Preuve. En utilisant les inégalités classiques liant les $\lambda_j(T_1\dots T_k)$ et les produits des $\lambda_j(T_i)$, il est aisé de voir que si

$$\frac{1}{p_1}+\dots+\frac{1}{p_k}=\frac{1}{p}\leqslant 1$$

le produit $T_1\dots T_k\in\mathscr{C}^p(H)$ et l'application multilinéaire

$$(T_1,\dots,T_k)\to T_1\dots T_k$$

est continue de $\mathscr{C}^{p_1}(H)\oplus\dots\oplus\mathscr{C}^{p_2}(H)$ dans $\mathscr{C}^p(H)$.

Prouvons le a).

En vertu de cette continuité et de la densité de $\mathfrak{F}(H)$ dans $\mathscr{C}^{p_j}(H)$ la Proposition 4 montre que la fonction définie sur $[0,1]^k$:

$$a\mapsto \operatorname{Log}\sup|\operatorname{Tr}(T_1,\dots,T_k)|$$

est convexe sur $[0,1]^k$. Pour obtenir l'inégalité de a), il suffit de la prouver. Dans le cas particulier $p_j=\delta_j^i$ (δ_j^i symbole de Kronecker), $i=1,\dots,k$ ce qui est évident.

Prouvons le b).

Soit p' le conjugué de p. En vertu de a)

$$\sup_{\substack{S\in\mathscr{C}^{p'}(H)\\ |S|_{p'}\leqslant 1}}|\operatorname{Tr}(ST_1,\dots,T_k)|\leqslant |T_1|_{p_1},\dots,|T_k|_{p_k}.$$

Or, en vertu du Théorème 1,

$$|T_1,\dots,T_k|_p=\sup_{\substack{S\in\mathscr{C}^{p'}(H)\\ |S|_{p'}\leqslant 1}}|\operatorname{tr}(ST_1,\dots,T_k)|$$

d'où le résultat.

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On weighted $\mathscr{L}_1$-approximation by polynomials

by

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Abstract. The Jackson theorem for the whole real line is formulated and proved for $\mathscr{L}_1$-approximation with weight $\exp\{-\frac{1}{2}x^{2k}\}$ (Theorem 1). The main tool is a functional analytic duality principle of S. M. Nikolski (Lemma 4). This is combined with a suitable new H. Bohr-type inequality (Theorem 4).

1. Introduction. Let us denote by $\mathfrak{P}_n$ the set of polynomials of degree n at most and let

$$W_k(x)=\exp\{-\tfrac{1}{2}x^{2k}\}\quad(k=1,2,\dots).\tag{1}$$

For an arbitrary measurable function satisfying $FW_k\in\mathscr{L}_1(-\infty,\infty)$ we define

$$\varepsilon_n^{(1)}(W_k;F)=\inf_{\varphi_n\in\mathfrak{P}_n}\|(F-\varphi_n)W_k\|.\tag{2}$$

In (2) as well as in the whole of this paper we denote by $\|.\|$ the $\mathscr{L}_1(-\infty,\infty)$ norm.

We introduce the expression

$$\omega(\mathscr{L}_1;W_k;F;\delta)=\max_{0\leqslant t\leqslant\delta}\int_{-\infty}^{\infty}|F(x+t)W_k(x+t)-F(x)W_k(x)|\,dx+\int_{-\infty}^{\infty}[\tau(\delta^{\frac{1}{2k-1}}x)]^{2k-1}|F(x)|W_k(x)\,dx\quad(\delta>0)\tag{3}$$

and call it the generalized $\mathscr{L}_1$-continuity modulus with respect to the weight W_k. The $\tau(x)$ figuring in (3) is defined by

$$\tau(x)=\begin{cases}|x| & (|x|\leqslant 1),\\ 1 & (|x|>1).\end{cases}\tag{4}$$

We observe that $W_kF\in\mathscr{L}_1(-\infty,\infty)$ implies $\lim\limits_{\delta\to0}\omega(\mathscr{L}_1;W_k;F;\delta)=0$.

Moreover, we observe that

$$\tau(\theta x)\leqslant\theta\tau(x)\quad(\theta\geqslant1)\tag{5}$$

and consequently

$$\omega(\mathscr{L}_1; W_k; F; 2\delta) \leqslant 2\omega(\mathscr{L}_1; W_k; F; \delta). \tag{6}$$

In all our paper we denote by $c_1, c_2, \ldots$ positive numbers which are either absolute constants or depend on k but attaining numerical constant values for every fixed k ($k = 1, 2, \ldots$).

The main result of the present paper is as follows:

THEOREM 1. *Let $F(x)$ be the r-times iterated integral function of $F^{(r)}(x)$ and let $F^{(r)}W_k \in \mathscr{L}_1(-\infty, \infty)$ then*

$$\varepsilon_n^{(1)}(W_k; F) \leqslant c_1 e^{c_2 r} n^{-r\left(1-\frac{1}{2k}\right)} \omega\left(\mathscr{L}_1; W_k; F^{(r)}; n^{-1+\frac{1}{2k}}\right) \quad (n > 2r;\ r = 0, 1, \ldots).$$

Theorem 1 will be an obvious consequence of Theorem 6, Theorem 7 (see below) and of (4).

A tool of principal interest which we are going to apply is a Bohr-type inequality (29).

For the case $k = 1$ Theorem 1 was proved in our paper [1], where also the weighted $\mathscr{L}_p(-\infty, \infty)$-approximation problem was settled for this special weight. We do hope to return soon on the Bernstein-type converse of Theorem 1.

2. An inequality for polynomials. Let

$$\varrho_n(\eta) = \sum_{r=0}^{n-1} \frac{\eta^r}{r!} \quad (n = 1, 2, \ldots). \tag{7}$$

LEMMA 1. *We have for $n = 2, 3, \ldots$*

$$\varrho_n(\eta) > \tfrac{1}{2} e^{\eta} \quad (0 \leqslant \eta \leqslant c_3 n), \tag{8}$$

$$|\varrho_n(\eta)| < c_4 e^{\eta} \quad (-c_5 n \leqslant \eta \leqslant 0) \tag{9}$$

and

$$|\varrho_n(\eta)| > \frac{1}{2}\frac{|\eta|^{n-1}}{(n-1)!} - 1 > \frac{1}{4}\frac{|\eta|^{n-1}}{(n-1)!} \quad (\eta < -2n). \tag{10}$$

The proofs of (8), (5) and (10) are elementary. We indicate here only the proof of (9). Taking $0 \geqslant \eta \geqslant -e^{-2}n$ we have by Taylor's formula

$$|e^{\eta} - \varrho_n(\eta)| \leqslant \frac{|\eta|^n}{n!} \leqslant c_6 \frac{(e^{-2}n)^n}{n^n e^{-n}} = c_6 e^{-n} < c_6 e^{\eta};$$

this implies (9).

We refer to some well-known facts, needed in the sequel, concerning the theory of orthogonal polynomials (see e.g. [2]).

Let $0 \leqslant W(x) \in \mathscr{L}_1$, a weight function, $p_r(W; x)$ ($r = 0, 1, \ldots$) the sequence of orthogonal polynomials with respect to W and let

$$\lambda_{\nu+1}(W; x) = \left\{\sum_{r=0}^{\nu} p_r^2(W; x)\right\}^{-1}. \tag{11}$$

The expression (11), called Christoffel-function, has the following extremal property:

$$\lambda_{\nu+1}(W; x) = \min_{\substack{\varphi_n \in \mathfrak{P}_n \\ \varphi_n(x) \neq 0}} \varphi_n^{-2}(x) \int_{-\infty}^{\infty} \varphi_n^2(t) W(t)\, dt. \tag{12}$$

We denote by $U_r(x)$, as usual, the second kind Chebyshev polynomial of degree r. Clearly

$$U_r(x) = \left|\frac{(x+\sqrt{x^2-1})^{r+1} - (x-\sqrt{x^2-1})^{r+1}}{2\sqrt{x^2-1}}\right| \leqslant 2(2|x|)^r \quad (|x| \geqslant 2). \tag{13}$$

LEMMA 2. *We have for every $\psi_\nu \in \mathfrak{P}_\nu$ and $|x| \geqslant 2$*

$$\psi_\nu^2(x) \leqslant 2^{2\nu+2}(\nu+1)|x|^{2\nu} \cdot \int_{-1}^{1} \psi_\nu^2(t)\sqrt{1-t^2}\, dt \leqslant 2^{2\nu+2}(\nu+1)|x|^{2\nu} \int_{-1}^{1} \psi_\nu^2(t)\, dt. \tag{14}$$

Proof. Let $W_1(t) = \sqrt{1-t^2}$ for $|t| \leqslant 1$ and $W_1(t) = 0$ for $|t| > 1$. The polynomials orthogonal to $W_1(t)$ are

$$p_r(W_1; x) = \sqrt{\frac{2}{\pi}}\, U_r(x) \quad (r = 0, 1, \ldots).$$

We infer from (11), (12)

$$\int_{-1}^{1} \psi_\nu^2(t)\sqrt{1-t^2}\, dt \geqslant \lambda_{\nu+1}(W_1; x)\psi_\nu^2(x) = \left[\frac{2}{\pi}\sum_{r=0}^{\nu} U_r^2(x)\right]^{-1} \psi_\nu^2(x); \tag{15}$$

Lemma 2 is a consequence of (13) and (15).

THEOREM 2. *We have for a proper choice of c_7 and c_9 and for every $\chi_n \in \mathfrak{P}_n$ ($n = 2, 3, \ldots$)*

$$\int_{c_7 n^{1/2k}}^{\infty} \chi_n^2(\xi) e^{-\xi^{2k}}\, d\xi \leqslant e^{-c_8 n} \int_{-c_9 n^{1/2k}}^{c_9 n^{1/2k}} \chi_n^2(x) e^{-x^{2k}}\, dx. \tag{16}$$

Proof. We apply Lemma 2 with $\nu = n + 2k(n-1)$ and inserting $\psi_\nu(x) = \chi_n(c_9 n^{1/2k} x)\, \varrho_n(-c_9^{2k} n x^{2k}/2) \in \mathfrak{P}_\nu$.

$$\chi_n^2(c_9 n^{1/2k} x)\, \varrho_n^2(-c_9^{2k} n x^{2k}/2) \leqslant 2^{2\nu+2}(\nu+1)|x|^{2\nu} \int_{-1}^{1} \chi_n^2(c_9 n^{1/2k} t)\, \varrho_n^2(-c_9^{2k} n t^{2k}/2)\, dt$$

$$= c_9^{-1} n^{-1/2k} 2^{2\nu+2}(\nu+1)|x|^{2\nu} \int_{-c_9 n^{1/2k}}^{c_9 n^{1/2k}} \chi_n^2(t)\, \varrho_n^2(-t^{2k}/2)\, dt \quad (|x| \geqslant 2)$$

i.e.

$$\text{(17)}\qquad \chi_n^2(\xi)\,\varrho_n^2(-\xi^{2k}/2) \le c_9^{-1}n^{-1/2k}2^{2\nu+2}(\nu+1)\cdot|c_9^{-1}n^{-1/2k}\xi|^{2\nu}\int\limits_{-c_9n^{1/2k}}^{c_9n^{1/2k}}\chi_n^2(t)\,\varrho_n^2(-t^{2k}/2)\,dt \qquad (|\xi|\le 2c_9n^{1/2k}).$$

We insert $\eta = -\xi^{2k}/2$ in (10) and apply it on the left-hand side of (17). This is allowed provided that $|\xi| > c_{10}n^{1/2k}$ where $c_{10} > 2c_9$ is chosen sufficiently large. At the same time, in the integrand of the right-hand side, (9) is applicable with $\eta = -t^{2k}/2$. Choosing, in the case of necessity, c_9 even smaller:

$$\chi_n^2(\xi) \le c_{11}2^{2\nu+2n}|\xi|^{2n}\frac{n^{2(n-1)}}{[(n-1)!]^2}(\nu+1)n^{-n/k}c_9^{-2\nu}\cdot\int\limits_{-c_9n^{1/2k}}^{c_9n^{1/2k}}\chi_n^2(t)\,e^{-t^{2k}}dt$$

$$\le e^{c_{12}n}n^{-n/k}|\xi|^{2n}\int\limits_{-c_9n^{1/2k}}^{c_9n^{1/2k}}\chi_n^2(t)\,e^{-t^{2k}}dt.$$

From this we obtain (16) by a straightforward calculation if only c_7 is choosen sufficiently large.

3. Estimation of the Christoffel functions. In this paragraph we will need a sequence $\{K_\mu(s,t)\}$ with the following properties:

(a) for a fixed value of s, $-1\le s\le 1$, $K_\mu(s,t)$ is a polynomial of degree μ at most with respect to t;

(b) $K_\mu(s,s) = 1$

(c) $\int_{-1}^{1}K_\mu^2(s,t)\,dt \le c_{13}\mu^{-1}$ $\qquad (\mu = 1,2,\ldots)$.

A possible definition of such polynomials is to be found in [2], V. (6.3)

THEOREM 3. *We have for* $\nu = 1,2,\ldots$

$$\text{(18)}\qquad \lambda_{\nu+1}(W_k^2;\,\xi) \le c_{14}\nu^{-1+1/2k}W_k^2(\xi) \qquad (|\xi| < c_{15}\nu^{1/2k}).$$

Remark. Replacing x by a new variable $2^{-1/2k}x$ we obtain from (18)

$$\text{(19)}\qquad \lambda_{\nu+1}(W_k;\,\xi) \le c_{16}\nu^{-1+1/2k}W_k(\xi) \qquad (|\xi| < c_{17}\nu^{1/2k}).$$

Proof of Theorem 3. We conclude from (12) and (16) that for every $\varphi_\nu\in\mathfrak{P}_\nu$, $\varphi_\nu(x)\ne 0$

$$\text{(20)}\qquad \lambda_{\nu+1}(W_k^2;\,x) \le c_{18}\varphi_\nu^{-2}(x)\int\limits_{-c_9\nu^{1/2k}}^{c_9\nu^{1/2k}}\varphi_\nu^2(t)W_k^2(t)\,dt.$$

Let us put in (20)

$$\text{(21)}\qquad \varphi_\nu(t) = K_{[\frac{\nu}{2}]}(c_9^{-1}\nu^{-1/2k}x;\,c_9^{-1}\nu^{-1/2k}t)\,\varrho_{[\frac{\nu}{4k}]}(t^{2k}/2).$$

Then we have by (b) and (8)

$$\text{(22)}\qquad \varphi_\nu(x) \ge \tfrac12 W_k^{-1}(x) \qquad (|x|\le c_{19}\nu^{1/2k})$$

and then by the evident inequality $\varrho_m(t^{2k}/2)\le W_k^{-1}(t)$

$$\text{(23)}\qquad \int\limits_{-c_9\nu^{1/2k}}^{c_9\nu^{1/2k}}\varphi_\nu^2(t)W_k^2(t)\,dt \le \int\limits_{-c_9\nu^{1/2k}}^{c_9\nu^{1/2k}}K_{[\frac{\nu}{2}]}^2(c_9^{-1}\nu^{-1/2k}x;\,c_9^{-1}\nu^{-1/2k}t)\,dt$$

$$= c_9\nu^{1/2k}\int\limits_{-1}^{1}K_{[\frac{\nu}{2}]}^2(c_9^{-1}\nu^{-1/2k}x;\,t)\,dt \le 2c_9c_{13}\nu^{-1+1/2k}.$$

We see from (22), (23) and (20) that (18) holds for $c_{15} = \min(c_9, c_{19})$ with $c_{14} = 8c_9c_{13}c_{18}$.

4. A Harald Bohr-type inequality. Let

$$\text{(24)}\qquad \Gamma(x,t) = \begin{cases}1 & (x\le t),\\ 0 & (x>t).\end{cases}$$

We consider $\Gamma(x,t)$ and—in what follows—$K_n(x,t)$ as a function of x for each fix value of the parameter t.

LEMMA 3. *We have for every fixed positive integer k and* $n = 1,2,\ldots$

$$\text{(25)}\qquad \varepsilon_n^{(1)}[W_k;\,\Gamma(x,t)] \le c_{20}n^{-1+1/2k}W_k(t) \qquad (-\infty<t<\infty).$$

Proof. By the Markov–Stieltjes construction we obtain a $K_n(x,t)\in\mathfrak{P}_n$ so that (see e. g. [2], § I. 5).

$$\text{(26)}\qquad \int\limits_{-\infty}^{\infty}|\Gamma(x,t)-K_n(x,t)|W_k(x)\,dx \le \lambda_{[n/2]+1}(W_k;\,t).$$

We see from (22) and (19) that (25) is valid for $|t|\le c_{21}n^{1/2k}$. For $t > c_{21}n^{1/2k}$ we approximate $\Gamma(x,t)$ by $\zeta_1(x)\equiv 1\in\mathfrak{P}_n$ and for $t<-c_{21}n^{1/2k}$ by $\zeta_0(x)\equiv 0\in\mathfrak{P}_n$. In this way (25) is true for every real t.

THEOREM 4. *Let $G(x)$ be measurable and*

$$\text{(27)}\qquad \operatorname*{vrai\,max}_{-\infty<x<\infty}|G(x)|W_k(x)\le 1$$

and for every $\varphi_n\in\mathfrak{P}_n$ let

$$\text{(28)}\qquad \int\limits_{-\infty}^{\infty}G(x)\varphi_n(x)W_k^2(x)\,dx = 0 \qquad (\varphi_n\in\mathfrak{P}_n)$$

then we have

$$\text{(29)}\qquad \Big|\int\limits_{-\infty}^{t}G(x)W_k^2(x)\,dx\Big| \le c_{22}n^{-1+1/2k}W_k(t) \qquad (-\infty<t<\infty).$$

Proof. Denoting the integral between the modulus signs in (29) by $H(t)$ we have

$$H(t) = \int_{-\infty}^{\infty} \Gamma(x,t) G(x) W_k^2(x)\, dx \tag{30}$$

$$= \int_{-\infty}^{\infty} [\Gamma(x,t) - \varphi_n(x)] G(x) W_k^2(x)\, dx,$$

where φ_n is any element of $\mathfrak{P}_n$. We choose φ_n in accordance with Lemma 3 so that

$$\int_{-\infty}^{\infty} |\Gamma(x,t) - \varphi_n(x)| W_k(x)\, dx \leqslant c_{20} n^{-1+1/2k} W_k(t);$$

consequently (29) holds.

5. A weighted L_1-variant of Jackson's inequality.

LEMMA 4. *We have*

$$\varepsilon_n^1(W_k; F) = \sup_{G \in \boldsymbol{G}} \int_{-\infty}^{\infty} F(x) G(x) W_k^2(x)\, dx$$

where $\boldsymbol{G}$ *is the set of measurable functions* $G(x)$ *satisfying* (27) *and* (28) (*as far as* (28) *is considered*, $\varphi_n \in \mathfrak{P}_n$ *arbitrarily*).

Lemma 4 is a transcription of S. M. Nikolski's duality theorem ([3], Corollary 2).

THEOREM 5. *Let* $F(x)$ *be of bounded variation in every finite interval, then we have—provided that the integral on the right-hand side of* (31) *is finite—*

$$\varepsilon_n^{(1)}(W_k; F) \leqslant c_{23} n^{-1+1/2k} \int_{-\infty}^{\infty} W_k(x) |dF(x)|. \tag{31}$$

Remark. In particular, if we replace in (31) $F(x)$ by a function $g(x)$ which is absolutely continuous in every finite interval then

$$\varepsilon_n^{(1)}(W_k; g) \leqslant c_{23} n^{-1+1/2k} \int_{-\infty}^{\infty} |g'(x)| W_k(x)\, dx. \tag{32}$$

Proof of Theorem 5. First we show that from the existence of the integral in (31) it follows

$$\lim_{|t| \to \infty} F(t) W_k(t) = 0. \tag{33}$$

For an arbitrary $\varepsilon > 0$ let us choose $\Omega > 0$ so large that

$$\int_{\Omega}^{\infty} W_k(x) |dF(x)| < \varepsilon.$$

Thus we have for $t > \Omega$

$$|F(t)| \leqslant |F(\Omega)| + \left| \int_{\Omega}^{t} W_k^{-1}(x) W_k(x) |dF(x)| \right| \leqslant |F(\Omega)| + W_k^{-1}(t)\varepsilon.$$

This shows the validity of (33) for $t \to \infty$; for $t \to -\infty$ the proof is the same.

Secondly, we observe that taking $G \in \boldsymbol{G}$, $H(t) = \int_{-\infty}^{t} G(x) W_k^2(x)\, dx$ we have by Theorem 4

$$|H(t)| \leqslant c_{22} n^{-1+1/2k} W_k(t). \tag{34}$$

Taking (33) in consideration we obtain by a partial integration

$$\int_{-\infty}^{\infty} F(x) G(x) W_k^2(x)\, dx = - \int_{-\infty}^{\infty} H(x)\, dF(x). \tag{35}$$

Theorem 5 follows from Lemma 4, (35) and (34).

THEOREM 6. *Let* f *be absolutely continuous and* $f' W_k \in L_1$, *then*

$$\varepsilon_n^{(1)}(W_k; f) \leqslant c_{24} n^{-1+1/2k} \varepsilon_{n-1}^{(1)}(W_k; f'). \tag{36}$$

Proof. Let $\varphi_{n-1} \in \mathfrak{P}_{n-1}$ and let

$$\|(f' - \varphi_{n-1}) W_k\| \leqslant 2\varepsilon_{n-1}^{(1)}(W_k; f').$$

Seting $\psi_n(x) = f(0) + \int_0^x \varphi_{n-1}(t)\, dt \in \mathfrak{P}_n$ we apply (32) for $F = f - \psi_n$:

$$\begin{aligned} \varepsilon_n^{(1)}(W_k; f) &= \varepsilon_n^{(1)}(W_k; f - \psi_n) \\ &\leqslant c_{23} n^{-1+1/2k} \int_{-\infty}^{\infty} |f'(x) - \varphi_{n-1}(x)| W_k(x)\, dx \\ &\leqslant 2c_{23} n^{-1+1/2k} \varepsilon_{n-1}^{(1)}(W_k; f'). \end{aligned}$$

6. Estimation of the L_1-approximation by the generalized continuity modulus. Let $f W_k \in L_1$ and let $h_n = n^{-1+1/2k}$

We set

$$\Phi_n(x) = \begin{cases} \Phi(x) = F(x) W_k(x) & (|x| \leqslant n^{1/2k}), \\ 0 & (|x| > n^{1/2k}) \end{cases} \tag{37}$$

and

$$F_n(x) = W_k^{-1}(x) h_n^{-1} \int_x^{x+h_n} \Phi_n(t)\, dt. \tag{38}$$

LEMMA 5. *We have*

$$\|(F - F_n) W_k\| \leqslant c_{25}\, \omega(L_1; W_k; F; h_n) \tag{39}$$

and

$$\|F_n' W_k\| \leqslant c_{26} h_n^{-1} \omega(L_1; W_k; F; h_n). \tag{40}$$

Proof. Obviously

(41) $$\|(F-F_n)W_k\| \leqslant \int\limits_{-\frac{1}{2}n^{1/2k}}^{\frac{1}{2}n^{1/2k}} |F(x)-F_n(x)|W_k(x)\,dx + \int\limits_{|x|>\frac{1}{2}n^{1/2k}} |F(x)|W_k(x)\,dx + \int\limits_{|x|>\frac{1}{2}n^{1/2k}} |F_n(x)|W_k(x)\,dx$$

and

$$\int\limits_{\frac{1}{2}n^{1/2k}}^{\infty} |F_n(x)|W_k(x)\,dx \leqslant h_n^{-1}\int\limits_0^{h_n}\Big\{\int\limits_{t+\frac{1}{4}n^{1/2k}}^{\infty}|F(y)|W_k(y)\,dy\Big\}\,dt \leqslant \int\limits_{\frac{1}{4}n^{1/2k}}^{\infty}|F(y)|W_k(y)\,dy.$$

In the last part of this chain of inequalities we used that $h_n < \frac{1}{4}n^{1/2k}$ if $n \geqslant 4$. We have a similar estimation for the integral between $-\infty$ and $-\frac{1}{2}n^{1/2k}$ therefore by virtue of (4) and (5)

(42) $$\int\limits_{|x|>\frac{1}{2}n^{1/2k}} |F_n(x)|W_k(x)\,dx + \int\limits_{|x|>\frac{1}{2}n^{1/2k}} |F(x)|W_k(x)\,dx \leqslant c_{27}\int\limits_{-\infty}^{\infty}[\tau(n^{-1/2k}x)]^{2k-1}|F(x)|W_k(x)\,dx.$$

Moreover

(43) $$\int\limits_{-\frac{1}{2}n^{1/2k}}^{\frac{1}{2}n^{1/2k}} |F_n(t)-F(t)|W_k(t)\,dt \leqslant h_n^{-1}\int\limits_{-\frac{1}{2}n^{1/2k}}^{\frac{1}{2}n^{1/2k}}\Big\{\int\limits_0^{h_n}|F(x+t)W_k(x+t)-F(x)W_k(x)|\,dt\Big\}\,dx \leqslant \max_{0\leqslant t\leqslant h_n}\int\limits_{-\infty}^{\infty}|F(x+t)W_k(x+t)-F(x)W_k(x)|\,dx.$$

The statement (39) is a consequence of (41), (42), (43) and the definition (3) of $\omega(L_1; W_k)$.

Now we turn to the proof of (40). By differentiation

(44) $$F_n'(x) = -kx^{2k-1}F_n(x)+h_n^{-1}W_k^{-1}(x)[\Phi_n(x+h_n)-\Phi_n(x)].$$

We consider now the weighted integrals of these two terms. For $|x| > 2n^{1/2k}$, $F_n(x)$ vanishes as a consequence of definition. It follows

(45) $$\int\limits_{-\infty}^{\infty}|x|^{2k-1}|F_n(x)|W_k(x)\,dx \leqslant \int\limits_{-2n^{1/2k}}^{2n^{1/2k}}|x|^{2k-1}|F_n(x)-F(x)|W_k(x)\,dx + \int\limits_{-2n^{1/2k}}^{2n^{1/2k}}|x|^{2k-1}|F(x)|W_k(x)\,dx \leqslant c_{28}h_n^{-1}\|(F_n-F)W_k\| + c_{29}h_n^{-1}\int\limits_{-\infty}^{\infty}[\tau(n^{-1/2k}x)]^{2k-1}|F(x)|W_k(x)\,dx.$$

Taking the just proved inequality (39) in consideration we see that the weighted integral of the first term in (44) is smaller than $c_{30}h_n^{-1}\omega(L_1; W_k; F; h_n)$. We estimate the contribution of the second term in (44) by

(46) $$h_n^{-1}\int\limits_{-\infty}^{\infty}|\Phi_n(x+h_n)-\Phi_n(x)|\,dx \leqslant 2h_n^{-1}\int\limits_{-\infty}^{\infty}|\Phi_n(x)-F(x)W_k(x)|\,dx + h_n^{-1}\int\limits_{-\infty}^{\infty}|F(x+h_n)W_k(x+h_n)-F(x)W_k(x)|\,dx.$$

The integrand of the first integral on the right-hand side vanishes for $|x| \leqslant n^{1/2k}$, consequently this term is smaller than

$$c_{31}h_n^{-1}\int\limits_{-\infty}^{\infty}[\tau(n^{-1/2k}x)]^{2k-1}|F(x)|W_k(x)\,dx.$$

The second term on the right of (46) is evidently smaller than $h_n^{-1}\omega(L_1; W_k; F; h_n)$. All in all, we obtain the desired second inequality (40).

THEOREM 7. *We have for every $FW_k \in L_1$ with $h_n = n^{-1+1/2k}$*

(47) $$\varepsilon_n^{(1)}(W_k; F) \leqslant c_{32}\,\omega(L_1: W_k; F; h_n) \quad (n = 1, 2, \ldots).$$

Proof. By (39)

$$\varepsilon_n^{(1)}(W_k; F) \leqslant \varepsilon_n^{(1)}(W_k; F_n) + c_{25}\,\omega(L_1; W_k; F; h_n).$$

Applying (32) with $g = F_n$ and by virtue of (40) it results inequality (47).

As it has been already pointed out in the introduction, our main statement Theorem 1 is obtained by the combination of Theorem 6 and Theorem 7.

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