$

From this and the induction argument we finally get

$$\begin{aligned} \left| \sum_{\nu=1}^n \sum_{\mu=1}^{\nu} a_{\mu} b_{\nu} \right| &\leq \left| \sum_{\nu=1}^{n-1} \sum_{\mu=1}^{\nu} c_{\mu} d_{\nu} \right| + |a_{k+1} b_k| \\ &\leq \varphi_{-1}(\varrho_{\varphi}^*(c)) \varphi_{-1}(\varrho_{\varphi^\sim}^*(d)) + \sum_{\nu=1}^{n-2} \varphi_{-1}\left(\frac{1}{\nu} \varrho_{\varphi}^*(c)\right) \varphi_{-1}\left(\frac{1}{\nu} \varrho_{\varphi^\sim}^*(d)\right) + \\ &\quad + \varphi_{-1}\left(\frac{1}{n-1} \varrho_{\varphi}^*(a)\right) \varphi_{-1}\left(\frac{1}{n-1} \varrho_{\varphi^\sim}^*(b)\right) \\ &\leq \varphi_{-1}(\varrho_{\varphi}(a)) \varphi_{-1}(\varrho_{\varphi^\sim}(b)) + \sum_{\nu=1}^{n-1} \varphi_{-1}\left(\frac{1}{\nu} \varrho_{\varphi}^*(a)\right) \varphi_{-1}\left(\frac{1}{\nu} \varrho_{\varphi^\sim}^*(b)\right). \end{aligned}$$

**3.15.** Let  $\varphi$  and  $\varphi^\sim$  be two log-convex  $\varphi$ -functions and let  $x$  and  $y$  be two functions defined on  $[a, b]$  and, moreover, let  $x(a) = 0$ . Then the following estimation of Riemann–Stieltjes sums is true

$$|R_{\pi}^0(x, y)| \leq \varphi_{-1}(V_{\varphi}(x)) \varphi_{-1}(V_{\varphi^\sim}(y)) + \sum_{\nu=1}^{\infty} \varphi_{-1}\left(\frac{1}{\nu} V_{\varphi}(x)\right) \varphi_{-1}\left(\frac{1}{\nu} V_{\varphi^\sim}(y)\right).$$

*Proof.* In view of 2.2 and the fact that  $x(a) = 0$  we have

$$\begin{aligned} R_{\pi}^0(x, y) &= \sum_{\nu=1}^n x(t_{\nu}) (y(t_{\nu}) - y(t_{\nu-1})) = \sum_{\nu=1}^n x_{\pi}^0(t_{\nu}) (y(t_{\nu}) - y(t_{\nu-1})) \\ &= \sum_{\nu=1}^n \sum_{\mu=1}^{\nu} (x_{\pi}^0(t_{\mu}) - x_{\pi}^0(t_{\mu-1})) (y(t_{\nu}) - y(t_{\nu-1})). \end{aligned}$$

Now we take into account two sequences  $a = \{a_{\nu}\}$  and  $b = \{b_{\nu}\}$  where  $a_{\nu} = x_{\pi}^0(t_{\nu}) - x_{\pi}^0(t_{\nu-1})$  for  $\nu = 1, 2, \dots, n$  and  $a_{\nu} = 0$  for  $\nu = n+1, n+2, \dots$ ,  $b_{\nu} = y(t_{\nu}) - y(t_{\nu-1})$  for  $\nu = 1, 2, \dots, n$  and  $b_{\nu} = 0$  for  $\nu = n+1, n+2, \dots$ . Let us notice that  $\varrho_{\varphi}^*(a) \leq V_{\varphi}(x_{\pi}^0)$  and  $\varrho_{\varphi^\sim}^*(b) \leq V_{\varphi^\sim}(y)$ . In virtue of 2.21 we have  $V_{\varphi}(x_{\pi}^0) \leq V_{\varphi}(x)$ . Thus, in view of 3.14., we obtain the required estimation of  $R_{\pi}^0(x, y)$ .

**3.21.** Now we shall estimate the Riemann–Stieltjes sums using the Haar functions defined as follows:

$$\chi_1(t) = 1 \quad \text{for } t \in [0, 1]$$

and

$$\chi_{2^n+k}(t) = \begin{cases} \sqrt{2^n} & \text{for } t \in \left(\frac{2k-2}{2^{n+1}}, \frac{2k-1}{2^{n+1}}\right), \\ -\sqrt{2^n} & \text{for } t \in \left(\frac{2k-1}{2^{n+1}}, \frac{2k}{2^{n+1}}\right), \\ 0 & \text{for other } t \text{ in } [0, 1], \end{cases}$$

( $n = 0, 1, \dots, k = 1, 2, \dots, 2^n$ ).

For a function  $x$  integrable on  $[0, 1]$  we define the Fourier coefficients with respect to the Haar system:

$$a_1(x) = \int_0^1 x(t) \chi_1(t) dt = \int_0^1 x(t) dt$$

and

$$\begin{aligned} a_{2^n+k}(x) &= \int_0^1 x(t) \chi_{2^n+k}(t) dt \\ &= \frac{1}{2\sqrt{2^n}} \int_0^1 \left( x\left(\frac{t}{2^{n+1}} + \frac{2k-2}{2^{n+1}}\right) - x\left(\frac{t}{2^{n+1}} + \frac{2k-1}{2^{n+1}}\right) \right) dt \end{aligned}$$

for  $n = 0, 1, \dots$  and  $k = 1, 2, \dots, 2^n$ .

Further, using the Haar functions we define the polygonal Schauder functions

$$\omega_0(t) = 1 \quad \text{and} \quad \omega_m(t) = \int_0^t \chi_m(\tau) d\tau \quad \text{for } m = 1, 2, \dots (t \in [0, 1]).$$

The system  $\{\omega_m\}$  is, as is well known [1], a Schauder basis in the space of all continuous functions on  $[0, 1]$ , i.e. every continuous function  $y$  on  $[0, 1]$  can be expanded into the following uniformly convergent series

$$y(t) = \sum_{m=0}^{\infty} c_m \omega_m(t),$$

where the coefficients  $c_m = c_m(y)$  are given by

$$c_0(y) = y(0), \quad c_1(y) = \int_0^1 \chi_1(t) dy(t) = y(1) - y(0)$$

and

$$c_{2^n+k}(y) = \int_0^1 \chi_{2^n+k}(t) dy(t) = -\sqrt{2^n} \left( y\left(\frac{2k-2}{2^{n+1}}\right) - 2y\left(\frac{2k-1}{2^{n+1}}\right) + y\left(\frac{2k}{2^{n+1}}\right) \right)$$

for  $n = 0, 1, \dots$  and  $k = 1, 2, \dots, 2^n$ .

From this we deduce that if  $x$  is a function of finite (ordinary) variation and  $y$  a continuous function on  $[0, 1]$  then

$$\int_0^1 x(t) dy(t) = \sum_{m=1}^{\infty} a_m(x) c_m(y).$$

Indeed, by well-known theorems concerning Riemann-Stieltjes integral we get

$$\begin{aligned} \int_0^1 x(t) dy(t) &= \lim_{k \rightarrow \infty} \int_0^1 x(t) d\left(\sum_{m=0}^k c_m(y) \omega_m(t)\right) = \sum_{m=0}^{\infty} c_m(y) \int_0^1 x(t) d\omega_m(t) \\ &= \sum_{m=1}^{\infty} c_m(y) \int_0^1 x(t) \chi_m(t) dt = \sum_{m=1}^{\infty} a_m(x) c_m(y). \end{aligned}$$

**3.22.** Let  $\varphi$  and  $\varphi^*$  be two convex  $\varphi$ -functions and  $x$  and  $y$  two functions defined on the interval  $[0, 1]$ . If the function  $x$  is of finite (ordinary) variation on  $[0, 1]$  and satisfies the condition  $x(0) = 0$ , and if for the function  $y$  the inequality

$$\varphi^*(|y(t_1) - y(t_2)|) \leq K(t_1 - t_2)$$

holds for  $0 \leq t_2 \leq t_1 \leq 1$  ( $K$  is some constant,  $K \geq 0$ ), then the Riemann-Stieltjes integral for the function  $x$  with respect to the function  $y$  exists and, moreover, the following inequality is true:

$$\left| \int_0^1 x(t) dy(t) \right| \leq \varphi_{-1}(V_{\varphi}(x)) \varphi_{-1}^*(K) + \sum_{n=0}^{\infty} 2^n \varphi_{-1}^*\left(\frac{1}{2^n} V_{\varphi}(x)\right) \varphi_{-1}^*\left(\frac{K}{2^{n+1}}\right).$$

**Proof.** The existence of the integral in question follows immediately from the fact that  $x$  is a function of finite variation and  $y$  is a continuous function on  $[0, 1]$ . To prove the desired inequality we notice that since  $x(0) = 0$  and  $\varphi$  is a convex  $\varphi$ -function, we have

$$|a_1(x)| = \varphi_{-1}\left(\varphi\left(\left|\int_0^1 (x(t) - x(0)) dt\right|\right)\right) \leq \varphi_{-1}\left(\int_0^1 \varphi(|x(t) - x(0)|) dt\right) \leq \varphi_{-1}(V_{\varphi}(x))$$

and

$$\begin{aligned} &\varphi\left(\frac{2\sqrt{2}^n}{2^n} \sum_{k=1}^{2^n} |a_{2^{n+k}}(x)|\right) \\ &= \varphi\left(\frac{1}{2^n} \sum_{k=1}^{2^n} \left|\int_0^1 \left(x\left(\frac{t}{2^{n+1}} + \frac{2k-2}{2^{n+1}}\right) - x\left(\frac{t}{2^{n+1}} + \frac{2k-1}{2^{n+1}}\right)\right) dt\right|\right) \\ &\leq \int_0^1 \frac{1}{2^n} \sum_{k=1}^{2^n} \varphi\left(\left|x\left(\frac{t}{2^{n+1}} + \frac{2k-2}{2^{n+1}}\right) - x\left(\frac{t}{2^{n+1}} + \frac{2k-1}{2^{n+1}}\right)\right|\right) dt \leq \frac{1}{2^n} V_{\varphi}(x). \end{aligned}$$

This implies that

$$\sum_{k=1}^{2^n} |a_{2^{n+k}}(x)| \leq \frac{1}{2} \sqrt{2}^n \varphi_{-1}\left(\frac{1}{2^n} V_{\varphi}(x)\right) \quad \text{for } n = 0, 1, \dots$$

Taking into account the property of  $y$  we obtain

$$|c_1(y)| = |y(1) - y(0)| \leq \varphi_{-1}^*(K)$$

and

$$\begin{aligned} |c_{2^{n+k}}(y)| &\leq \sqrt{2}^n \left| y\left(\frac{2k-2}{2^{n+1}}\right) - y\left(\frac{2k-1}{2^{n+1}}\right) \right| + \left| y\left(\frac{2k-1}{2^{n+1}}\right) - y\left(\frac{2k}{2^{n+1}}\right) \right| \\ &\leq 2\sqrt{2}^n \varphi_{-1}^*\left(\frac{K}{2^{n+1}}\right) \end{aligned}$$

for  $n = 0, 1, \dots$  and  $k = 1, 2, \dots, 2^n$ .

Hence, in view of the following inequality, which is a simple consequence of 3.21:

$$\left| \int_0^1 x(t) dy(t) \right| \leq \sum_{m=1}^{\infty} |a_m(x)| |c_m(y)| = |a_1(x)| |c_1(y)| + \sum_{k=0}^{\infty} \sum_{m=1}^{2^n} |a_{2^{n+k}}(x)| |c_{2^{n+k}}(y)|,$$

we get the desired inequality.

**3.23.** Let  $\varphi$  and  $\varphi^*$  be two convex  $\varphi$ -functions and  $x$  and  $y$  two functions defined on  $[a, b]$  and let  $x(a) = 0$ . Then the following estimation for the Riemann-Stieltjes sums is true:

$$\begin{aligned} |R_{\pi}^0(x, y)| &\leq \varphi_{-1}(V_{\varphi}(x)) \varphi_{-1}^*(V_{\varphi^*}(y)) + \sum_{n=0}^{\infty} 2^n \varphi_{-1}\left(\frac{1}{2^n} V_{\varphi}(x)\right) \varphi_{-1}^*\left(\frac{1}{2^{n+1}} V_{\varphi^*}(y)\right) \\ &\leq 2 \sum_{n=1}^{\infty} \varphi_{-1}\left(\frac{1}{2^n} V_{\varphi}(x)\right) \varphi_{-1}^*\left(\frac{1}{2^n} V_{\varphi^*}(y)\right). \end{aligned}$$

**Proof.** In virtue of 2.2 the Riemann-Stieltjes sum can be written in the form of an integral,

$$\begin{aligned} R_{\pi}^0(x, y) &= \sum_{r=1}^n x(t_r) (y(t_r) - y(t_{r-1})) = \sum_{r=1}^n x_{\pi}^0(t_r) (y_{\pi}(t_r) - y_{\pi}(t_{r+1})) \\ &= \int_a^b x_{\pi}^0(t) dy_{\pi}(t). \end{aligned}$$

When the polygonal function  $y_{\pi}$  is constant on the whole interval  $[a, b]$ , i.e. when  $V_{\varphi^*}(y_{\pi}) = 0$ , then the required inequality is obvious. Let us exclude this case and suppose that  $y_{\pi}$  is not constant on any subinterval of the interval  $[a, b]$ . Then by 2.11, 2.12 and 2.14 we see that the function

$v(t) = V_{\varphi^{-1}}(y_n; a, t)$  is strictly increasing and continuous on  $[a, b]$  and so there exists its inverse function  $v_{-1}$ . The function  $q(s) = v_{-1}(V_{\varphi^{-1}}(y_n; s))$  is strictly increasing and continuous on  $[0, 1]$  and, moreover,  $q(0) = a$  and  $q(1) = b$ .

Thus  $V_{\varphi^{-1}}(x_n^0(q(\cdot)); 0, 1) = V_{\varphi^{-1}}(x_n^0; a, b) = V_{\varphi^{-1}}(x_n^0)$  and

$$R_n^0(x, y) = \int_a^b x_n^0(t) dy_n(t) = \int_0^1 x_n^0(q(s)) dy_n(q(s)).$$

The function  $x_n^0(q(\cdot))$  is a step function on  $[0, 1]$  and besides  $x_n^0(q(0)) = x(a) = 0$ , and by 2.1.1, the function  $y_n(q(\cdot))$  satisfies for  $0 \leq s_1 \leq s_2 \leq 1$  the inequality

$$\begin{aligned} \varphi^{-1}(|y_n(q(s_2)) - y_n(q(s_1))|) &\leq V_{\varphi^{-1}}(y_n; q(s_1), q(s_2)) \\ &\leq V_{\varphi^{-1}}(y_n; a, q(s_2)) - V_{\varphi^{-1}}(y_n; a, q(s_1)) \\ &= v(q(s_2)) - v(q(s_1)) = V_{\varphi^{-1}}(y_n)(s_2 - s_1). \end{aligned}$$

Thus, in view of 3.22, 2.21 and 2.22 we get the required estimation of  $R_n^0(x, y)$ .

To complete the proof we must consider the case when  $y_n$  is not constant on  $[a, b]$  but is constant on some subintervals of  $[a, b]$ . Then the polygonal function

$$y_n^-(t) = \varphi_{-1}(V_{\varphi^{-1}}(y_n)) \frac{1}{b-a} \int_a^t (1 - \text{sign } |y'(s)|) ds, \quad (a \leq t \leq b),$$

where  $y'(s) = \lim_{h \rightarrow 0+} (y_n(s+h) - y_n(s))/h$  for  $a \leq s < b$ , is strictly increasing on these subintervals. Moreover, since  $y_n^-$  is a nondecreasing function on  $[a, b]$  and  $\varphi^{-1}$  a convex  $\varphi$ -function, so (see [9], 1.03)  $V_{\varphi^{-1}}(y_n^-) = \varphi^{-1}(y_n^-(b) - y_n^-(a)) \leq V_{\varphi^{-1}}(y_n)$ . Hence for arbitrary  $0 < \varepsilon < 1$  the polygonal function  $(1-\varepsilon)y_n + \varepsilon y_n^-$  is not constant on any subinterval of  $[a, b]$ . Besides, since  $\varphi^{-1}$  is a convex  $\varphi$ -function, so in view of 2.22,  $V_{\varphi^{-1}}((1-\varepsilon)y_n + \varepsilon y_n^-) \leq (1-\varepsilon)V_{\varphi^{-1}}(y_n) + \varepsilon V(y_n^-) \leq V_{\varphi^{-1}}(y_n) \leq V_{\varphi^{-1}}(y)$ . By what has already been proved we get

$$|(1-\varepsilon)R_n^0(x, y) + \varepsilon R_n^0(x, y_n^-)| = |R_n^0(x; (1-\varepsilon)y_n + \varepsilon y_n^-)|$$

$$\leq \varphi_{-1}(V_{\varphi^{-1}}(x)) \varphi_{-1}(V_{\varphi^{-1}}(y)) + \sum_{n=0}^{\infty} 2^n \varphi_{-1}\left(\frac{1}{2^n} V_{\varphi^{-1}}(x)\right) \varphi_{-1}\left(\frac{1}{2^{n+1}} V_{\varphi^{-1}}(y)\right),$$

and letting  $\varepsilon$  tend to 0 we obtain the desired estimation of  $R_n^0(x, y)$ . To complete the proof we write, in view of 1.12 and the fact that

$\varphi_{-1}(2v) \leq 2\varphi_{-1}(v)$  for  $v \geq 0$ , the inequality

$$\begin{aligned} &\varphi_{-1}(V_{\varphi^{-1}}(x)) \varphi_{-1}(V_{\varphi^{-1}}(y)) + \sum_{n=0}^{\infty} 2^n \varphi_{-1}\left(\frac{1}{2^n} V_{\varphi^{-1}}(x)\right) \varphi_{-1}\left(\frac{1}{2^{n+1}} V_{\varphi^{-1}}(y)\right) \\ &\leq \varphi_{-1}(V_{\varphi^{-1}}(x)) \varphi_{-1}(V_{\varphi^{-1}}(y)) + \sum_{n=0}^{\infty} 2^{n+1} \varphi_{-1}\left(\frac{1}{2^{n+1}} V_{\varphi^{-1}}(x)\right) \varphi_{-1}\left(\frac{1}{2^{n+1}} V_{\varphi^{-1}}(y)\right) \\ &\leq 2 \sum_{n=1}^{\infty} \varphi_{-1}\left(\frac{1}{2^n} V_{\varphi^{-1}}(x)\right) \varphi_{-1}\left(\frac{1}{2^n} V_{\varphi^{-1}}(y)\right). \end{aligned}$$

**4. Riemann–Stieltjes integral.** In this section we shall be concerned with the problem of existence of the Riemann–Stieltjes integral (RS-integral) of a function  $x \in \mathcal{V}^{*\varphi}$  with respect to a function  $y \in \mathcal{V}^{*\varphi^{-1}}$ . Besides we shall prove a generalization of Helly's theorem on passing to the limit under the sign of RS-integral. From the estimation of Riemann–Stieltjes sums given in 3.15 we derive the following

**4.01.** If  $\varphi$  and  $\varphi^{-1}$  are log-convex  $\varphi$ -functions and the RS-integral of a function  $x \in \mathcal{V}^{*\varphi}$  with respect to a function  $y \in \mathcal{V}^{*\varphi^{-1}}$  exists then the following estimation holds:

$$\left| \int_a^b x(t) dy(t) \right| \leq \varphi_{-1}(V_{\varphi^{-1}}(x)) \varphi_{-1}(V_{\varphi^{-1}}(y)) + \sum_{n=1}^{\infty} \varphi_{-1}\left(\frac{1}{2^n} V_{\varphi^{-1}}(x)\right) \varphi_{-1}\left(\frac{1}{2^n} V_{\varphi^{-1}}(y)\right).$$

From this we get

**4.02.** If  $\varphi(u) = \psi(u^s)$  and  $\varphi^{-1}(u) = \psi^{-1}(u^{s^{-1}})$ , ( $u \geq 0$ ) where  $\psi$  and  $\psi^{-1}$  are convex  $\varphi$ -functions and  $0 < s, s^{-1} \leq 1$ , and if the RS-integral of a function  $x \in \mathcal{V}^{*\varphi}$  with respect to a function  $y \in \mathcal{V}^{*\varphi^{-1}}$  exists then the following estimation holds:

$$\left| \int_a^b x(t) dy(t) \right| \leq \left( \varphi_{-1}(1) \varphi_{-1}(1) + \sum_{n=1}^{\infty} \varphi_{-1}\left(\frac{1}{2^n}\right) \varphi_{-1}\left(\frac{1}{2^n}\right) \right) \|x\|_{\mathcal{V}^{*\varphi}}^{1/s} \|y\|_{\mathcal{V}^{*\varphi^{-1}}}^{1/s^{-1}}.$$

**4.1.1.** If  $\varphi$  and  $\varphi^{-1}$  are log-convex  $\varphi$ -functions such that their L. O. Young's series is convergent then the RS-integral  $\int_a^b x(t) dy(t)$  for functions  $x \in \mathcal{C} \mathcal{V}^{*\varphi}$  and  $y \in \mathcal{V}^{*\varphi^{-1}}$  exists.

**Proof.** With no loss of generality we may assume that  $x \in \mathcal{C} \mathcal{V}^{*\varphi}$  and  $y \in \mathcal{V}^{*\varphi^{-1}}$ . From the hypothesis, by 1.13, we deduce that there exists a convex  $\varphi$ -function  $\psi$  satisfying condition (0<sub>1</sub>) and such that the L. O. Young's series for log-convex functions  $\varphi^+(u) = \psi(\varphi(u))$  and  $\varphi^-(u) = \psi(\varphi^{-1}(u))$  ( $u \geq 0$ ) is convergent. We notice here that the functions  $\chi^+$  and  $\chi^-$  defined by  $\chi^+(u) = \varphi^+(u)/\varphi(u)$  and  $\chi^-(u) = \varphi^-(u)/\varphi^{-1}(u)$  for  $u > 0$  and  $\chi^+(0) = \chi^-(0) = 0$  are non-decreasing and continuous for  $u \geq 0$ .

Now, let  $\pi_1$  and  $\pi_2$  be two partitions of  $[a, b]$ ,  $\pi_1: a = t_0^{(1)} < t_1^{(1)} < \dots < t_{n_1}^{(1)} = b$ ,  $\pi_2: a = t_0^{(2)} < t_1^{(2)} < \dots < t_{n_2}^{(2)} = b$  and let  $\pi$  be a partition of  $[a, b]$ :  $a = t_0 < t_1 < \dots < t_n = b$  such that all parting points of  $\pi_1$  and  $\pi_2$  are parting points for  $\pi$ , and no else parting points of  $\pi$  exist, i.e.  $\pi = \pi_1 \cup \pi_2$ . We observe that

$$R_{\pi_1}^{0_1}(x, y) = \sum_{\nu=1}^{n_1} x_{\pi_1}^{0_1}(t_\nu^{(1)}) (y(t_\nu^{(1)}) - y(t_{\nu-1}^{(1)})) = \sum_{\nu=1}^n x_{\pi_1}^{0_1}(t_\nu) (y(t_\nu) - y(t_{\nu-1}))$$

and, analogously,

$$R_{\pi_2}^{0_2}(x, y) = \sum_{\nu=1}^n x_{\pi_2}^{0_2}(t_\nu) (y(t_\nu) - y(t_{\nu-1})).$$

Hence we have

$$R_{\pi_1}^{0_1}(x, y) - R_{\pi_2}^{0_2}(x, y) = 2 \sum_{\nu=1}^n \frac{1}{2} (x_{\pi_1}^{0_1}(t_\nu) - x_{\pi_2}^{0_2}(t_\nu)) (y(t_\nu) - y(t_{\nu-1})).$$

The sum on the right-hand side of the above equation is the RS-sum of the functions  $\frac{1}{2}(x_{\pi_1}^{0_1} - x_{\pi_2}^{0_2})$  and  $y$  with respect to the partition  $\pi$ . Thus, in view of 3.15 applied to the log-convex functions  $\varphi^+$  and  $\varphi^-$ , we get

$$|R_{\pi_1}^{0_1}(x, y) - R_{\pi_2}^{0_2}(x, y)| \leq 4 \sum_{\nu=1}^n \varphi_{-1} \left( \frac{1}{\nu} V_{\varphi^+} \left( \frac{1}{2} (x_{\pi_1}^{0_1} - x_{\pi_2}^{0_2}) \right) \right) \varphi_{-1} \left( \frac{1}{\nu} V_{\varphi^-}(y) \right).$$

By 2.02, 2.01 and 2.21 we have here

$$V_{\varphi^+}(y) \leq \chi^-(2d(y)) V_{\varphi^-}(y) \leq \chi^-(2\varphi_{-1}(V_{\varphi^-}(y))) V_{\varphi^-}(y) < \infty$$

and

$$\begin{aligned} V_{\varphi^+} \left( \frac{1}{2} (x_{\pi_1}^{0_1} - x_{\pi_2}^{0_2}) \right) &\leq \chi^-(d(x_{\pi_1}^{0_1} - x_{\pi_2}^{0_2})) V_{\varphi} \left( \frac{1}{2} (x_{\pi_1}^{0_1} - x_{\pi_2}^{0_2}) \right) \\ &\leq \chi^-(d(x_{\pi_1}^{0_1} - x_{\pi_2}^{0_2})) (V_{\varphi}(x_{\pi_1}^{0_1}) + V_{\varphi}(x_{\pi_2}^{0_2})) \leq \chi^-(d(x_{\pi_1}^{0_1} - x) + d(x_{\pi_2}^{0_2} - x)) \cdot 2 V_{\varphi}(x). \end{aligned}$$

Now we take an arbitrary number  $\eta > 0$ . Since the function  $x$ , being continuous, is uniformly continuous on  $[a, b]$ , so there exists a  $\delta > 0$  such that for the partition  $\pi_1$  of the mesh  $\delta_{\pi_1} = \sup\{t_\nu^{(1)} - t_{\nu-1}^{(1)}: \nu = 1, \dots, n_1\}$  the following holds true:

$$\begin{aligned} d(x_{\pi_1}^{0_1} - x) &= \sup_{a \leq t \leq b} |x_{\pi_1}^{0_1}(t) - x(t)| \\ &= \sup \left\{ \sup_{t_{\nu-1}^{(1)} < t \leq t_\nu^{(1)}} |x(t_\nu^{(1)}) - x(t)|: \nu = 1, \dots, n_1 \right\} < \frac{1}{2} \eta \end{aligned}$$

and similarly for the partition  $\pi_2$  of the mesh  $\delta_{\pi_2} = \sup\{t_\nu^{(2)} - t_{\nu-1}^{(2)}: \nu = 1, \dots, n_2\}$ ,  $d(x_{\pi_2}^{0_2} - x) < \frac{1}{2} \eta$ .

Thus we have

$$|R_{\pi_1}^{0_1}(x, y) - R_{\pi_2}^{0_2}(x, y)| \leq 4 \sum_{\nu=1}^n \varphi_{-1} \left( \frac{1}{\nu} 2 \chi^-(\eta) V_{\varphi}(x) \right) \varphi_{-1} \left( \frac{1}{\nu} V_{\varphi^-}(y) \right).$$

The right-hand side of this inequality tends to 0 when  $\eta \rightarrow 0+$ , since for  $\eta = 1$  the series converges, by 1.11, and when  $\eta \rightarrow 0+$  the terms of this series converge monotonically to 0. From this we conclude the existence of the integral postulated in the theorem.

**4.12.** When  $\limsup_{u \rightarrow 0+} u^{-1} \varphi(u) = \infty$  Theorem 4.11 is trivial.

Indeed, it is known ([2]) that then the only function belonging to  $\mathcal{E}\mathcal{V}^{*\varphi}$  is identically equal 0 on  $[a, b]$ .

In case when  $\varphi^-$  is a  $\varphi$ -function such that  $\liminf_{u \rightarrow 0+} u^{-1} \varphi^-(u) > 0$  it is known ([9]) that the inclusion  $\mathcal{V}^{*\varphi^-} \subset \mathcal{V}$  holds, where  $\mathcal{V}$  denotes the class of functions of finite (ordinary) variation on  $[a, b]$  and equal zero at the point  $t = a$ . Thus then the RS-integral  $\int_a^b x(t) dy(t)$  exists for every function  $x$  continuous on  $[a, b]$  and for every function  $y \in \mathcal{V}^{*\varphi^-}$ . Theorem 4.11 in this case does not always decide on the existence of the RS-integral since L. C. Young's series is then not always convergent. E. g., for convex  $\varphi$ -functions  $\varphi(u) = \exp(-u^{-1})$  for  $0 < u \leq \frac{1}{2}$  and  $\varphi(u) = e^{-2}(4u-1)$  for  $u > \frac{1}{2}$  and  $\varphi^-(u) = u$  for  $u \geq 0$  the L. C. Young's series is divergent.

The existence of an RS-integral  $\int_a^b x(t) dy(t)$  always implies the existence of the RS-integral  $\int_a^b y(t) dx(t)$  and we always have the relation

$$\int_a^b y(t) dx(t) = (x(b)y(b) - x(a)y(a)) - \int_a^b x(t) dy(t).$$

Thus Theorem 4.11 is also true for  $x \in \mathcal{V}^{*\varphi}$  and  $y \in \mathcal{E}\mathcal{V}^{*\varphi^-}$ . This symmetry in the problem of existence of the RS-integral for functions  $x$  and  $y$  and symmetry of L. C. Young's series for  $\varphi$  and  $\varphi^-$  suggests to attempt proving a theorem analogous to 4.11 where no assumption of continuity of  $y$  is done. Such a theorem would have a symmetric form. This can be done when the sets of discontinuity of functions  $x$  and  $y$  are disjoint. We shall, however, limit ourselves to remarking that it is then convenient to use the following theorem.

**4.13.** Let  $\varphi$  and  $\varphi^-$  be convex  $\varphi$ -functions such that their L. C. Young's series (\*) is convergent. If the sequences  $\{x_n\} \subset \mathcal{V}^{*\varphi}$  and  $\{y_n\} \subset \mathcal{V}^{*\varphi^-}$  are uniformly convergent on  $[a, b]$  respectively to functions  $x$  and  $y$ , and such that  $\sup V_{\varphi}(\lambda_1 x_n) < \infty$  and  $\sup V_{\varphi^-}(\lambda_2 y_n) < \infty$  for some constants  $\lambda_1, \lambda_2 > 0$ , and if the RS-integrals of functions  $x_n$  with respect to functions  $y_n$  exist for  $n = 1, 2, \dots$  then the RS-integral of the function  $x$  with respect to the function  $y$  exists and

$$\int_a^b x(t) dy(t) = \lim_{n \rightarrow \infty} \int_a^b x_n(t) dy_n(t).$$



Proof. The proof will remain general if we assume that  $\{x_n\} \subset \mathcal{V}^p$  and  $\sup_n V_{\varphi}(x_n) = M_1 < \infty$ ,  $\{y_n\} \subset \mathcal{V}^{p^*}$  and  $\sup_n V_{\varphi^*}(y_n) = M_2 < \infty$ . By 2.03 we then have  $V_{\varphi}(x) \leq M_1$  and  $V_{\varphi^*}(y) \leq M_2$ . Now, in view of 1.13, we introduce, as in the proof of 4.11, functions  $\varphi^{\wedge}$ ,  $\varphi^{\vee}$ ,  $\chi^{\wedge}$  and  $\chi^{\vee}$ . Further, in virtue of 3.15, we obtain that

$$\begin{aligned} & |R_{\pi}^0(x, y) - R_{\pi}^0(x_n, y_n)| \\ & \leq 2 |R_{\pi}^0(\tfrac{1}{2}(x+x_n), \tfrac{1}{2}(y-y_n))| + 2 |R_{\pi}^0(\tfrac{1}{2}(x-x_n), \tfrac{1}{2}(y+y_n))| \\ & \leq 4 \sum_{\nu=1}^{\infty} \varphi_{\nu}^{\wedge} \left( \frac{1}{\nu} V_{\varphi^{\wedge}} \left( \frac{1}{2}(x+x_n) \right) \right) \varphi_{\nu}^{\vee} \left( \frac{1}{\nu} V_{\varphi^{\vee}} \left( \frac{1}{2}(y-y_n) \right) \right) \\ & \quad + 4 \sum_{\nu=1}^{\infty} \varphi_{\nu}^{\wedge} \left( \frac{1}{\nu} V_{\varphi^{\wedge}} \left( \frac{1}{2}(x-x_n) \right) \right) \varphi_{\nu}^{\vee} \left( \frac{1}{\nu} V_{\varphi^{\vee}} \left( \frac{1}{2}(y+y_n) \right) \right). \end{aligned}$$

The right-hand side of this inequality tends to 0 when  $n \rightarrow \infty$ , since by 2.02 and 2.01 and by the uniform convergence of the sequences  $\{x_n\}$  and  $\{y_n\}$  we have

$$\begin{aligned} \sup_n V_{\varphi^{\wedge}}(\tfrac{1}{2}(x+x_n)) & \leq V_{\varphi^{\wedge}}(x) + \sup_n V_{\varphi^{\wedge}}(x_n) \\ & \leq \chi^{\wedge}(2\varphi_{-1}(V_{\varphi}(x))) V_{\varphi}(x) + \sup_n \chi^{\wedge}(2\varphi_{-1}(V_{\varphi}(x_n))) V_{\varphi}(x_n) \\ & \leq 2\chi^{\wedge}(2\varphi_{-1}(M_1)) M_1 = M_1^{\wedge} < \infty, \\ V_{\varphi^{\vee}}(\tfrac{1}{2}(y-y_n)) & \leq \chi^{\vee}(d(y-y_n)) V_{\varphi^{\vee}}(\tfrac{1}{2}(y-y_n)) \leq \chi^{\vee}(d(y-y_n)) (V_{\varphi^{\vee}}(y) + V_{\varphi^{\vee}}(y_n)) \\ & \leq \chi^{\vee}(d(y-y_n)) \cdot 2M_2 \rightarrow 0 \quad \text{with } n \rightarrow \infty. \end{aligned}$$

Similarly,

$$\begin{aligned} \sup_n V_{\varphi^{\vee}}(\tfrac{1}{2}(y+y_n)) & \leq 2\chi^{\vee}(2\varphi_{-1}(M_2)) M_2 = M_2^{\vee} < \infty, \\ V_{\varphi^{\wedge}}(\tfrac{1}{2}(x-x_n)) & \leq \chi^{\wedge}(d(x-x_n)) \cdot 2M_1 \rightarrow 0 \quad \text{with } n \rightarrow \infty. \end{aligned}$$

From these inequalities it follows that for an  $\varepsilon > 0$  there exists an  $n_0$  such that for  $n \geq n_0$

$$(+)$$

$$|R_{\pi}^0(x, y) - R_{\pi}^0(x_n, y_n)| < \frac{\varepsilon}{3}$$

regardless of the partition  $\pi$  of  $[a, b]$  and of the system of points  $\theta$ .

Since the RS-integral of  $x_{n_0}$  with respect to  $y_{n_0}$  exists, hence for a given  $\varepsilon > 0$  there exists a  $\delta > 0$  such that for two partitions  $\pi_1$  and  $\pi_2$  of  $[a, b]$  whose meshes are, respectively,  $\delta_{\pi_1}, \delta_{\pi_2} < \delta$  we have

$$|R_{\pi_1}^0(x_{n_0}, y_{n_0}) - R_{\pi_2}^0(x_{n_0}, y_{n_0})| < \frac{\varepsilon}{3}.$$

From this and from (+) we get for the partitions  $\pi_1$  and  $\pi_2$  of  $[a, b]$

$$\begin{aligned} & |R_{\pi_1}^0(x, y) - R_{\pi_2}^0(x, y)| \leq |R_{\pi_1}^0(x, y) - R_{\pi_1}^0(x_{n_0}, y_{n_0})| + |R_{\pi_1}^0(x_{n_0}, y_{n_0}) - \\ & - R_{\pi_2}^0(x_{n_0}, y_{n_0})| + |R_{\pi_2}^0(x_{n_0}, y_{n_0}) - R_{\pi_2}^0(x, y)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

and this implies the existence of the RS-integral of  $x$  with respect to  $y$ . Now, taking in (+) a sequence  $\{\pi_m\}$  of partitions of  $[a, b]$  with meshes  $\delta_{\pi_m} \rightarrow 0$  ( $m \rightarrow \infty$ ) we obtain the inequality

$$\left| \int_a^b x(t) dy(t) - \int_a^b x_n(t) dy_n(t) \right| \leq \frac{\varepsilon}{3} \quad \text{for } n \geq n_0,$$

and this immediately gives our assertion.

The following theorem concerns the case of the interval  $[0, 1]$ .

**4.14.** If  $\varphi$  and  $\varphi^*$  are convex  $\varphi$ -functions such that their L. O. Young's series (\*) is convergent then for functions  $x \in \mathcal{V}^{*\varphi}$  and  $y \in \mathcal{C}\mathcal{V}^{*\varphi^*}$  the following relation holds:

$$\int_0^1 x(t) dy(t) = \sum_{m=1}^{\infty} a_m(x) c_m(y),$$

where  $a_m(x)$  and  $c_m(y)$  are defined in 3.21.

Proof. We may assume here that  $y \in \mathcal{C}\mathcal{V}^{*\varphi^*}$ . From the proof of 2.22 and in view of properties of polygonal Schauder functions  $\omega_m$  we see that  $V_{\varphi^*}(\omega_m) \leq \varphi^*(2d(\omega_m)) < \infty$  and  $\omega_m(0) = 0$  for  $m = 1, 2, \dots$  and further, that  $\omega_m \in \mathcal{C}\mathcal{V}^{*\varphi^*}$ . Thus, by 4.11 and 4.12, the RS-integrals of  $x$  with respect to  $\omega_m$  exist and

$$\int_0^1 x(t) d\omega_m(t) = \int_0^1 x(t) \chi_m(t) dt = a_m(x).$$

Now we observe that the function

$$y_n(t) = \sum_{m=1}^n c_m(y) \omega_m(t) \quad (0 \leq t \leq 1)$$

is in fact a polygonal function  $y_n$  for a partition  $\pi$  of  $[0, 1]$  by the points  $t_{\nu} = \nu/2^{r+1}$  for  $\nu = 0, 1, \dots, 2k$  and  $t_{\nu} = (\nu-k)/2^r$  for  $\nu = 2k+1, \dots, 2^r+k$ , when  $n = 2^r+k$ ,  $0 \leq k < 2^r$ . Hence, by 2.22, we have  $V_{\varphi^*}(y_n) \leq V_{\varphi^*}(y)$ . The sequence  $\{y_n\}$  converges uniformly on  $[0, 1]$  to a function  $y$ . Thus, by 4.13, we get

$$\begin{aligned} \int_0^1 x(t) dy(t) & = \lim_{n \rightarrow \infty} \int_0^1 x(t) d \left( \sum_{m=1}^n c_m(y) \omega_m(t) \right) \\ & = \sum_{m=1}^{\infty} c_m(y) \int_0^1 x(t) d\omega_m(t) = \sum_{m=1}^{\infty} c_m(y) a_m(x). \end{aligned}$$



**4.15.** Let  $\varphi$  and  $\varphi^-$  be log-convex  $\varphi$ -functions such that their L. O. Young's series (\*) is convergent. If  $x \in C\mathcal{V}^{*\varphi}$  and  $\{y_n\} \subset \mathcal{V}^{*\varphi^-}$  is a sequence of functions converging in a dense subset of  $[a, b]$  containing the point  $b$  to a function  $y \in \mathcal{V}^{*\varphi}$  and if  $\sup_n V_{\varphi^-}(\lambda y_n) < \infty$  for some constant  $\lambda > 0$ , then

$$\lim_{n \rightarrow \infty} \int_a^b x(t) dy_n(t) = \int_a^b x(t) dy(t).$$

*Proof.* Without affecting generality of the proof we may assume that  $x \in C\mathcal{V}^\varphi$  and that  $\{y_n\} \subset \mathcal{V}^{\varphi^-}$ ,  $\sup_n V_{\varphi^-}(y_n) = M < \infty$  and  $y_n(t) \rightarrow 0$  as  $n \rightarrow \infty$  for  $t \in D$ , where  $D$  is a dense subset of  $[a, b]$  containing the point  $b$ . Also the point  $a$  belongs to  $D$  since  $y_n(a) = 0$  for  $n = 1, \dots$ . Replacing in the proof of 4.11 the function  $y$  by functions  $y_n$  ( $n = 1, 2, \dots$ ) and taking into account that

$$\sup_n V_{\varphi^-}(y_n) \leq \sup_n \left( \chi^-(2\varphi_{-1}^-(V_{\varphi^-}(y_n))) V_{\varphi^-}(y_n) \right) \leq \chi^-(2\varphi_{-1}^-(M)) M < \infty,$$

we conclude that for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that for arbitrary partitions  $\pi_1$  and  $\pi_2$  of  $[a, b]$  with meshes  $\delta_{\pi_1}, \delta_{\pi_2} < \delta$  we have

$$|R_{\pi_1}^{01}(x, y_n) - R_{\pi_2}^{02}(x, y_n)| < \frac{\varepsilon}{3}, \quad \text{regardless of } n.$$

Hence it follows that for an arbitrary partition  $\pi$  of  $[a, b]$  of mesh  $\delta_\pi < \delta$  we have

$$\left| \int_a^b x(t) dy_n(t) - R_\pi^0(x, y_n) \right| < \frac{\varepsilon}{2}, \quad \text{regardless of } n.$$

The set  $D$  is dense in  $[a, b]$ , contains the points  $a$  and  $b$ , so we can find a partition  $\pi$  of  $[a, b]$ :  $a = t_0 < t_1 < \dots < t_m = b$  whose mesh is  $\delta_\pi < \delta$  such that all its points  $t_\nu$  ( $\nu = 0, \dots, m$ ) belong to  $D$ . Since  $y_n(t_\nu) \rightarrow 0$  when  $n \rightarrow \infty$  for  $\nu = 0, 1, \dots, m$ , thus for a given  $\varepsilon > 0$  there exists an  $n_0$  such that

$$|R_\pi^0(x, y_n)| = \left| \sum_{\nu=1}^m x(\theta_\nu) (y_n(t_\nu) - y_n(t_{\nu-1})) \right| \leq 2d(x) \sum_{\nu=0}^m |y_n(t_\nu)| < \frac{\varepsilon}{2} \quad \text{for } n \geq n_0.$$

Hence for  $n \geq n_0$  we get the inequality

$$\left| \int_a^b x(t) dy_n(t) \right| \leq \left| \int_a^b x(t) dy_n(t) - R_\pi^0(x, y_n) \right| + |R_\pi^0(x, y_n)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

from which it follows that

$$\lim_{n \rightarrow \infty} \int_a^b x(t) dy_n(t) = 0.$$

Theorem 4.15 generalizes Helly's theorem on passing to the limit under the sign of RS-integral.

**4.21.** If  $\varphi$  and  $\varphi^-$  are convex  $\varphi$ -functions such that  $1 < s_\varphi \leq \sigma_\varphi < \infty$  and  $1 < s_{\varphi^-} \leq \sigma_{\varphi^-} < \infty$  and their L. O. Young's series is divergent, then there exist functions  $x \in \mathcal{C}\mathcal{V}^{*\varphi}$  and  $y \in \mathcal{C}\mathcal{V}^{*\varphi^-}$  such that their RS-integral does not exist.

*Proof.* Hypotheses  $1 < s_\varphi \leq \sigma_\varphi < \infty$  and  $1 < s_{\varphi^-} \leq \sigma_{\varphi^-} < \infty$  are equivalent to the statement that for some constants  $c, d, c^-, d^- > 1$  and  $u_0 > 0$  the following inequalities hold for  $0 \leq u \leq u_0$ :

$$(\Delta_2) \quad \varphi(2u) \leq c\varphi(u), \quad \varphi^-(2u) \leq c^-\varphi^-(u),$$

$$(\nabla_2) \quad 2\varphi(u) \leq d^{-1}\varphi(du), \quad 2\varphi^-(u) \leq d^{-1}\varphi^-(d^-u).$$

From these for  $0 \leq v \leq v_0 = \inf\{\varphi(u_0), \varphi^-(u_0)\}$  and for a positive integer  $m$  such that  $\sup\{c, 2d, c^-, 2d^-, v_0^{-1}, 4\} \leq m$  we get

$$(1) \quad \varphi_{-1}(v) \leq \frac{1}{2}\varphi_{-1}(mv), \quad (1^-) \quad \varphi_{-1}^-(v) \leq \frac{1}{2}\varphi_{-1}^-(mv),$$

$$(2) \quad \varphi_{-1}(mv) \leq \frac{1}{2}m\varphi_{-1}(v), \quad (2^-) \quad \varphi_{-1}^-(mv) \leq \frac{1}{2}m\varphi_{-1}^-(v).$$

We take into account the functions  $x$  and  $y$  defined on the interval  $[0, 1]$  in the following fashion:

$$x(t) = \frac{1}{8\pi} \sum_{k=1}^{\infty} \varphi_{-1}\left(\frac{1}{m^k}\right) \sin(2\pi m^k t),$$

$$y(t) = \frac{1}{8\pi} \sum_{k=1}^{\infty} \varphi_{-1}^-\left(\frac{1}{m^k}\right) (1 - \cos(2\pi m^k t)).$$

Both series in the above definition are uniformly convergent, since by (1)

we have  $\varphi_{-1}\left(\frac{1}{m^k}\right) \leq \frac{1}{2}\varphi_{-1}\left(\frac{1}{m^{k-1}}\right) \leq \dots \leq \frac{1}{2^k}\varphi_{-1}(1)$ , and similarly by (1<sup>-</sup>)

$\varphi_{-1}^-\left(\frac{1}{m^k}\right) \leq \frac{1}{2^k}\varphi_{-1}^-(1)$  for  $k = 1, 2, \dots$ . The functions  $x$  and  $y$  are continuous on  $[0, 1]$  and their value at  $t = 0$  is 0. We shall show that  $V_\varphi(x) \leq 1$  and  $V_{\varphi^-}(y) \leq 1$ . For  $0 < h \leq 1$  and  $0 \leq t < 1 - h$  we have

$$|x(t+h) - x(t)| \leq \frac{1}{8\pi} \sum_{k=1}^{\infty} \varphi_{-1}\left(\frac{1}{m^k}\right) \cdot 2 \left| \cos(\pi m^k(2t+h)) \right| |\sin(\pi m^k h)|$$

$$\leq \frac{1}{4\pi} \left\{ \sum_{k=1}^{l-1} \varphi_{-1}\left(\frac{1}{m^k}\right) \pi m^k h + \sum_{k=l}^{\infty} \varphi_{-1}\left(\frac{1}{m^k}\right) \right\}$$

with  $l = \sup\{k: m^{k-1}h \leq 1\}$ . When  $l = 1$ , the first term in the outer brackets equals 0. By (2), for  $k = 1, \dots, l-1$  ( $l \geq 2$ ) we get the inequality

$$\begin{aligned} \varphi_{-1}\left(\frac{1}{m^k}\right) &= \varphi_{-1}\left(m^{l-k-1} \frac{1}{m^{l-1}}\right) \leq \frac{m}{2} \varphi_{-1}\left(m^{l-k-2} \frac{1}{m^{l-1}}\right) \leq \dots \\ &\leq \left(\frac{m}{2}\right)^{l-k-1} \varphi_{-1}\left(\frac{1}{m^{l-1}}\right) = \left(\frac{1}{2}\right)^{l-k-1} \frac{1}{m^k h} m^{l-1} h \varphi_{-1}\left(\frac{1}{m^{l-1}}\right) \\ &\leq \left(\frac{1}{2}\right)^{l-k-1} \frac{1}{m^k h} \varphi_{-1}(h) \end{aligned}$$

and, by (1), for  $k = l, l+1, \dots$  ( $l \geq 1$ ) the inequality

$$\varphi_{-1}\left(\frac{1}{m^k}\right) \leq \frac{1}{2} \varphi_{-1}\left(\frac{1}{m^{k-1}}\right) \leq \dots \leq \left(\frac{1}{2}\right)^{k-l} \varphi_{-1}\left(\frac{1}{m^l}\right) \leq \left(\frac{1}{2}\right)^{k-l} \varphi_{-1}(h).$$

Thus

$$|x(t+h) - x(t)| \leq \frac{1}{4\pi} \left\{ \pi \sum_{k=l}^{l-1} \left(\frac{1}{2}\right)^{l-k-1} + \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{k-l} \right\} \varphi_{-1}(h) \leq \varphi_{-1}(h).$$

From this we deduce that  $V_{\varphi}(x) \leq 1$ . For the function  $y$  we get a similar estimation

$$\begin{aligned} |y(t+h) - y(t)| &\leq \frac{1}{8\pi} \sum_{k=1}^{\infty} \varphi_{-1}\left(\frac{1}{m^k}\right) 2 |\sin(\pi m^k(2t+h))| |\sin(\pi m^k h)| \\ &\leq \frac{1}{4\pi} \left\{ \sum_{k=1}^{l-1} \varphi_{-1}\left(\frac{1}{m^k}\right) \pi m^k h + \sum_{k=l}^{\infty} \varphi_{-1}\left(\frac{1}{m^k}\right) \right\} \end{aligned}$$

with  $l = \sup\{k: m^{k-1}h \leq 1\}$ . Using now inequalities (1 $\sim$ ) and (2 $\sim$ ) in a similar manner as we used (1) and (2) we see that  $|y(t+h) - y(t)| \leq \varphi_{-1}(h)$  and hence we deduce that  $V_{\varphi}(y) \leq 1$ . Thus we have shown that  $x \in \mathcal{C}^{\varphi}$  and  $y \in \mathcal{C}^{\varphi}$ .

Let us now consider the Riemann-Stieltjes sums of the form

$$R_n = \sum_{\nu=1}^{m^n} x\left(\frac{\nu}{m^n}\right) \left( y\left(\frac{\nu}{m^n}\right) - y\left(\frac{\nu-1}{m^n}\right) \right).$$

We notice that

$$\begin{aligned} R_n &= \sum_{\nu=1}^{m^n} \left( \frac{1}{8\pi} \sum_{k=1}^{n-1} \varphi_{-1}\left(\frac{1}{m^k}\right) \sin \frac{2\pi m^k \nu}{m^n} \right) \times \\ &\quad \times \left( \frac{1}{8\pi} \sum_{l=1}^{n-1} \varphi_{-1}\left(\frac{1}{m^l}\right) \left( \cos \frac{2\pi m^l(\nu-1)}{m^n} - \cos \frac{2\pi m^l \nu}{m^n} \right) \right) \\ &= \left( \frac{1}{8\pi} \right)^2 \sum_{k=1}^{n-1} \sum_{l=1}^{n-1} \varphi_{-1}\left(\frac{1}{m^k}\right) \varphi_{-1}\left(\frac{1}{m^l}\right) \times \\ &\quad \times \left( \sum_{\nu=1}^{m^n} \sin \frac{2\pi m^k \nu}{m^n} \left( \cos \frac{2\pi m^l(\nu-1)}{m^n} - \cos \frac{2\pi m^l \nu}{m^n} \right) \right). \end{aligned}$$

By the identities

$$\frac{1}{2} + \sum_{\nu=1}^n \cos \nu t = \frac{\sin(n + \frac{1}{2})t}{2 \sin \frac{1}{2}t}, \quad \sum_{\nu=1}^n \sin \nu t = \frac{\cos \frac{1}{2}t - \cos(n + \frac{1}{2})t}{2 \sin \frac{1}{2}t},$$

( $0 < t < 2\pi$ ),

well known in the theory of trigonometric series, we get for  $k \neq l$

$$\begin{aligned} &\sum_{\nu=1}^{m^n} \sin \frac{2\pi m^k \nu}{m^n} \left( \cos \frac{2\pi m^l(\nu-1)}{m^n} - \cos \frac{2\pi m^l \nu}{m^n} \right) \\ &= \sum_{\nu=1}^{m^n} \left\{ \frac{1}{2} \left( \cos \frac{2\pi m^l}{m^n} - 1 \right) \left( \sin \frac{2\pi(m^k + m^l)\nu}{m^n} + \sin \frac{2\pi(m^k - m^l)\nu}{m^n} \right) + \right. \\ &\quad \left. + \frac{1}{2} \sin \frac{2\pi m^l}{m^n} \left( \cos \frac{2\pi(m^k - m^l)\nu}{m^n} - \cos \frac{2\pi(m^k + m^l)\nu}{m^n} \right) \right\} = 0 \end{aligned}$$

and for  $k = l = 1, 2, \dots, n-1$

$$\begin{aligned} &\sum_{\nu=1}^{m^n} \sin \frac{2\pi m^k \nu}{m^n} \left( \cos \frac{2\pi m^k(\nu-1)}{m^n} - \cos \frac{2\pi m^k \nu}{m^n} \right) \\ &= \sum_{\nu=1}^{m^n} \left\{ \frac{1}{2} \left( \cos \frac{2\pi m^k}{m^n} - 1 \right) \sin \frac{4\pi m^k \nu}{m^n} + \frac{1}{2} \sin \frac{2\pi m^k}{m^n} \left( 1 - \cos \frac{4\pi m^k \nu}{m^n} \right) \right\} \\ &= m^n \cdot \frac{1}{2} \sin \frac{2\pi}{m^{n-k}} \geq m^n \cdot \frac{1}{2} \cdot \frac{2}{\pi} \cdot \frac{2\pi}{m^{n-k}} = 2m^k. \end{aligned}$$

Thus

$$R_n \geq \frac{1}{32\pi^2} \sum_{k=1}^{n-1} m^k \varphi_{-1} \left( \frac{1}{m^k} \right) \tilde{\varphi}_{-1} \left( \frac{1}{m^k} \right).$$

The assumption that L. C. Young's series for  $\varphi$  and  $\varphi^-$  is divergent then implies by 1.12, that  $R_n \rightarrow \infty$  when  $n \rightarrow \infty$ . This proves that the RS-integral for  $x$  and  $y$  does not exist.

From 4.21 follows the following:

**4.22.** If  $\varphi$  is a convex  $\varphi$ -function satisfying conditions  $(0_1)$  and  $(\infty_1)$  and such that  $1 < s_\varphi \leq \sigma_\varphi < \infty$  then there exist functions  $w \in \mathcal{C}\mathcal{V}^{*\varphi}$  and  $y \in \mathcal{C}\mathcal{V}^{*\varphi}$  such that their RS-integral does not exist.

**5. Vector functions of finite  $\varphi$ -variation.** In this section we shall attempt to generalize the notion of a function of finite  $\varphi$ -variation to vector functions.

Let  $X$  be a Banach space and  $\mathcal{E}$  its dual, and let  $\mathcal{E}_0 = \{\xi \in \mathcal{E}: \|\xi\| \leq 1\}$ . Further let  $\varphi$  be a  $\varphi$ -function and  $x$  a function whose domain is  $[a, b]$  and range is in  $X$ . We define

$$V_\varphi^X(x) = \sup \{V_\varphi(\xi x): \xi \in \mathcal{E}_0\}.$$

We call  $V_\varphi^X(x)$  the  $\varphi$ -variation of  $x$  on  $[a, b]$ . In the sequel we shall denote by  $\mathcal{V}_X^\varphi$  the class of all functions  $x: [a, b] \rightarrow X$  of finite  $\varphi$ -variation on  $[a, b]$  and satisfying the condition  $x(a) = 0$ , and by  $\mathcal{V}_X^{*\varphi}$  the class of all functions  $x$  such that  $\lambda x \in \mathcal{V}_X^\varphi$  for some number  $\lambda > 0$  (depending, in general, on  $x$ ). Further, by  $\mathcal{C}\mathcal{V}_X^\varphi$  we shall denote the class of all continuous functions on  $[a, b]$  belonging to  $\mathcal{V}_X^\varphi$  and by  $\mathcal{C}\mathcal{V}_X^{*\varphi}$  the class of all continuous functions on  $[a, b]$  belonging to  $\mathcal{V}_X^{*\varphi}$ .  $\mathcal{V}_X^{*\varphi}$  is a vector subspace of the space consisting of all functions  $x: [a, b] \rightarrow X$  satisfying the condition  $x(a) = 0$ , and  $\mathcal{C}\mathcal{V}_X^{*\varphi}$  is a vector subspace of  $\mathcal{V}_X^{*\varphi}$ .

**5.01.** If a function  $x: [a, b] \rightarrow X$  satisfies the condition  $x(a) = 0$  then

$$\varphi(d^X(x)) \leq V_\varphi^X(x), \quad \text{where } d^X(x) = \sup \{\|x(t)\|: a \leq t \leq b\}.$$

Proof. By 2.01 we have

$$\begin{aligned} \varphi(\sup \{d(\xi x): \xi \in \mathcal{E}_0\}) &= \sup \{\varphi(d(\xi x)): \xi \in \mathcal{E}_0\} \\ &\leq \sup \{V_\varphi(\xi x): \xi \in \mathcal{E}_0\} = V_\varphi^X(x) \end{aligned}$$

and this yields the desired inequality, since

$$\begin{aligned} \sup \{d(\xi x): \xi \in \mathcal{E}_0\} &= \sup \{\sup \{|\xi x(t)|: \xi \in \mathcal{E}_0\}: a \leq t \leq b\} \\ &= \sup \{\|x(t)\|: a \leq t \leq b\} = d^X(x). \end{aligned}$$

**5.02.** If  $\varphi$  and  $\varphi^-$  are  $\varphi$ -functions such that the function  $\chi$  defined by  $\chi(u) = \varphi^-(u)/\varphi(u)$  for  $u > 0$  and  $\chi(0) = 0$  is nondecreasing for  $u \geq 0$ , then

$$V_{\varphi^-}^X(x) \leq \chi(2d^X(x)) V_\varphi^X(x),$$

where  $d^X(x)$  is defined as in 5.01.

**5.03.** If  $\{x_n\}$  is a sequence of functions from  $[a, b]$  to  $X$  which is weakly convergent on  $[a, b]$  to a function  $x$ , then  $V_\varphi^X(x) \leq \liminf_{n \rightarrow \infty} V_\varphi^X(x_n)$ .

We omit easy proofs of 5.02 and 5.03 since one needs only apply 2.02 or 2.03 to the function  $\xi x$ , where  $\xi \in \mathcal{E}_0$ , in the obvious manner.

Similarly as for scalar functions we may also define by formulae 2.2 the step functions  $x_n^0$  and polygonal functions  $x_n$  for a vector function  $x: [a, b] \rightarrow X$ .

Since for every  $\xi \in \mathcal{E}$  and, in particular, for every  $\xi \in \mathcal{E}_0$  we have  $\xi x_n^0 = (\xi x)_n^0$  and  $\xi x_n = (\xi x)_n$ , hence, by 2.21 and 2.22, respectively, we obtain the following lemmas:

$$\mathbf{5.04.} \quad V_\varphi^X(x_n^0) \leq V_\varphi^X(x).$$

$$\mathbf{5.05.} \quad \text{If } \varphi \text{ is a convex } \varphi\text{-function then } V_\varphi^X(x_n) \leq V_\varphi^X(x).$$

$$\mathbf{5.06.} \quad x \in \mathcal{V}_X^{*\varphi} \text{ if and only if } \xi x \in \mathcal{V}^{*\varphi} \text{ for every } \xi \in \mathcal{E}.$$

Proof. Let  $x \in \mathcal{V}_X^{*\varphi}$ . Then we have  $x(a) = 0$  and  $V_\varphi^X(\lambda x) < \infty$  for some number  $\lambda > 0$ . Hence, for arbitrary  $\xi \in \mathcal{E}$ ,  $\xi \neq 0$ , we get  $\xi x(a) = \xi 0 = 0$  and  $V_\varphi((\lambda/\|\xi\|)\xi x) \leq V_\varphi^X(\lambda x) < \infty$ . Thus  $\xi x \in \mathcal{V}^{*\varphi}$  for every  $\xi \in \mathcal{E}$ . Conversely, let  $\xi x \in \mathcal{V}^{*\varphi}$  for every  $\xi \in \mathcal{E}$ . Then for any  $\xi \in \mathcal{E}$  we get  $V_\varphi(\lambda \xi x) < \infty$  for some  $\lambda > 0$ , and so for a positive integer  $n$  such that  $\sup \{\lambda^{-1}, V_\varphi(\lambda \xi x)\} \leq n$  we have  $V_\varphi\left(\frac{1}{n}\xi x\right) \leq V_\varphi(\lambda \xi x) \leq n$ . This implies that

$$\mathcal{E} = \bigcup_{n=1}^{\infty} \mathcal{E}_n, \quad \text{where } \mathcal{E}_n = \left\{ \xi \in \mathcal{E}: V_\varphi\left(\frac{1}{n}\xi x\right) \leq n \right\} \text{ for } n = 1, 2, \dots$$

By 5.03 we conclude that the sets  $\mathcal{E}_n$  are closed in  $\mathcal{E}$ . Since the space  $\mathcal{E}$  is of the second category, hence some set  $\mathcal{E}_{n_0}$  contains a ball  $K(\xi_0; r)$  =  $\{\xi \in \mathcal{E}: \|\xi - \xi_0\| \leq r\}$ ,  $r > 0$ . For  $\xi \in K(\xi_0; r)$  we have  $V_\varphi\left(\frac{1}{n_0}\xi x\right) \leq n_0$ .

Thus for  $\xi \in \mathcal{E}_0$  we get

$$\begin{aligned} V_\varphi\left(\frac{r}{2n_0}\xi x\right) &= V_\varphi\left(\frac{1}{2}\left(\frac{1}{n_0}(r\xi + \xi_0)x - \frac{1}{n_0}\xi_0 x\right)\right) \\ &\leq V_\varphi\left(\frac{1}{n_0}(r\xi + \xi_0)x\right) + V_\varphi\left(\frac{1}{n_0}\xi_0 x\right) \leq n_0 + n_0 = 2n_0. \end{aligned}$$

Hence  $V_\varphi^X\left(\frac{r}{2n_0}x\right) \leq 2n_0$ . From this and since  $\xi x(a) = 0$  for every  $\xi \in \mathcal{E}$  implies  $x(a) = 0$ , we get  $x \in \mathcal{V}_X^{*\varphi}$ .

**5.07.**  $\lim_{\lambda \rightarrow 0+} V_\varphi^X(\lambda x) = 0$  if and only if  $\lim_{\lambda \rightarrow 0+} V_\varphi(\lambda \xi x) = 0$  for every  $\xi \in \mathcal{E}$ .

*Proof.* If  $\lim_{\lambda \rightarrow 0+} V_\varphi^X(\lambda x) = 0$  then for any  $\xi \in \mathcal{E}$ ,  $\xi \neq 0$ , in view of the inequality  $V_\varphi((\lambda/\|\xi\|)\xi x) \leq V_\varphi^X(\lambda x)$ , we get  $\lim_{\lambda \rightarrow 0+} V_\varphi(\lambda \xi x) = 0$ . Conversely, let  $\lim_{\lambda \rightarrow 0+} V_\varphi(\lambda \xi x) = 0$  for every  $\xi \in \mathcal{E}$ . Then for arbitrary number  $\varepsilon > 0$  we have

$$\mathcal{E} = \bigcup_{n=1}^{\infty} E_n, \quad \text{where } E_n = \left\{ \xi \in \mathcal{E} : V_\varphi\left(\frac{1}{n}\xi x\right) \leq \frac{\varepsilon}{2} \right\} \text{ for } n = 1, 2, \dots$$

In virtue of 5.03 the sets  $E_n$  are closed in  $\mathcal{E}$ . Thus, similarly as in the proof of 5.06, certain set  $E_{n_0}$  contains a ball  $K(\xi_0; r)$ ,  $r > 0$ . For  $\xi \in K(\xi_0; r)$  we have  $V_\varphi\left(\frac{1}{n_0}\xi x\right) \leq \frac{\varepsilon}{2}$ . For  $0 < \lambda \leq \frac{r}{2n_0}$  and  $\xi \in \mathcal{E}_0$  we now get

$$V_\varphi(\lambda \xi x) \leq V_\varphi\left(\frac{r}{2n_0}\xi x\right) \leq V_\varphi\left(\frac{1}{n_0}(r\xi + \xi_0)x\right) + V_\varphi\left(\frac{1}{n_0}\xi_0 x\right) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This means that  $V_\varphi^X(\lambda x) \leq \varepsilon$  for  $0 < \lambda \leq \frac{r}{2n_0}$ , and hence  $\lim_{\lambda \rightarrow 0+} V_\varphi^X(\lambda x) = 0$ .

**5.11.** For any  $\varphi$ -function  $\varphi$  the functional  $V_\varphi^X(\cdot)$  defined on the space of functions  $x: [a, b] \rightarrow X$  satisfying the condition  $x(a) = 0$  is a modular in the sense of [10]. Consequently, in the vector space

$$\mathcal{V}_X^{*\varphi} = \{x \in \mathcal{V}_X^{*\varphi} : \lim_{\lambda \rightarrow 0+} V_\varphi^X(\lambda x) = 0\}$$

we may define an  $F$ -norm

$$\|x\|_\varphi^X = \inf\{\varepsilon > 0 : V_\varphi^X(x/\varepsilon) \leq \varepsilon\}, \quad (x \in \mathcal{V}_X^{*\varphi}).$$

The space  $\mathcal{V}_X^{*\varphi}$  equipped with the topology induced by this norm is complete.

*Proof.* Let  $\{x_n\}$  be a sequence of elements of  $\mathcal{V}_X^{*\varphi}$  such that for any  $\varepsilon > 0$  there exists an  $n_0$  such that  $\|x_m - x_n\|_\varphi^X < \varepsilon$  for  $n, m \geq n_0$ . Then we also have

$$(+)\quad V_\varphi^X((x_n - x_m)/\varepsilon) \leq \varepsilon \quad \text{for } n, m \geq n_0.$$

From this, by 5.01, we get  $\varphi(d^X(x_n - x_m)/\varepsilon) \leq \varepsilon$  for  $n, m \geq n_0$ . This, together with the completeness of  $X$ , implies that the sequence  $\{x_n(\cdot)\}$  is uniformly convergent on  $[a, b]$  to a function  $x(\cdot)$ . Obviously, this function satisfies the condition  $x(a) = 0$ . Now let  $m \rightarrow \infty$  in (+); then, by 5.03, we get

$$(+)\quad V_\varphi^X((x_n - x)/\varepsilon) \leq \varepsilon \quad \text{for } n \geq n_0.$$

Hence we get for the function  $x$

$$\lim_{\lambda \rightarrow 0+} V_\varphi^X(\lambda x) \leq \lim_{\lambda \rightarrow 0+} (V_\varphi^X(2\lambda(x - x_{n_0})) + V_\varphi^X(2\lambda x_{n_0})) \leq V_\varphi^X((x - x_{n_0})/\varepsilon) \leq \varepsilon$$

and, since  $\varepsilon > 0$  was an arbitrary number, this implies that  $\lim_{\lambda \rightarrow 0+} V_\varphi^X(\lambda x) = 0$ .

Hence  $x \in \mathcal{V}_X^{*\varphi}$ . Now, from (++) we get that  $\|x_n - x\|_\varphi^X \leq \varepsilon$  for  $n \geq n_0$  and this ends the proof of completeness of  $\mathcal{V}_X^{*\varphi}$  with respect to the norm  $\|\cdot\|_\varphi^X$ .

We shall also show that

$$\|x\|_\varphi^X = \sup\{\|\xi x\|_\varphi : \xi \in \mathcal{E}_0\} \quad \text{for every } x \in \mathcal{V}_X^{*\varphi}.$$

Indeed, let  $\|x\|_\varphi^X < \varepsilon$ . Then  $V_\varphi^X(x/\varepsilon) \leq \varepsilon$  and for any  $\xi \in \mathcal{E}_0$  we get  $V_\varphi(\xi x/\varepsilon) \leq \varepsilon$  and further  $\|\xi x\|_\varphi \leq \varepsilon$ . This implies that  $\|\xi x\|_\varphi \leq \|x\|_\varphi^X$  for any  $\xi \in \mathcal{E}_0$ . Thus  $\sup\{\|\xi x\|_\varphi : \xi \in \mathcal{E}_0\} \leq \|x\|_\varphi^X$ . Now let  $\sup\{\|\xi x\|_\varphi : \xi \in \mathcal{E}_0\} < \varepsilon$ . Then for every  $\xi \in \mathcal{E}_0$  we have  $V_\varphi(\xi x/\varepsilon) \leq \varepsilon$ . This means that  $V_\varphi^X(x/\varepsilon) \leq \varepsilon$  and  $\|x\|_\varphi^X \leq \varepsilon$ . Hence  $\|x\|_\varphi^X \leq \sup\{\|\xi x\|_\varphi : \xi \in \mathcal{E}_0\}$ .

From 5.07 it follows that the relation  $\mathcal{V}_X^{*\varphi} = \mathcal{V}_X^{*\varphi}$  holds if and only if  $\mathcal{V}_X^{*\varphi} = \mathcal{V}_X^{*\varphi}$ . This implies, in view of 2.32, that  $\mathcal{V}_X^{*\varphi} = \mathcal{V}_X^{*\varphi}$  when  $\varphi$  satisfies (V<sub>2</sub>), and, in particular, when  $\varphi(u) = \psi(u^s)$  for  $u \geq 0$ , where  $\psi$  is a convex  $\varphi$ -function and  $s > 0$ .

**5.12.** When  $\varphi(u) = \psi(u^s)$  for  $u \geq 0$ , where  $\psi$  is a convex  $\varphi$ -function and  $s$  is a number such that  $0 < s \leq 1$  we may equip the space  $\mathcal{V}_X^{*\varphi}$  with an  $s$ -homogeneous norm

$$\|x\|_{s\varphi}^X = \inf\{\varepsilon > 0 : V_\varphi^X(x/\varepsilon^{1/s}) \leq 1\} \quad (x \in \mathcal{V}_X^{*\varphi}).$$

Similarly as in 5.11 we have here

$$\|x\|_{s\varphi}^X = \sup\{\|\xi x\|_{s\varphi} : \xi \in \mathcal{E}_0\} \quad \text{for every } x \in \mathcal{V}_X^{*\varphi}.$$

This, together with 2.33 yields the inequality

$$(\|x\|_\varphi^X)^{1+s} \leq \|x\|_{s\varphi}^X \leq (\|x\|_\varphi^X)^s \quad \text{for } x \in \mathcal{V}_X^{*\varphi} \text{ such that } V_\varphi^X(x) \leq 1.$$

From this we see that the norms  $\|\cdot\|_\varphi^X$  and  $\|\cdot\|_{s\varphi}^X$  are equivalent.

**5.13.** When  $\varphi$  is a convex  $\varphi$ -function satisfying conditions (0<sub>1</sub>) and ( $\infty_1$ ) the space  $\mathcal{V}_X^{*\varphi}$  can be furnished with, besides the homogeneous norm  $\|\cdot\|_\varphi^X$ , the homogeneous norm

$$\|x\|_{(\varphi)}^X = \sup \left\| \sum_{r=1}^n a_r (x(t_r) - x(t_{r-1})) \right\|, \quad (x \in \mathcal{V}_X^{*\varphi}),$$

where the supremum is taken over all partitions  $\pi$  of  $[a, b]$  by points  $a = t_0 < t_1 < \dots < t_n = b$  and over all  $n$ -tuples  $(a_1, \dots, a_n)$  such that  $\sum_{r=1}^n \varphi^*(|a_r|) \leq 1$ .

Clearly

$$\|x\|_{(\varphi)}^X = \sup\{\|\xi x\|_{(\varphi)} : \xi \in \mathcal{E}_0\} \quad \text{for every } x \in \mathcal{V}_X^{*\varphi}.$$

From this and 2.34 we get the inequality

$$\|x\|_{1\varphi}^X \leq \|x\|_{(\varphi)}^X \leq 2 \|x\|_{1\varphi}^X \quad \text{for every } x \in \mathcal{V}_{\varphi}^{*q},$$

from which it follows that the norms  $\|\cdot\|_{1\varphi}^X$  and  $\|\cdot\|_{(\varphi)}^X$  are equivalent.

**5.2.** We shall now be concerned with the problem of existence of the RS-integral for vector functions. We shall consider two cases: (a)  $x$  is a scalar function and  $y$  — a vector function, (b)  $x$  is a vector function and  $y$  — a scalar function. In both these cases we define the Riemann-Stieltjes sums analogously to what we have done in Section 3 and by the RS-integral  $\int_a^b x(t) dy(t)$  we mean an element  $z \in X$  possessing the property that for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that for any partition  $\pi$  of  $[a, b]$  whose mesh is  $\delta_\pi < \delta$  we have  $\|R_\pi^0(x, y) - z\| < \varepsilon$ , regardless of  $\theta$ .

**5.21.** Let  $\varphi$  and  $\varphi^*$  be log-convex  $\varphi$ -functions. Then, if we assume  $x(a) = 0$ , the following estimations of the RS-sums  $R_\pi^0(x, y)$  are true for cases (a) and (b) respectively:

$$(a) \quad \|R_\pi^0(x, y)\| \leq \varphi_{-1}(V_\varphi(x)) \varphi_{-1}(V_{\varphi^*}(y)) + \sum_{\nu=1}^{\infty} \varphi_{-1}\left(\frac{1}{\nu} V_\varphi(x)\right) \varphi_{-1}\left(\frac{1}{\nu} V_{\varphi^*}(y)\right),$$

$$(b) \quad \|R_\pi^0(x, y)\| \leq \varphi_{-1}(V_{\varphi^*}(x)) \varphi_{-1}(V_\varphi(y)) + \sum_{\nu=1}^{\infty} \varphi_{-1}\left(\frac{1}{\nu} V_{\varphi^*}(x)\right) \varphi_{-1}\left(\frac{1}{\nu} V_\varphi(y)\right).$$

Moreover, if the RS-integral  $\int_a^b x(t) dy(t)$  exists then analogical estimations hold also for this integral.

**Proof.** We shall consider only case (a) since for (b) the proof is analogous.

By 3.15 we get for arbitrary  $\xi \in E_0$

$$\begin{aligned} |\xi R_\pi^0(x, y)| &= |R_\pi^0(x, \xi y)| \leq \varphi_{-1}(V_\varphi(x)) \varphi_{-1}(V_{\varphi^*}(\xi y)) + \\ &+ \sum_{\nu=1}^{\infty} \varphi_{-1}\left(\frac{1}{\nu} V_\varphi(x)\right) \varphi_{-1}\left(\frac{1}{\nu} V_{\varphi^*}(\xi y)\right) \\ &\leq \varphi_{-1}(V_\varphi(x)) \varphi_{-1}(V_{\varphi^*}(y)) + \sum_{\nu=1}^{\infty} \varphi_{-1}\left(\frac{1}{\nu} V_\varphi(x)\right) \varphi_{-1}\left(\frac{1}{\nu} V_{\varphi^*}(y)\right). \end{aligned}$$

and this yields our inequality since  $\sup\{|\xi R_\pi^0(x, y)| : \xi \in E_0\} = \|R_\pi^0(x, y)\|$ . From 5.21 we derive the following result

**5.22.** If  $\varphi(u) = \psi(u^s)$  and  $\varphi^*(u) = \psi^*(u^{s^*})$ , ( $u \geq 0$ ), where  $\psi$  and  $\psi^*$  are convex  $\varphi$ -functions and  $s, s^*$  are numbers such that  $0 < s, s^* \leq 1$  and

if the RS-integral  $\int_a^b x(t) dy(t)$  of the functions (a)  $x \in \mathcal{V}^{*q}$ ,  $y \in \mathcal{V}_{\varphi^*}^{*q}$ , (b)  $x \in \mathcal{V}_{\varphi^*}^{*q}$ ,  $y \in \mathcal{V}^{*q}$ , exists then the following estimation is true:

$$(a) \quad \left\| \int_a^b x(t) dy(t) \right\| \leq \left( \varphi_{-1}(1) \varphi_{-1}(1) + \sum_{\nu=1}^{\infty} \varphi_{-1}\left(\frac{1}{\nu}\right) \varphi_{-1}\left(\frac{1}{\nu}\right) \right) \|x\|_{\varphi^*}^{1/s} (\|y\|_{\varphi^*}^X)^{1/s^*},$$

$$(b) \quad \left\| \int_a^b x(t) dy(t) \right\| \leq \left( \varphi_{-1}(1) \varphi_{-1}(1) + \sum_{\nu=1}^{\infty} \varphi_{-1}\left(\frac{1}{\nu}\right) \varphi_{-1}\left(\frac{1}{\nu}\right) \right) (\|x\|_{\varphi^*}^X)^{1/s} \|y\|_{\varphi^*}^{1/s^*}.$$

Let us now look again at Theorems 4.11, 4.13 and 4.15. Since we may apply the corresponding methods in the vector case, these theorems are valid also for vector functions. Thus we have three theorems:

**5.23.** If  $\varphi$  and  $\varphi^*$  are log-convex  $\varphi$ -functions such that their L. C. Young's series (\*) is convergent then in both cases (a)  $x \in \mathcal{C} \mathcal{V}^{*q}$ ,  $y \in \mathcal{C} \mathcal{V}_{\varphi^*}^{*q}$  and (b)  $x \in \mathcal{C} \mathcal{V}_{\varphi^*}^{*q}$ ,  $y \in \mathcal{C} \mathcal{V}^{*q}$ , the RS-integrals  $\int_a^b x(t) dy(t)$  exist.

The existence of the vector RS-integral  $\int_a^b x(t) dy(t)$  always implies, as is well known, the existence of the RS-integral  $\int_a^b y(t) dx(t)$  and the following relation holds:

$$\int_a^b y(t) dx(t) = (x(b)y(b) - x(a)y(a)) - \int_a^b x(t) dy(t).$$

Thus Theorem 5.23 is also true in any of the cases (a)  $x \in \mathcal{V}^{*q}$ ,  $y \in \mathcal{C} \mathcal{V}_{\varphi^*}^{*q}$  and (b)  $x \in \mathcal{V}_{\varphi^*}^{*q}$ ,  $y \in \mathcal{C} \mathcal{V}^{*q}$ .

**5.24.** Let  $\varphi$  and  $\varphi^*$  be log-convex  $\varphi$ -functions such that their L. C. Young's series (\*) is convergent. Then in any of the cases:

$$(a) \quad \{x_n\} \subset \mathcal{V}^{*q} \text{ and } \{y_n\} \subset \mathcal{V}_{\varphi^*}^{*q} \text{ are sequences such that } \sup_n V_{\varphi^*}(\lambda_1 x_n) < \infty \text{ and } \sup_n V_{\varphi^*}(\lambda_2 y_n) < \infty \text{ for some constants } \lambda_1, \lambda_2 > 0,$$

$$(b) \quad \{x_n\} \subset \mathcal{V}_{\varphi^*}^{*q} \text{ and } \{y_n\} \subset \mathcal{V}^{*q} \text{ are sequences such that } \sup_n V_{\varphi^*}(\lambda_1 x_n) < \infty \text{ and } \sup_n V_{\varphi^*}(\lambda_2 y_n) < \infty \text{ for some constants } \lambda_1, \lambda_2 > 0,$$

if these sequences are uniformly convergent on  $[a, b]$  to, respectively,  $x$  and  $y$  and if the RS-integrals  $\int_a^b x_n(t) dy_n(t)$  exist for  $n = 1, 2, \dots$  then the RS-integral  $\int_a^b x(t) dy(t)$  exists and

$$\int_a^b x(t) dy(t) = \lim_{n \rightarrow \infty} \int_a^b x_n(t) dy_n(t).$$

**5.25.** Let  $\varphi$  and  $\varphi^\sim$  be log-convex  $\varphi$ -functions such that their L. C. Young's series is convergent. Then in both cases:

(a)  $x \in \mathcal{C}\mathcal{V}^{*\varphi}$  and  $\{y_n\} \subset \mathcal{V}_X^{*\varphi^\sim}$  is a sequence convergent in a dense subset of  $[a, b]$  containing the point  $b$  to a function  $y \in \mathcal{V}_X^{*\varphi^\sim}$ , and such that  $\sup_n V_{\varphi^\sim}^X(\lambda y_n) < \infty$  for some constant  $\lambda > 0$ ,

(b)  $x \in \mathcal{C}\mathcal{V}_X^{*\varphi}$  and  $\{y_n\} \subset \mathcal{V}^{*\varphi^\sim}$  is a sequence convergent in a dense subset of  $[a, b]$  containing the point  $b$  to a function  $y \in \mathcal{V}^{*\varphi^\sim}$ , and such that  $\sup_n V_{\varphi^\sim}(\lambda y_n) < \infty$  for some constant  $\lambda > 0$ ,

we have

$$\int_a^b x(t) dy(t) = \lim_{n \rightarrow \infty} \int_a^b x(t) dy_n(t).$$

**5.26.** Let  $\varphi$  and  $\varphi^\sim$  be log-convex  $\varphi$ -functions such that their L. C. Young's series is convergent. If  $x \in \mathcal{C}\mathcal{V}^{*\varphi}$  and  $\{y_n\} \subset \mathcal{V}_X^{*\varphi^\sim}$  is a sequence of functions weakly convergent to a function  $y \in \mathcal{V}_X^{*\varphi^\sim}$  in a dense subset of  $[a, b]$  containing the point  $b$  and such that  $\sup_n V_{\varphi^\sim}^X(\lambda y_n) < \infty$  for some constant  $\lambda > 0$ , then

$$\int_a^b x(t) dy(t) = w\text{-}\lim_{n \rightarrow \infty} \int_a^b x(t) dy_n(t),$$

where  $w\text{-}\lim$  is the weak limit.

**Proof.** From assumptions made on  $\{y_n\}$  it follows that for every  $\xi \in \Xi$  the sequence  $\{\xi y_n\} \subset \mathcal{V}^{*\varphi^\sim}$  is convergent to  $\xi y \in \mathcal{V}^{*\varphi^\sim}$  in a dense subset of  $[a, b]$  containing the point  $b$ . Besides, for this sequence we have  $\sup_n V_{\varphi^\sim}(\lambda \xi y_n) < \infty$  for some constant  $\lambda > 0$ . Hence by 4.15 we get

$$\xi \int_a^b x(t) dy(t) = \int_a^b x(t) d\xi y(t) = \lim_{n \rightarrow \infty} \int_a^b x(t) d\xi y_n(t) = \lim_{n \rightarrow \infty} \xi \int_a^b x(t) dy_n(t)$$

which was to be proved.

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