STUDIA MATHEMATICA T. XLV. (1973)

Convolution of singular measures

by

RAOUF DOSS (Stony Brook, N. Y.)

Abstract. On a compact abelian group G, we prove that if $\mu \neq 0$ is any continuous (possibly singular) measure on G, then there is a real singular measure ν such that the convolution $\mu * \nu$ has an absolutely convergent Fourier series. As a consequence we prove that a multiplier from singular measures to singular measures is necessarily given by convolution with a discrete measure. Also, we prove that on the circle group T, if S is an uncountable Borel set of measure zero, then there is a Borel null set S' such that $S+S' = T$.

In this paper G is a compact abelian group with dual Γ. $M_s = M_s(G)$ is the Banach space of singular measures on G. Every $\mu \in M_s$ has a unique decomposition into a continuous part μ_c and a discrete part μ_d: $\mu = \mu_c + \mu_d$.

If $\nu \in M_s$ is supported by the null set S', then $\mu_d * \nu$ is supported by an enumerable union of translates of S' and hence $\mu_d * \nu$ is singular. The counterpart of this statement is the following theorem:

THEOREM 1. *Let G be a compact abelian group. Let $\mu \neq 0$ be any continuous (possibly singular) measure on G. Then there is a real singular measure ν such that the convolution $\mu * \nu$ has a non-vanishing absolutely convergent Fourier series.*

LEMMA 1. *Let μ be a continuous measure such that $\hat{\mu}(0) = 1$. Then there is a sequence of characters $\gamma_n \in \Gamma$ such that*

$$|\hat{\mu}(\pm\gamma_1)| < 8^{-1} \tag{1}$$

and such that for every $n \geqslant 1$ and every sequence $c_1, \ldots, c_n$, $c_i = 0, \pm 1, \pm 2, \pm 3$, we have

$$|\hat{\mu}(\pm\gamma_{n+1} + c_1\gamma_1 + \ldots + c_n\gamma_n)| < 8^{-(n+1)} \tag{2}$$

and consequently the γ_n are distinct.

Proof. We use the following theorem about a continuous measure μ: "To every $\varepsilon > 0$ there corresponds a symmetric neighborhood V of 0 in G, such that for any continuous positive definite function f, with compact support lying in V and such that $f(0) = 1$ we have

$$\sum_{\gamma \in \Gamma} \hat{f}(\gamma)\, |\hat{\mu}(\gamma)|^2 < \varepsilon."$$

See e.g. Rudin [3], p. 118.

The function g defined by $g(x) = 2^{-1}\{f(x)+f(-x)\}$ has the same properties as f and $\hat{g}(\gamma) = 2^{-1}\{\hat{f}(\gamma)+\hat{f}(-\gamma)\}$. Hence

$$\sum_{\gamma} \hat{f}(\gamma)[|\hat{\mu}(\gamma)|^2+|\hat{\mu}(-\gamma)|^2] = 2\sum_{\gamma} \hat{g}(\gamma)|\hat{\mu}(\gamma)|^2 < 2\varepsilon.$$

Choose $2\varepsilon = 8^{-2}$; since $\sum \hat{f}(\gamma) = f(0) = 1$ there is a $\gamma_1 \epsilon \Gamma$ such that $|\hat{\mu}(\gamma_1)|^2+|\hat{\mu}(-\gamma_1)|^2 < 8^{-2}$, i.e.

$$|\hat{\mu}(\pm\gamma_1)| < 8^{-1}.$$

Assume the first k elements of the desired sequence have been chosen satisfying (1) and, in case $k > 1$, also (2) for $n \leqslant k-1$. We show how to choose γ_{k+1} so that (2) holds for $n = k$.

Denote by F_k the finite set of elements of the form $c_1\gamma_1+\ldots+c_k\gamma_k$, where $c_1, \ldots, c_k = 0, \pm1, \pm2, \pm3$. F_k has at most 7^k elements, and F_k is symmetric: $F_k = -F_k$.

Consider the finite set of continuous measures $\bar{\delta}\mu$, where $\delta \epsilon F_k$. Then $(\bar{\delta}\mu)\hat{}\,(\gamma) = \hat{\mu}(\gamma+\delta)$ for $\gamma \epsilon \Gamma$.

Choose a symmetric neighborhood V of 0 in G, and a continuous positive definite function f whose compact support lies in V and such that $f(0) = 1$ and

$$\sum_{\gamma} \hat{f}(\gamma)[|\hat{\mu}(-\gamma+\delta)|^2+|\hat{\mu}(\gamma+\delta)|^2] < 7^{-k}8^{-2(k+1)} \quad \text{for every } \delta \epsilon F_k.$$

Then

$$\sum_{\delta \epsilon F_k}\sum_{\gamma} \hat{f}(\gamma)[|\hat{\mu}(\gamma+\delta)|^2+|\hat{\mu}(-\gamma+\delta)|^2] < 8^{-2(k+1)}$$

that is

$$\sum_{\gamma} \hat{f}(\gamma)\sum_{\delta \epsilon F_k}[|\hat{\mu}(\gamma+\delta)|^2+|\hat{\mu}(-\gamma+\delta)|^2] < 8^{-2(k+1)}.$$

Since $\sum_{\gamma \epsilon \Gamma} \hat{f}(\gamma) = 1$ there is $\gamma_{k+1} \epsilon \Gamma$ such that

$$\sum_{\delta \epsilon F_k}[|\hat{\mu}(\gamma_{k+1}+\delta)|^2+|\hat{\mu}(-\gamma_{k+1}+\delta)|^2] < 8^{-2(k+1)},$$

that is

$$|\hat{\mu}(\pm\gamma_{k+1}+\delta)| < 8^{-(k+1)}, \qquad \delta \epsilon F_k$$

which is (2) for $n = k$.

LEMMA 2. *Suppose $\{\gamma_n\}$ is an infinite sequence of distinct elements in Γ. Then there is a real singular measure ν such that* (a) $\hat{\nu}(0) \geqslant \frac{1}{2}$, (b) $\hat{\nu}(\gamma) = 0$ *if γ is not of the form $\pm\gamma_{n+1}+c_1\gamma_1+\ldots+c_n\gamma_n$, where $c_i = 0, \pm1, \pm2, \pm3$, with at most seven exceptional values of such γ's.*

Proof. It is evident that if $\{\delta_n\}$ is a subsequence of $\{\gamma_n\}$ and if ν is a measure satisfying the conditions of Lemma 2, with γ_n replaced by δ_n, then ν will also satisfy these conditions for the original sequence $\{\gamma_n\}$. Hence there is no problem, if, considering a subsequence of $\{\gamma_n\}$ we still call it $\{\gamma_n\}$.

We consider two cases:

Case I. The set $\{2\gamma_n\}$ is infinite.

Then we can extract from γ_n a subsequence, still denoted $\{\gamma_n\}$, such that, for $n = 1, 2, \ldots$

$$(3) \qquad 2\gamma_{n+1} \neq c_1\gamma_1+\ldots+c_n\gamma_n, \qquad c_i = 0, \pm1, \pm2, \pm3.$$

This is possible, since for any fixed n, the set

$$\{c_1\gamma_1+\ldots+c_n\gamma_n\colon\ c_i = 0, \pm1, \pm2, \pm3\}$$

is finite.

Put

$$P_N(x) = \prod_{k=1}^{N}\{1+\tfrac{1}{2}(x, \gamma_k)+\tfrac{1}{2}(x, -\gamma_k)\}.$$

Then $\|P_N\|_1 = 1$. Since, for every N, $\hat{P}_N(\gamma) = 0$ except for an enumerable (even finite) set of values of γ, then a subsequence of $\{P_N\}$ converges in the weak *-topology of $M(G)$, to a measure $\lambda \epsilon M(G)$.

Denote by A the set of finite sums of the form

$$a_1\gamma_1+\ldots+a_n\gamma_n, \qquad a_i = 0, \pm1, \qquad n = 1, 2, \ldots$$

Clearly

$$\hat{\lambda}(\gamma_k) \geqslant \tfrac{1}{2}$$

while

$$\hat{\lambda}(\gamma) = 0 \qquad \text{if } \gamma \notin A.$$

Put

$$\lambda_k = \gamma_k\lambda.$$

Then $\hat{\lambda}_k(\gamma) = \hat{\lambda}(\gamma+\gamma_k)$. Again, for every k, $\hat{\lambda}_k(\gamma) = 0$ except for an enumerable set of values of γ. Hence a subsequence of λ_k converges in the weak *-topology of $M(G)$ to a measure $\nu_0 \epsilon M(G)$. Using a device due to Helson (see [1]), we show that ν_0 is singular.

In fact, let $\lambda = \lambda'+\lambda''$ be the Lebesgue decomposition of λ, where λ'' is singular (with respect to Haar measure). The Fourier coefficients of λ' vanish at infinity; hence the absolutely continuous translates $\bar{\gamma}_k\lambda'$ converge weak* to zero. Therefore ν_0 is a weak* limit of the singular translates $\bar{\gamma}_k\lambda''$. For every f continuous on G we have

$$\left|\int f d\nu_0\right| \leqslant \int |f|\, d|\lambda''|$$

which shows that ν_0 is absolutely continuous with respect to λ''; since λ'' is singular so is ν_0.

Since $\hat{\lambda}(\gamma_k) \geqslant \frac{1}{2}$, then $\hat{\lambda}_k(0) \geqslant \frac{1}{2}$ and therefore

(a) $$\hat{\nu}_0(0) \geqslant \tfrac{1}{2}.$$

Now, assume $\gamma \neq 0$, $\gamma \neq \pm\gamma_1$, and γ is not of the form

(4) $$\gamma = \pm\gamma_{n+1} + c_1\gamma_1 + \ldots + c_n\gamma_n, \qquad c_i = 0, \pm 1, \pm 2, \pm 3,$$

If, for some $k_1 > 1$ we have

$$\gamma + \gamma_{k_1} = a_1\gamma_1 + \ldots + a_n\gamma_n \in A, \qquad a_i = 0, \pm 1,$$

then we may suppose $a_n \neq 0$, for the relation $\gamma + \gamma_{k_1} = 0$ is impossible by assumption (see (4)). Also $n = k_1$, otherwise γ would be of the excluded form (4). Moreover the relation $a_n = a_{k_1} = 1$ would contradict the assumption on γ. Thus $a_{k_1} = -1$ and therefore

(5) $$\gamma = a_1\gamma_1 + \ldots - 2\gamma_{k_1}, \qquad a_i = 0, \pm 1.$$

If for some $k_2 > k_1$ we have also

$$\gamma + \gamma_{k_2} = a'_1\gamma_1 + \ldots + a'_m\gamma_m \in A, \qquad a'_i = 0, \pm 1,$$

then

(6) $$\gamma = a'_1\gamma_1 + \ldots - 2\gamma_{k_2}.$$

Relations (5) and (6) are incompatible with (3). Therefore, there is at most one integer k such that $\gamma + \gamma_k \in A$. Hence, $\hat{\lambda}(\gamma + \gamma_k) = 0$ for large k and therefore $\hat{\nu}_{A_0}(\gamma) = 0$.

Now ν_0 is not necessarily real. Put $\nu = \frac{1}{2}\{\nu_0 + \bar{\nu}_0\}$; ν is real; if $\gamma \neq 0$, $\gamma \neq \pm\gamma_1$ and if γ is not of the form (4), then $\hat{\nu}_0(\gamma) = \overline{\hat{\nu}_0(-\gamma)} = 0$ and therefore $\hat{\nu}(\gamma) = 0$. The singular measure ν has now all the required properties.

Case II. The set $\{2\gamma_n\}$ is finite.

Then there is an infinite subset of $\{\gamma_n\}$, still denoted $\{\gamma_n\}$, such that $2\gamma_n = 2\gamma_1$ for all n. (The case $2\gamma_1 = 0$ is not excluded.)

Define P_N, λ, $\{\lambda_k\}$ and ν_0 as in Case I. Then, as before

$$\hat{\lambda}(\gamma_k) \geqslant \tfrac{1}{2}, \qquad \hat{\lambda}(\gamma) = 0, \qquad \text{if } \gamma \notin A,$$

ν_0 is singular and $\hat{\nu}_0(0) \geqslant \frac{1}{2}$.

Assume that $\gamma \neq 0$, $\gamma \neq \pm\gamma_1$, $\gamma \neq \pm 2\gamma_1$, $\gamma \neq \pm 3\gamma_1$ (seven exceptional values for γ) and that γ is not of the form

(4) $$\gamma = \pm\gamma_{n+1} + c_1\gamma_1 + \ldots + c_n\gamma_n, \qquad c_i = 0, \pm 1, \pm 2, \pm 3.$$

If $(k \geqslant 2)$

$$\gamma + \gamma_k = a_1\gamma_1 + \ldots + a_n\gamma_n \in A, \qquad a_i = 0, \pm 1$$

then as before (see (5))

$$\gamma = a_1\gamma_1 + \ldots + a_{k-1}\gamma_{k-1} - 2\gamma_k,$$

that is

$$\gamma = (a_1 - 2)\gamma_1 + \ldots + a_{k-1}\gamma_{k-1}.$$

This contradicts the assumption about γ. Therefore

$$\gamma + \gamma_k \notin A, \qquad \hat{\lambda}(\gamma + \gamma_k) = 0, \qquad \hat{\nu}_0(\gamma) = 0.$$

Again, replace ν_0 by $\nu = \frac{1}{2}\{\nu + \bar{\nu}\}$. Then ν will have all the properties mentioned in the lemma.

The proof of Lemma 2 is now complete.

For results related to Lemma 2, see [2].

Proof of Theorem 1. By translation and multiplication by a scalar we may assume that $\hat{\mu}(0) = 1$.

Let $\{\gamma_n\}$ be the sequence given by Lemma 1 and let ν be the real singular measure given by Lemma 2.

Consider now $\mu * \nu$.

Suppose γ is not one of the seven exceptional values mentioned in Lemma 2. If γ is not of the form

(4) $$\gamma = \pm\gamma_{n+1} + c_1\gamma_1 + \ldots + c_n\gamma_n, \qquad c_i = 0, \pm 1, \pm 2, \pm 3,$$

then $\hat{\mu}(\gamma)\hat{\nu}(\gamma) = 0$. For any fixed n, the set E_n of elements of the form (4) has at most $2\cdot 7^n$ elements and for these

$$\sum_{\gamma \in E_n} |\hat{\mu}(\gamma)\hat{\nu}(\gamma)| \leqslant 2\cdot 7^n 8^{-(n+1)} \|\nu\| < (\tfrac{7}{8})^n \|\nu\|.$$

Hence $\sum_{\gamma \in \Gamma} |\hat{\mu}(\gamma)\hat{\nu}(\gamma)| < \infty$. Since $\hat{\mu}(0) = 1$ and $\hat{\nu}(0) \geqslant \frac{1}{2}$ then $\mu * \nu$ has a nonvanishing absolutely convergent Fourier series and Theorem 1 is proved.

Corollary 1. *Let μ be any singular measure with nonvanishing continuous part. Then there is a singular measure ν such that the convolution $\mu * \nu$ is not singular.*

Proof. Put $\mu = \mu_c + \mu_d$ where μ_c is continuous and μ_d is discrete. Since μ_c is nonvanishing, there exists, by Theorem 1, a singular measure ν such that $\mu_c * \nu$ has a non-singular part. On the other hand, $\mu_d * \nu$ is singular. Hence $\mu * \nu$ is not singular and the Corollary is proved.

APPLICATION 1

Definition. Denote by $M_s = M_s(G)$ the Banach space of singular measures on G. A function φ on Γ is called a *singular multiplier* if for every $\nu \in M_s$, the function $\varphi\hat{\nu}$ is the transform of some $\lambda \in M_s$: $\varphi\hat{\nu} = \hat{\lambda}$.

THEOREM 2. *Let G be a compact abelian group. A function φ is a singular multiplier if and only if φ is the transform of a discrete measure μ: $\varphi = \hat{\mu}$.*

Proof. Taking ν_0 to be the unit mass at the origin, we have $\hat{\nu}_0 = 1$. Hence $\varphi = \varphi\hat{\nu}_0$ is the transform of some singular measure μ: $\varphi = \hat{\mu}$ where $\mu \in M_s$. Hence

$$\varphi\hat{\nu} = \hat{\mu}\hat{\nu} = (\mu*\nu)\hat{} \quad \text{for every } \nu \in M_s.$$

If now μ has a nonvanishing continuous part, then, by Corollary 1, there is a singular measure ν such that $\mu*\nu$ is not singular. Hence $\varphi\hat{\nu}$ is not the transform of a singular measure. This contradiction shows that the continuous part of μ is zero and therefore μ must be discrete.

APPLICATION 2

We specialize G to the unit circle T.

THEOREM 3. *Let S be an uncountable analytic* (in particular Borel) *set in T of measure* 0. *Then there is a Borel set S' of measure zero such that $S+S' = T$.*

Proof. By a theorem of Souslin, see e.g. [4], p. 224, the analytic set S contains a non-empty perfect set P. Since $m(P) = 0$ then P is totally disconnected. A classical construction of Lebesgue gives a real nonvanishing continuous singular measure μ on P. By Theorem 1 there is a real singular measure ν, concentrated on a null set P' such that $\mu*\nu$ has an absolutely convergent Fourier series (and hence $d\mu*\nu = \varphi(x)dx$ with φ real continuous). Since ν is regular, there is a sequence of compact sets $K_n \subset P'$ such that $|\nu|(K_n) \to |\nu|(P')$. Hence, replacing P' by $\bigcup K_n$ we may assume that P' is an F_σ-set and therefore that $P+P'$ is measurable (Borel).

Let x_0 be a point such that $\varphi(x_0) = \alpha \neq 0$. We may assume $\alpha > 0$; hence there is a non-degenerate interval I such that $\varphi(x) > \alpha/2$ for $x \in I$. We shall show that the set $A = I \setminus (P+P')$ has measure 0.

For assume $m(A) > 0$. We have

$$p \in P,\ p' \in P' \Rightarrow p+p' \notin A \Rightarrow p \notin A-p',$$

that is

$$\chi_{A-p'}(p) = 0 \quad \text{for } p \in P,\ p' \in P'.$$

Therefore

$$\mu*\nu(A) = \int_{P'} \mu(A-p')\,d\nu(p') = \int_{P'}\int_{P} \chi_{A-p'}(p)\,d\mu(p)\,d\nu(p') = 0.$$

But

$$\mu*\nu(A) = \int_A \varphi(x)\,dx \geqslant \tfrac{1}{2}\alpha m(A) > 0.$$

This contradiction shows that $m(A) = 0$, i.e. that $P+P'$ covers almost all I. If P'' is a suitable finite union of translates of P', then $P+P''$ covers almost all T:

$$m\big(T \setminus (P+P'')\big) = 0, \quad m(P'') = 0.$$

If now x_1 is any element of P and if we put

$$S' = P'' \cup \Big(\big(T \setminus (P+P'')\big) - x_1\Big)$$

then S' is a null set and

$$P+S' \supset P+P'' \cup T \setminus (P+P'') = T.$$

A fortiori $S+S' \supset T$ and the theorem is proved.

A direct proof of Theorem 3 would be desirable.

References

[1] H. Helson, *Proof of a conjecture of Steinhaus*, Proc. Nat. Acad. Sci. U.S.A. 40 (1954), pp. 205–206.

[2] E. Hewitt and H. Zuckerman, *Singular measures with absolutely continuous convolution squares*, Proc. Cambridge Phil. Soc. 62 (1966), pp. 399–420.

[3] W. Rudin, *Fourier analysis on groups*, N. Y. 1963.

[4] W. Sierpiński, *General Topology*, Toronto 1956.

STATE UNIVERSITY OF NEW YORK
STONY BROOK, NEW YORK

Received May 19, 1971 (344)